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THE Q_α-CONVOLUTION OF ARITHMETIC FUNCTIONS AND SOME OF ITS PROPERTIES

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Abstract

Let α be an arithmetic function such that $\alpha(n) \neq 0$ $(n \in \mathbb{N})$. The Q_α-convolution of two arithmetic functions is defined as

$$(f \diamond g)(n) = \sum_{ij=n} \frac{\alpha(n)}{\alpha(i)\alpha(j)} f(i)g(j).$$

Basic properties of the Q_α-convolution and characterizations of completely multiplicative functions using Q_α-convolution are derived. The solubility of the equation

$$T_\alpha g := a_d \diamond g^{\diamond d} + a_{d-1} \diamond g^{\diamond(d-1)} + \cdots + a_1 \diamond g + a_0 = 0$$

with fixed arithmetic functions $a_d(\neq 0), a_{d-1}, \ldots, a_1, a_0$ is investigated.

1 Introduction

An arithmetic function is a complex-valued function whose domain is the set of positive integers, $\mathbb{N}$, and whose range is a subset of $\mathbb{C}$. Let $\mathcal{A}$ be the set of arithmetic functions equipped with addition, (usual) multiplication and Dirichlet

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convolution defined over $\mathbb{N}$, respectively, by

$$(f+g)(n) = f(n) + g(n),\ fg(n) = f(n)g(n),\ (f*g)(n) = \sum_{xy=n} f(x)g(y).$$

The usual multiplication identity of $\mathcal{A}$ is the unit function u defined by $u(n) = 1$ for all $n \in \mathbb{N}$. The Dirichlet convolution identity $I \in \mathcal{A}$ is defined by $I(1) = 1$ and $I(n) = 0$ for $n > 1$. An arithmetic function $f \in \mathcal{A}$ is said to be multiplicative if

$$f(mn) = f(m)f(n) \tag{1}$$

whenever $\gcd(m,n) = 1$, and is said to be completely multiplicative if (1) holds for all $m, n \in \mathbb{N}$. For brevity, put

$$\mathcal{M} = \{f \in \mathcal{A}\ ;\ f \text{ is multiplicative}\}\,,\ \mathcal{C} = \{f \in \mathcal{A}\ ;\ f \text{ is completely multiplicative}\}\,.$$

In [4], Lambek, see also [1], gave the following characterization of completely multiplicative functions: f is completely multiplicative if and only if $f(g*h) = fg*fh$ for every pair $g, h \in \mathcal{A}$.

In [7], Tóth and Haukkanen introduced the binomial convolution which is defined for $f, g \in \mathcal{A}$ by

$$(f \circ g)(n) = \sum_{xy=n} \frac{\xi(n)}{\xi(x)\xi(y)} f(x)g(y),$$

where $\xi(n) = \prod_{prime\, p} \nu_p(n)!$; $\nu_p(n)$ being the highest power of p dividing n. Tóth-Haukkanen proved that a similar characterization under the Dirichlet convolution holds for the binomial convolution; note that the arithmetic function I is also the identity with respect to binomial convolution.

In this work, we define a new kind of convolution called the Q_α-convolution, extending the binomial convolution. Let $\alpha \in \mathcal{A}$ be such that $\alpha(n) \neq 0$ for all $n \in \mathbb{N}$. The Q_α-convolution of $f, g \in \mathcal{A}$ is defined as

$$(f \diamond g)(n) = \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} f(x)g(y). \tag{2}$$

Clearly, the Q_α-convolution can be expressed in term of the Dirichlet convolution as

$$f \diamond g = \alpha\left(\frac{f}{\alpha} * \frac{g}{\alpha}\right). \tag{3}$$

Our objectives are first to establish basic properties of Q_α-convolution, second to use it to characterize completely multiplicative functions and finally to solve a polynomial Q_α-convolution equation similar to the one in [2].

2 Basic properties

Our first information deals with basic structure, whose proof is just an easy exercise.

Theorem 1. *The algebras* $(\mathcal{A}, +, *, \mathbb{C})$ *and* $(\mathcal{A}, +, \diamond, \mathbb{C})$ *are isomorphic under the map* $f \mapsto f/\alpha$.

