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ON THE ASYMPTOTIC STABILITY OF CERTAIN SETS OF PRIME IDEALS

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Abstract

Let (R, m) be a Noetherian local ring, I, J two ideals of R, and M a finitely generated $R-$module. It is shown that the set $\mathrm{Ass}_R(H_I^{d-1}(N_n)) \cup \{m\}$ is stable for large n, where we use N_n to denote $R-$module $J^nM/J^{n+1}M$ or $R-$module M/J^nM and d is the eventual value of $\dim N_n$. We also give a counter-example for stability of certain sets of associated prime ideals of Ext $-$modules and of attached prime ideals of local cohomology modules w.r.t. m.

1 Introduction

Let (R, m) be a Noetherian local ring, I, J two ideals of R, and M a finitely generated $R-$module. The asymptotic behavior of sets of associated prime ideals $\mathrm{Ass}_R(R/I^n)$ was given by L. J. Ratliff [14] in 1976. In 1979, M. Brodmann [1, 2] proved that the sets $\mathrm{Ass}_R(J^nM/J^{n+1}M)$ and $\mathrm{Ass}_R(M/J^nM)$ are stable for large n. As dual results, in 1986, R. Y. Sharp [15] proved that the sets of attached prime ideals $\mathrm{Att}_R(0 :_A I^n)$, $\mathrm{Att}_R((0 :_A I^{n+1})/(0 :_A I^n))$ are stable for large n, where A is an Artinian $R-$module. Then L. Melkersson-P. Schenzel [13] proved corresponding stability results for the associated, respectively attached primes, for the derived functors Tor_i of the tensor product, respectively Ext^i of the Hom-functor.

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In 1990, C. Huneke [8, Problem 4] asked whether the set of associated primes of $H^i_I(M)$ is finite for all finitely generated modules M and all ideal I. Although A. Singh [16] and M. Katzmann [9] provided examples of finitely generated modules M having some local cohomology with infinite associated prime ideals, the problem is still true in many situations. Hence, it is an important problem in the theory of local cohomology to study the question when the sets $\mathrm{Ass}_R(H^i_I(M))$ are finite.

In [10], T. Marley used the Ratliff's result [14] about the stability of set of associated prime ideals to prove that the set $\mathrm{Supp}_R(H_I^{\dim R-1}(M))$ is finite (see [10, Corollary 2.5]). Hence, Marley had given the positive answer to the question of Huneke [8, Problem 4] in the case the dimension of R less than or equal 3. By replacing R by $R/\operatorname{ann}_R(M)$, we can show that $\mathrm{Supp}_R(H_I^{\dim M-1}(M))$ is finite (cf. [4, Lemma 2.6]), so that $\mathrm{Ass}_R(H_I^{\dim M-1}(M))$ is finite. From this we obtain that $\mathrm{Ass}_R(H_I^{d-1}(N_n))$ is finite for large n, where we use N_n to denote $R-$module $J^nM/J^{n+1}M$ or $R-$module M/J^nM and d is the eventual value of $\dim N_n$. So, it is natural to ask the following question.

Question 1. *Is the set* $\mathrm{Ass}_R(H_I^{d-1}(N_n))$ *stable for large* n*?*

In the first result of this paper, we show that except the maximal ideal m the set in the above question is stable for large n.

In 1993, Melkersson and Schenzel proved that both of the sets $\mathrm{Ass}_R \mathrm{Tor}_i^R(R/I^n, M)$ and $\mathrm{Ass}_R \mathrm{Tor}_i^R(I^n/I^{n+1}, M)$ are stable for large n (see [13]). Then, also in this paper, they asked the following question.

Question 2. *Does the sets* $\mathrm{Ass}_R(\mathrm{Ext}^i_R(R/I^n, M)$ *become for large* n *independent of* n*?*

The second result of this paper shows that the answer for this question is not affirmative in general.

It is well-known that the modules $H^i_m(M)$ are Artinian for all i. Therefore, the set $\mathrm{Att}_R(H^i_m(M))$ of attached prime ideals of $H^i_m(M)$ is a finite set. Now, we consider a sequence $x_1, \ldots, x_n$ of elements in R. Then $\mathrm{Att}_R(H^i_m(M/(x_1^n, \ldots, x_r^n)M))$ is finite for all n and all i. So, we have the following question.

Question 3. *Is the set* $\mathrm{Att}_R(H^i_m(M/(x_1^n, \ldots, x_r^n)M))$ *for all* i *stable for large* n*?*

In fact, we will show that the set $\mathrm{Att}_R(H^i_m(M/(x_1^n, \ldots, x_r^n)M))$ in general is not stable for large n.

