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COMMUTATIVITY OF PRIME NEAR RINGS

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Abstract

The purpose of this paper is to study and generalize some results of [1] and [6] on commutativity in prime near-rings. Let N be a prime near-ring with multiplicative centre Z. Let σ and τ be automorphisms of N and δ be a (σ, τ)-derivation of N such that $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, for all $a \in N$. The following results are proved:

1. If N is 2-torsion free and $\delta(N) \subseteq Z$, or $\delta(x)\delta(y) = \delta(y)\delta(x)$, for all $x, y \in N$, then N is a commutative ring.
2. If N is 2-torsion free, δ_1 is a derivation of N, δ_2 is a (σ, τ)-derivation of N such that τ commutes with δ_1 and δ_2, then $\delta_1\delta_2(N) = 0$ implies $\delta_1 = 0$ or $\delta_2 = 0$.

1 Introduction

Throughout this article N denotes a zero symmetric left near-ring with multiplicative centre Z. N is called a prime near ring if x N y = $\{0\}$ implies x = 0 or y = 0. An element $x \in N$ is said to be distributive if (y + z)x = yx + zx for all $x, y, z \in N$. N is called zero symmetric if 0x = 0 for all $x \in N$ (left distributivity implies x0 = 0). An additive mapping $\delta : N \to N$ is said to be derivation on N if $\delta(xy) = \delta(x)y + x\delta(y)$; for all $x, y \in N$ or equivalently $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in N$. An additive mapping $\delta : N \to N$ is called (σ, τ)-derivation if there exists automorphisms $\sigma, \tau : N \to N$ such that

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$\delta(xy) = \delta(x)\sigma(y) + \tau(x)\delta(y)$ for all $x, y \in N$. The symbol [x, y] denotes xy - yx and (x, y) denotes additive group commutator x + y - x - y. For all $x, y \in N$, we write $[x, y]_{\sigma,\tau} = \tau(x)y - y\sigma(x)$; in particular $[x, y]_{1,1} = [x, y]$, in usual sense. An element $c \in N$ for which $\delta(c) = 0$ is called a constant.

Some recent results on rings deal with commutativity of prime and semi-prime rings admitting suitably-constrained derivations. Here in this paper we look for comparable results on near-rings and this has been done in [1, 2, 3, 4, 5, 6, 7, 8].

2 Main Result

As the addition of a near-ring is not necessarily commutative, the following Lemma 2.1 has its own significance.

Lemma 2.1. *An additive endomorphism δ on a near-ring N is a (σ, τ)-derivation if and only if $\delta(xy) = \tau(x)\delta(y) + \delta(x)\sigma(y)$, for all $x, y \in N$.*

Proof Let δ be a (σ, τ)-derivation on a near-ring N. Since x(y + y) = xy + xy, we have

$$\delta(x(y+y)) = \delta(x)\sigma(y+y) + \tau(x)\delta(y+y) =$$
$$\delta(x)\sigma(y) + \delta(x)\sigma(y) + \tau(x)\delta(y) + \tau(x)\delta(y),$$

for all $x, y \in N$.

Also

$$\delta(xy + xy) = \delta(xy) + \delta(xy) = \delta(x)\sigma(y) + \tau(x)\delta(y) + \delta(x)\sigma(y) + \tau(x)\delta(y),$$

for all $x, y \in N$.

Comparing above, we have

$$\delta(x)\sigma(y) + \tau(x)\delta(y) = \tau(x)\delta(y) + \delta(x)\sigma(y)$$

for all $x, y \in N$. Hence, we have $\delta(xy) = \tau(x)\delta(y) + \delta(x)\sigma(y)$ for all $x, y \in N$.

Conversely, suppose that $\delta(xy) = \tau(x)\delta(y) + \delta(x)\sigma(y)$ for all $x, y \in N$. Then for all $x, y \in N$,

$$\delta(x(y+y)) = \tau(x)\delta(y+y) + \delta(x)\sigma(y+y) = \tau(x)\delta(y) + \tau(x)\delta(y) + \delta(x)\sigma(y) + \delta(x)\sigma(y).$$

Also

$$\delta(xy + xy) = \delta(xy) + \delta(xy) = \tau(x)\delta(y) + \delta(x)\sigma(y) + \tau(x)\delta(y) + \delta(x)\sigma(y)$$

for all $x, y \in N$.