We denote the inverses of $f \in \mathcal{A}$ under the Dirichlet convolution and the Q_α convolution by f^{-1*} and $f^{-1\diamond}$, respectively. It is easy to check that f^{-1*} and $f^{-1\diamond}$ exist if and only if $f(1) \neq 0$. Moreover, we have

Theorem 2. *If* $f \in \mathcal{A}$ *satisfies* $f(1) \neq 0$, *then*

$$f^{-1*} = \frac{(\alpha f)^{-1\diamond}}{\alpha}, \tag{4}$$

$$f^{-1\diamond} = \alpha \left(\frac{f}{\alpha} \right)^{-1*}. \tag{5}$$

Proof. Since $f * f^{-1*} = I$, from (2) we have $\alpha f \diamond \alpha f^{-1*} = \alpha I$. Thus,

$$\alpha f^{-1*} = (\alpha f)^{-1\diamond},$$

and (4) follows. On the other hand, from $f \diamond f^{-1\diamond} = \alpha I$, we get

$$\frac{f}{\alpha} * \frac{f^{-1\diamond}}{\alpha} = I,$$

from which (5) follows. □

Next, we give a characterization of completely multiplicative functions through the use of the distributivity with respect to Q_α-convolution.

Theorem 3. *Let* $f \in \mathcal{A}$ *be multiplicative. Then* $f \in \mathcal{C}$ *if and only if* $f(g \diamond h) = fg \diamond fh$ *for all* $g, h \in \mathcal{A}$.

Proof. Assume that $f \in \mathcal{C}$. Let $g, h \in \mathcal{A}$. Then

$$f(g \diamond h) = f\alpha \left(\frac{g}{\alpha} * \frac{h}{\alpha} \right) = \alpha \left(\frac{fg}{\alpha} * \frac{fh}{\alpha} \right) = fg \diamond fh.$$

Conversely, assume that $f(g \diamond h) = fg \diamond fh$ for all $g, h \in \mathcal{A}$. Then

$$\alpha f(g * h) = f(\alpha g \diamond \alpha h) = \alpha fg \diamond \alpha fh = \alpha \left(\frac{\alpha fg}{\alpha} * \frac{\alpha fh}{\alpha} \right) = \alpha(fg * fh)$$

and so $f(g * h) = fg * fh$. By Lembek's Theorem, [7], $f \in \mathcal{C}$. □

Theorem 4. *Let $f \in \mathcal{M}$. Then $f \in \mathcal{C}$ if and only if $(fg)^{-1\diamond} = fg^{-1\diamond}$ for all $g \in \mathcal{A}$ with $g(1) \neq 0$.*

Proof. If $f \in \mathcal{C}$, then by Theorem 3 we get

$$\alpha I = f\alpha I = f(g \diamond g^{-1\diamond}) = fg \diamond fg^{-1\diamond},$$

i.e., $(fg)^{-1\diamond} = fg^{-1\diamond}$. Conversely, if $(fg)^{-1\diamond} = fg^{-1\diamond}$ for all $g \in \mathcal{A}$ with $g(1) \neq 0$, then using also Theorem 2, we get

$$\alpha\, f^{-1(*)} = \alpha \left(\frac{f\alpha}{\alpha}\right)^{-1*} = (f\alpha)^{-1\diamond} = f\alpha^{-1\diamond} = f\left(\alpha\left(\frac{\alpha}{\alpha}\right)^{-1*}\right) = f\;\alpha\;\mu$$

i.e., $f^{-1*} = f\mu$. By a well-known characterization, see [1], $f \in \mathcal{C}$. $\square$

In [5], Langford used the concept of discriminative and partially discriminative products to derive necessary and sufficient conditions for complete multiplicativity. Our next aim is to extend these results through generalizing the Q_α-convolution. Before doing so, we need one some more definitions.