This paper is divided into three sections. In Section 2, we give a positive answer for Question 1. Section 3 devotes to give negative answers for Question 2 and 3.

2 The asymptotic stability of the set of associated prime ideals of local cohomology at $d-1$

Notation 2.1. Let (R, m) be a Noetherian local ring, I, J two ideals of R and M a finitely generated $R-$module. Let $\mathcal{R} = \oplus_{n \geq 0} R_n$ be a finitely generated standard graded algebra over local ring $R_0 = R$ and $\mathcal{M} = \oplus_{n \geq 0} M_n$ a finitely generated graded $\mathcal{R}-$module. For the convenience, we use N_n to denote either the $R-$module M_n or $R-$module $M/J^n M$.

Let situation as above, the Brodmann's theorem on the asymptotic stability of associated prime ideals [1] (see also [12, Theorem 3.1]) can be stated as follows.

Lemma 2.2. *The set* $\mathrm{Ass}_R(N_n)$ *is stable for large* n.

By Lemma 2.2, $\dim N_n$ takes constant value d for all large n. We call d is *the eventual value* of $\dim N_n$. The main result in this section is the following theorem.

Theorem 2.3. *Let* N_n *be defined as in Notation 2.1. For each non-negative integer* l*, the set* $\bigcup_{j \geq l} \mathrm{Supp}_R(H_I^j(N_n))$ *is stable for large* n*. In particular, the set* $\mathrm{Ass}_R(H_I^{d-1}(N_n)) \cup \{m\}$ *is stable for large* n*, where* d *is the eventual value of* $\dim N_n$.

In order to prove Theorem 2.3, we need some auxiliary lemmas as follows.

Lemma 2.4. *Let* M, N *be finitely generated* $R-$*modules and* I *an ideal of* R*. If* $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(N)$ *then for any non-negative integer* l *we have*

$$\mathrm{Supp}_R(H_I^l(M)) \subseteq \bigcup_{j \geq l} \mathrm{Supp}_R(H_I^j(N)).$$

Proof. We prove the lemma by descending induction on l. The theorem is trivial for $l > \dim M$ by Grothendieck's Vanishing Theorem. Assume that $l \leq \dim M$ and the theorem true for $l+1$. Since $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(N)$, so we get by Gruson's theorem [17, Theorem 4.1] that there exists a chain

$$0 = L_0 \subseteq L_1 \subseteq ... \subseteq L_t = M \tag{1}$$

of submodules of M, in which each of factors L_i/L_{i-1} is a homomorphic image of finitely many copies of N. For each $i = 1, ..., t$, by the exact sequence

$$0 \to L_{i-1} \to L_i \to L_i/L_{i-1} \to 0$$

we have the following exact sequence

$$H_I^l(L_{i-1}) \to H_I^l(L_i) \to H_I^l(L_i/L_{i-1}).$$

Hence $\operatorname{Supp}_R(H_I^l(L_i)) \subseteq \operatorname{Supp}_R(H_I^l(L_{i-1})) \cup \operatorname{Supp}_R(H_I^l(L_i/L_{i-1}))$. Therefore

$$\begin{aligned}
&\operatorname{Supp}_R(H_I^l(M)) \subseteq \operatorname{Supp}_R(H_I^l(L_{t-1})) \cup \operatorname{Supp}_R(H_I^l(L_t/L_{t-1}))\\
&\subseteq \operatorname{Supp}_R(H_I^l(L_{t-2})) \cup \operatorname{Supp}_R(H_I^l(L_{t-1}/L_{t-2})) \cup \operatorname{Supp}_R(H_I^l(L_t/L_{t-1}))\\
&\ldots\\
&\subseteq \bigcup_{1\le i\le t} \operatorname{Supp}_R(H_I^l(L_i/L_{i-1})).
\end{aligned}$$

By the property of the chain (1), for each $i \in \{1, \ldots, t\}$ we have the short exact sequence

$$0 \to K_i \to \mathop{\oplus}_{k=1}^{s_i} N \to L_i/L_{i-1} \to 0$$

for some finitely generated $R-$module K_i. This implies the following exact sequence

$$H_I^l(\mathop{\oplus}_{k=1}^{s_i} N) \to H_I^l(L_i/L_{i-1}) \to H_I^{l+1}(K_i).$$

Hence

$$\begin{aligned}
\operatorname{Supp}_R(H_I^l(L_i/L_{i-1})) &\subseteq \operatorname{Supp}_R(H_I^{l+1}(K_i)) \cup \operatorname{Supp}_R(H_I^l(\mathop{\oplus}_{k=1}^{s_i} N))\\
&= \operatorname{Supp}_R(H_I^{l+1}(K_i)) \cup \operatorname{Supp}_R(H_I^l(N)).
\end{aligned}$$