Let $r \in \mathbb{N},\ r \geq 2$ and let $g_1, g_2, \ldots, g_r \in \mathcal{A}\backslash\ \{0\}$. We say that the product $k = g_1 \diamond g_2 \diamond \cdots \diamond g_r$ is

- r-fold Q_α-discriminative, or r.q.d. for short, if the relation

$$\alpha(1)^{r-1}k(n) = \sum_{j=1}^{r} g_1(1)\cdots g_{j-1}(1)g_j(n)g_{j+1}(1)\cdots g_r(1) \tag{6}$$

  holds only when n is prime;

- r-fold Q_α-partially discriminative, or r.q.p.d. for short, if for every prime power p^i $(i \in \mathbb{N})$ the relation

$$\alpha(1)^{r-1}k(p^i) = \sum_{j=1}^{r} g_1(1)\cdots g_{j-1}(1)g_j(p^i)g_{j+1}(1)\cdots g_r(1) \tag{7}$$

  implies that $i = 1$;

- r-fold Q_α-semi-discriminative, or r.q.s.d. for short, if the relation

$$\alpha(1)^{r-1}k(n) = \sum_{j=1}^{r} g_1(1)\cdots g_{j-1}(1)g_j(n)g_{j+1}(1)\cdots g_r(1) \tag{8}$$

  holds only when $n = 1$ or n is prime.

Theorem 5. *Let $f \in \mathcal{A}$ with $f(1) \neq 0$. If f distributes over a an r.q.d. product, then $f(1)$ is an $(r-1)^{th}$ root of unity and $f(p_1\cdots p_m) = f(p_1)\cdots f(p_m)$ for any primes $p_1, \cdots, p_m$.*

Proof. Assume that f distributes over a an r.q.d. product $k = g_1 \diamond g_2 \diamond \cdots \diamond g_r$. We first show that $k(1) \neq 0$. If $k(1) = 0$, then

$$\frac{1}{\alpha(1)^{r-1}} \sum_{j=1}^{r} g_1(1) \cdots g_{j-1}(1) g(1) g_{j+1}(1) \cdots g_r(1) = 0 = k(1),$$

i.e.

$$\alpha(1)^{r-1} k(1) = \sum_{j=1}^{r} g_1(1) \cdots g_{j-1}(1) g_j(1) g_{j+1}(1) \cdots g_r(1),$$

so the equation (6) holds for $n = 1$, which is a contradiction. Thus $k(1) \neq 0$. Since

$$fk = f(g_1 \diamond g_2 \diamond \cdots \diamond g_r) = fg_1 \diamond fg_2 \diamond \cdots \diamond fg_r,$$

we get

$$\begin{aligned} f(1)k(1) &= \alpha(1) \frac{fg_1}{\alpha}(1) \frac{fg_2}{\alpha}(1) \cdots \frac{fg_r}{\alpha}(1) \\ &= f(1)^r \left(\frac{g_1(1) g_2(1) \cdots g_r(1)}{\alpha(1)^{r-1}} \right) \\ &= f(1)^r k(1). \end{aligned}$$

It follows from $f(1)k(1) \neq 0$ that $f(1)^{r-1} = 1$. Next we show that for all primes $p_1, \ldots, p_m$ (not necessary distinct),

$$f(p_1 \cdots p_m) = f(p_1) \cdots f(p_m). \tag{9}$$

We proceed by induction on m. This is trivial if $m = 1$, so assume that $n = p_1 \cdots p_m;\ m \geq 2$ and that (9) is true for all integers whose number of prime factors (not necessary distinct) is less then m. Since

$$f(g_1 \diamond g_2 \diamond \cdots \diamond g_r) = fg_1 \diamond fg_2 \diamond \cdots \diamond fg_r,$$

then

$$f(p_1 \cdots p_m) \alpha(p_1 \cdots p_m) \sum_{d_1 \cdots d_r = p_1 \cdots p_m} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_r}{\alpha}(d_r)$$

$$= \alpha(p_1 \cdots p_m) \sum_{d_1 \cdots d_r = p_1 \cdots p_m} \frac{fg_1}{\alpha}(d_1) \cdots \frac{fg_r}{\alpha}(d_r).$$

Using the induction hypothesis, we get

$$\begin{aligned}
&f(p_1\cdots p_m)\sum_{\substack{d_1\cdots d_r=p_1\cdots p_m\\ d_j\neq p_1\cdots p_m \text{ for all } j\in\{1,\dots,r\}}}\frac{g_1}{\alpha}(d_1)\cdots\frac{g_1}{\alpha}(d_r)\\
&\quad+f(p_1\cdots p_m)\sum_{\substack{d_1\cdots d_r=p_1\cdots p_m\\ d_j= p_1\cdots p_m \text{ for some } j\in\{1,\dots,r\}}}\frac{g_1}{\alpha}(d_1)\cdots\frac{g_1}{\alpha}(d_r)\\
&=f(p_1)\cdots f(p_m)\sum_{\substack{d_1\cdots d_r=p_1\cdots p_m\\ d_j\neq p_1\cdots p_m \text{ for all } j\in\{1,\dots,r\}}}\frac{g_1}{\alpha}(d_1)\cdots\frac{g_1}{\alpha}(d_r)\\
&\quad+f(p_1\cdots p_m)f(1)^{r-1}\sum_{\substack{d_1\cdots d_r=p_1\cdots p_m\\ d_j= p_1\cdots p_m \text{ for some } j\in\{1,\dots,r\}}}\frac{g_1}{\alpha}(d_1)\cdots\frac{g_1}{\alpha}(d_r).
\end{aligned}$$

But $f(1)^{r-1}=1$. Hence

$$[f(p_1\cdots p_m)-f(p_1)\cdots f(p_m)]\sum_{\substack{d_1\cdots d_r=p_1\cdots p_m\\ d_j\neq p_1\cdots p_m \text{ for all } j\in\{1,\dots,r\}}}\frac{g_1}{\alpha}(d_1)\cdots\frac{g_1}{\alpha}(d_r)=0.$$

Since

$$\begin{aligned}
&\sum_{\substack{d_1\cdots d_r=p_1\cdots p_m\\ d_j\neq p_1\cdots p_m \text{for all } j\in\{1,\dots,r\}}}\frac{g_1}{\alpha}(d_1)\cdots\frac{g_1}{\alpha}(d_r)\\
&=k(p_1\cdots p_m)-\sum_{\substack{d_1\cdots d_r=p_1\cdots p_m\\ d_j= p_1\cdots p_m \text{ for some } j\in\{1,\dots,r\}}}\frac{g_1}{\alpha}(d_1)\cdots\frac{g_1}{\alpha}(d_r)
\end{aligned}$$

and k is $\diamond-$r.d. product, it follows that

$$\sum_{\substack{d_1\cdots d_r=p_1\cdots p_m\\ d_j\neq p_1\cdots p_m \text{ for all } j\in\{1,\dots,r\}}}\frac{g_1}{\alpha}(d_1)\cdots\frac{g_1}{\alpha}(d_r)\neq 0.$$

This show that $f(p_1\cdots p_m)=f(p_1)\cdots f(p_m)$, as desired. □

If we take $f(1)=1$, in theorem 5, then we have the following corollary.

Corollary 1. *Suppose that $f(1)=1$. Then f is completely multiplicative if and only if it distributes over an r.q.d. product.*

Proof. If f is completely multiplicative then $f(g\diamond h)=fg\diamond fh$ for all $g,h\in\mathcal{A}$. Assume now that f distributes over an r.q.d. product. Then $f(p_1\cdots p_m)=f(p_1)\cdots f(p_m)$ for any primes $p_1,\cdots,p_m$, by Theorem 5. Hence f is completely multiplicative. □

Theorem 6. *Suppose that f is multiplicative. Then f is completely multiplicative if and only if it distributes over an r.q.p.d. product.*

Proof. If f is completely multiplicative then $f(g \diamond h) = fg \diamond fh$ for all $g, h \in \mathcal{A}$. Conversely, assume that f distributes over an r.q.p.d. product $k = g_1 \diamond g_2 \diamond \cdots \diamond g_r$. Since f is multiplicative, it suffices to show that for all primes p, $f(p^m) = f(p)^m$; for all $m \in \mathbb{N}$. The case $m = 1$ being trivial, so assume that $m \geq 2$ and $f(p^t) = f(p)^t$ holds for $t < m$. The remaining proof is similar to that of the last half of Theorem 5 by induction on m, but making use of k being an r.q.p.d. product. □

Corollary 2. *If we take $f(1) = 1$, then f is completely multiplicative if and only if it distributes over a an r.q.s.d. product.*