It notes that

$$\operatorname{Supp}_R(K_i) \subseteq \operatorname{Supp}_R(\mathop{\oplus}_{k=1}^{s_i} N) = \operatorname{Supp}_R(N).$$

Thus we get by the induction assumption that

$$\operatorname{Supp}_R(H_I^{l+1}(K_i)) \subseteq \bigcup_{j\ge l+1} \operatorname{Supp}_R(H_I^j(N)).$$

Therefore

$$\operatorname{Supp}_R(H_I^l(L_i/L_{i-1})) \subseteq \bigcup_{j\ge l} \operatorname{Supp}_R(H_I^j(N)),$$

and so that $\operatorname{Supp}_R(H_I^l(M)) \subseteq \bigcup_{j\ge l} \operatorname{Supp}_R(H_I^j(N))$. This completes the lemma. □

Lemma 2.5. *Let* $\dim M = d$. *Then*

$$\operatorname{Supp}_R(H_I^{d-1}(M)) \cup \{m\} = \operatorname{Ass}_R(H_I^{d-1}(M)) \cup \{m\}.$$

Proof. It is clear that

$$\operatorname{Supp}_R(H_I^{d-1}(M)) \cup \{m\} \supseteq \operatorname{Ass}_R(H_I^{d-1}(M)) \cup \{m\}.$$

By [4, Lemma 2.6], the set $\mathrm{Supp}_R(H_I^{d-1}(M))$ is finite. So $\dim(R/p) \leq 1$ for all $p \in \mathrm{Supp}(H_I^{d-1}(M))$ by Ratliff [11, Theorem 31.2]. Thus for any $p \in \mathrm{Supp}(H_I^{d-1}(M))$ we obtain that p is a minimal element of $\mathrm{Supp}(H_I^{d-1}(M))$ or $p = m$. It follows that

$$\mathrm{Supp}_R(H_I^{d-1}(M)) \subseteq \mathrm{Ass}_R(H_I^{d-1}(M)) \cup \{m\}.$$

Therefore the lemma follows. $\square$

Now, it is ready to prove Theorem 2.3.

Proof of Theorem 2.3. By Lemma 2.2, there exists an integer n_0 such that

$$\mathrm{Supp}_R(N_n) = \mathrm{Supp}_R(N_{n_0})$$

for all $n \geq n_0$. It follows by Lemma 2.4 that

$$\bigcup_{j\geq l} \mathrm{Supp}_R(H_I^j(N_n)) = \bigcup_{j\geq l} \mathrm{Supp}_R(H_I^j(N_{n_0}))$$

for all $n \geq n_0$. Therefore the first claim of the theorem follows.

For the last claim, it follows by Lemma 2.5 that

$$\begin{aligned}\{m\} \cup \mathrm{Ass}_R(H_I^{d-1}(N_n)) &= \{m\} \cup \mathrm{Supp}_R(H_I^{d-1}(N_n)) \\ &= \{m\} \cup \Big(\bigcup_{j\geq d-1} \mathrm{Supp}_R(H_I^j(N_n))\Big),\end{aligned}$$

which is stable for large n by the first claim. $\square$

It is shown that the set $\mathrm{Ass}_R(H_I^1(M/J^nM))$ is not stable for large n in general (see [6, Theorem 3.3, (ii)]). However, in [6, Theorem 3.3, (i)], by applying [5, Theorem 1.1] we also showed that the set $\mathrm{Ass}_R(H_I^1(M_n))$ is stable for large n, where M_n is defined in Notation 2.1.

The rest of this section, we consider furthermore some consequences of Theorem 2.3 for M_n. We first recall that the *cohomological dimension of M with respect to I* is defined by

$$\mathrm{cd}(I, M) = \sup\{i \in \mathbb{Z} | H_I^i(M) \neq 0\}.$$

It is easy to see that $\mathrm{cd}(I, M) \leq \dim M$. In [7, Theorem 1.4], T. Dibaei and S. Yassemi proved that if M and N are finitely generated $R-$modules such that $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(N)$ then $\mathrm{cd}(I, M) \leq \mathrm{cd}(I, N)$. From this and by the Lemma 2.2, we have the following lemma.

Lemma 2.6. $\mathrm{cd}(I, M_n)$ *takes a constant value for large* n.

Let d be the eventual value of $\dim M_n$. It is clear that $\mathrm{Ass}_R(H^i_I(M_n))$ is stable for large n, for $i = 0$ or $i > d$. By [6, Theorem 3.3], $\mathrm{Ass}_R(H^1_I(M_n))$ is stable for large n. Note that $H^d_I(M_n)$ is Artinian. This implies by Lemma 2.6 that the set $\mathrm{Ass}_R(H^d_I(M_n))$ is stable for large n. Moreover $\mathrm{Ass}_R(H^{d-1}_I(M_n)) \cup \{m\}$ is stable for large n by Theorem 2.3. Especially, in the case R is a ring of low dimension then we have the following result.

Corollary 2.7. *If $\dim R \leq 2$ then the set $\mathrm{Ass}_R(H^i_I(M_n))$ is stable for large n and all i.*

Corollary 2.8. *If $\dim R \leq 3$ then the set $\mathrm{Ass}_R(H^i_I(M_n)) \cup \{m\}$ is stable for large n and all i.*

3 The example for non stability of certain set of prime ideals

First of all, we recall a counter-example of Katzman which plays an important role in the rest of this paper.

Lemma 3.1. *(see [9, Corollary 1.3]) Let $S = k[x, y, s, t, u, v]$ be the polynomial ring of six variables over a field k and $f = sx^2v^2-(s+t)xyuv+ty^2u^2$. Denote by T the localization of S/fS at the irrelevant maximal ideal $m = (x, y, s, t, u, v)$. Then the set $\mathrm{Ass}_T(H^2_{(u,v)T}(T))$ is infinite.*

We get the following theorem which is an answer for Question 2 and Question 3.

Theorem 3.2. *Let (T, m) be the local ring as in Lemma 3.1 and $I = (u, v)T$. Then the following statements are true.*

(i) The set $\mathrm{Ass}_T(\mathrm{Ext}^2_T(T/I^n, T))$ is not stable for large n.

(ii) The set $\mathrm{Att}_T(H^i_m(T/(u^n, v^n)T)$ is not stable for large n for some $i \in \{1, 2, 3\}$.

Proof. (i) We have the following isomorphism

$$H^2_I(T) \cong \varinjlim_n \mathrm{Ext}^2_T(T/I^n, T)$$

Hence

$$\begin{aligned}\mathrm{Ass}_T(H^2_I(T)) &= \mathrm{Ass}_T(\varinjlim_n \mathrm{Ext}^2_T(T/I^n, T)) \\ &\subseteq \bigcup_{n\geq 0} \mathrm{Ass}_T(\mathrm{Ext}^2_T(T/I^n, T)).\end{aligned}$$

Note that $\mathrm{Ass}_T(H_I^2(T))$ is an infinite set by Lemma 3.1. Therefore we get by the above inclusion that the following set

$$\bigcup_{n\geq 0} \mathrm{Ass}_T(\mathrm{Ext}_T^2(T/I^n, T))$$

is infinite, and the statement (i) follows.

(ii) By [3, Theorem 5.2.9] we have the following isomorphism

$$H_I^2(T) \cong \varinjlim_n (T/(u^n, v^n)T).$$

Therefore

$$\begin{aligned}\mathrm{Ass}_T(H_I^2(T)) &= \mathrm{Ass}_T(\varinjlim_n T/(u^n, v^n)T) \\ &\subseteq \bigcup_{n\geq 0} \mathrm{Ass}_T(T/(u^n, v^n)T))\end{aligned}.$$

It is obvious that $\dim(T/(u^n, v^n)T) = 3$. Thus we get by Brodmann and Sharp [3, 11.3.9] that

$$\mathrm{Ass}_T(T/(u^n, v^n)T)) \subseteq \bigcup_{i=0}^{3} \mathrm{Att}_T(H_m^i(T/(u^n, v^n)T)).$$

It implies that

$$\begin{aligned}\mathrm{Ass}_T(H_I^2(T)) &\subseteq \bigcup_{n\geq 0} \mathrm{Ass}_T(T/(u^n, v^n)T) \\ &\subseteq \bigcup_{n\geq 0} \bigcup_{i=0}^{3} \mathrm{Att}_T(H_m^i(T/(u^n, v^n)T)) \\ &= \bigcup_{i=0}^{3} \bigcup_{n\geq 0} \mathrm{Att}_T(H_m^i(T/(u^n, v^n)T))\end{aligned}$$

Since $\mathrm{Ass}_T(H_I^2(T))$ is an infinite set by Lemma 3.1, so that the set

$$\bigcup_{i=0}^{3} \bigcup_{n\geq 0} \mathrm{Att}_T(H_m^i(T/(u^n, v^n)T))$$

is an infinite set. Therefore the set

$$\bigcup_{n\geq 0} \mathrm{Att}_T\, H_m^i(T/(u^n, v^n)T)$$

is infinite set for some $i \in \{1, 2, 3\}$, and the statement (ii) is followed. $\square$

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