3 Solving a polynomial Q_α-convolution equation

In [2], Glöckner, Lucht and Porubský solved the polynomial convolution equation

$$Tg = a_d * g^{*d} + a_{d-1} * g^{*(d-1)} + \cdots + a_1 * g + a_0 = 0 \tag{10}$$

with fixed coefficients $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$ and $a_d \neq 0$ by showing that it has a solution $g \in \mathcal{A}$ satisfying $g(1) = z_o$, if z_0 is a simple zero of the polynomial

$$f(z) = a_d(1)z^d + a_{d-1}(1)z^{d-1} + \cdots + a_1(1)z + a_0(1).$$

We show next that the polynomial Q_α-convolution equation

$$T_\alpha g = a_d \diamond g^{\diamond d} + a_{d-1} \diamond g^{\diamond(d-1)} + \cdots + a_1 \diamond g + a_0 = 0$$

has solution $g \in \mathcal{A}$ under similar conditions.

Theorem 7. *For $d \in \mathbb{N}$, let $T_\alpha : \mathcal{A} \to \mathcal{A}$ be defined by*

$$T_\alpha g = a_d \diamond g^{\diamond d} + a_{d-1} \diamond g^{\diamond(d-1)} + \cdots + a_1 \diamond g + a_0 \tag{11}$$

with $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$ and $a_d \neq 0$. If z_0 is a simple zero of the polynomial

$$f(z) = a_d(1)z^d + a_{d-1}(1)z^{d-1} + \cdots + a_1(1)z + a_0(1), \tag{12}$$

then there exists a unique solution $g \in \mathcal{A}$ to the convolution equation $T_\alpha g = 0$ satisfying $g(1) = \alpha(1)z_o$.

Proof. Assume that z_0 is a simple zero of $f(z)$. If $g \in \mathcal{A}$ satisfies $T_\alpha g = 0$, then

$$\begin{aligned} 0 &= \alpha(1)\left(\frac{a_d}{\alpha}(1)\left(\frac{g}{\alpha}\right)(1)^d + \frac{a_{d-1}}{\alpha}(1)\left(\frac{g}{\alpha}\right)(1)^{d-1} + \cdots + \frac{a_1}{\alpha}(1)\frac{g}{\alpha}(1)\right) + a_0(1) \\ &= f\left(\frac{g}{\alpha}(1)\right), \end{aligned}$$

so that $\frac{g}{\alpha}(1)$ is a zero of $f(z) = 0$. Define the starting value of $g \in \mathcal{A}$ by $\frac{g(1)}{\alpha(1)} = z_0$. Observe that

$$a_j \diamond g^{\diamond j}(n) = j\frac{a_j}{\alpha}(1)g(1)^{j-1}g(n) + \alpha(n) \sum_{\substack{ln_1\cdots n_j=n \\ n_1\cdots n_j<n}} \frac{a_j}{\alpha}(l)\frac{g}{\alpha}(n_1)\cdots\frac{g}{\alpha}(n_j) \quad (j \in \mathbb{N}).$$

We now show that the equation $T_\alpha g = 0$ uniquely and successively determines the values of $g(n)$ for $n \geq 2$. To this end, consider

$$\begin{aligned} 0 &= \sum_{1\leq j\leq d} a_j \diamond g^{\diamond j}(n) + a_0(n) \\ &= \sum_{1\leq j\leq d} j\frac{a_j}{\alpha}(1)g(1)^{j-1}g(n) + \alpha(n) \sum_{1\leq j\leq d} \sum_{\substack{ln_1\cdots n_j=n \\ n_1\cdots n_j<n}} \frac{a_j}{\alpha}(l)\frac{g}{\alpha}(n_1)\cdots\frac{g}{\alpha}(n_j) + a_0(n). \end{aligned}$$

Since $\sum_{1\leq j\leq d} j\frac{a_j}{\alpha}(1)g(1)^{j-1}g(n) = \frac{f'(g(1))}{\alpha(1)}g(n)$ and $f'(g(1)) \neq 0$, it follows that, for $n \geq 2$,

$$g(n) = -\frac{\alpha(1)}{f'(g(1))}\left\{\alpha(n) \sum_{1\leq j\leq d} \sum_{\substack{ln_1\cdots n_j=n \\ n_1\cdots n_j<n}} \frac{a_j}{\alpha}(l)\frac{g}{\alpha}(n_1)\cdots\frac{g}{\alpha}(n_j) + a_0(n)\right\},$$

i.e., the value of $g(n)$ can be uniquely and successively determined. $\square$

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