Comparing above, we have $\tau(x)\delta(y) + \delta(x)\sigma(y) = \delta(x)\sigma(y) + \tau(x)\delta(y)$, for all $x, y \in N$. Thus we have, $\delta(xy) = \delta(x)\sigma(y) + \tau(x)\delta(y)$ for all $x, y \in N$. □

Lemma 2.2. *Let δ be a (σ, τ)-derivation on a near-ring N. Then for all $x, y, z \in N$; $(\delta(x)\sigma(y) + \tau(x)\delta(y))\sigma(z) = \delta(x)\sigma(y)\sigma(z) + \tau(x)\delta(y)\sigma(z)$.*

Proof For all $x, y, z \in N$,
$\delta((xy)z) = \delta(xy)\sigma(z) + \tau(xy)\delta(z) = (\delta(x)\sigma(y) + \tau(x)\delta(y))\sigma(z) + \tau(x)\tau(y)\delta(z)$. Also, $\delta(x(yz)) = \delta(x)\sigma(yz) + \tau(x)\delta(yz) = \delta(x)\sigma(y)\sigma(z) + \tau(x)(\delta(y))\sigma(z) + \tau(y)\delta(z)) = \delta(x)\sigma(y)\sigma(z) + \tau(x)\delta(y))\sigma(z) + \tau(x)\tau(y)\delta(z)$ for all $x, y, z \in N$.

Combining these two facts, we get

$$(\delta(x)\sigma(y) + \tau(x)\delta(y))\sigma(z) = \delta(x)\sigma(y)\sigma(z) + \tau(x)\delta(y)\sigma(z)$$

for all $x, y, z \in N$. □

Lemma 2.3. *Let N be a prime near-ring with multiplicative centre Z. Then:*

(1) If z is a nonzero element of Z, then z is not a zero divisor.

(2) If there exists a nonzero element $z \in Z$ such that $z + z \in Z$, then (N, +) is abelian.

(3) Let δ be a nonzero (σ, τ)-derivation of N and $a \in N$. If $\delta(N)\sigma(a) = 0$, then $a = 0$, and if $a\delta(N) = 0$, then $a = 0$.

Proof The proofs of (1) and (2) are in [1]

(3) By hypothesis, $\delta(N)\sigma(a) = 0$. It follows that, for all $x, y \in N$, $\delta(xy)\sigma(a) = 0$. By 2.2, we have

$$\delta(x)\sigma(y)\sigma(a) + \tau(x)\delta(y)\sigma(a) = 0$$

i.e. $\delta(x)\sigma(y)\sigma(a) = 0$, or $\delta(x)N\sigma(a) = 0$. Since N is a prime near-ring, δ a nonzero (σ, τ)-derivation of N and σ is an automorphism, we get $a = 0$. Let $a\delta(N) = 0$. Then, for all $x, y \in N$, $a\delta(xy) = 0$, i.e.

$$a(\delta(x)\sigma(y) + \tau(x)\delta(y)) = 0$$

or

$$a\delta(x)\sigma(y) + a\tau(x)\delta(y) = 0.$$

Therefore, we have $a\tau(x)\delta(y) = 0$, for all $x, y \in N$. Since τ is an automorphism of N, it would imply that $(aN)\delta(N) = 0$. Moreover, N is prime and $\delta(N) \neq 0$ we infer that $a = 0$, and the proof is now complete. □

Lemma 2.4. *Let N be a 2-torsion free near-ring, and δ be a (σ, τ)-derivation of N. If $\delta^2 = 0$, and σ, τ both commute with δ, then $\delta = 0$.*

Proof For all $x, y \in N$, $\delta^2(xy) = 0$. So, we have $0 = \delta(\delta(xy)) = \delta(\delta(x)\sigma(y) + \tau(x)\delta(y)) = \delta(\delta(x)\sigma(y)) + \delta(\tau(x)\delta(y)) = \delta(\delta(x))\sigma(\sigma(y)) + \tau(\delta(x))\delta(\sigma(y)) + \delta(\tau(x))\sigma(\delta(y)) + \tau(\tau(x))\delta(\delta(y)) = \delta^2(x))\sigma^2(y)) + \tau(\delta(x))\delta(\sigma(y)) + \delta(\tau(x))\sigma(\delta(y)) + \tau^2(x))\delta^2(y)) = 2\delta(\tau(x))\delta(\sigma(y))$, by hypothesis.

Therefore, for all $x, y \in N$, $\delta(\tau(x))\delta(\sigma(y)) = 0$. Since N is a 2-torsion free near-ring and σ is an automorphism of N, we get $\delta(\tau(x))\delta(N) = 0$. It follows from 2.3 that $\delta = 0$. □

Theorem 2.5. *Let δ be a (σ, τ)-derivation of a near-ring N. If $a \in N$ is not a left zero divisor and $[a, \delta(a)]_{\sigma,\tau} = 0$, then (x, a) is constant (i.e. $\delta(x, a) = 0$) for all $x \in N$.*

Proof We have, $a(x + a) = ax + a^2$ and therefore, $\delta(a(x + a)) = \delta(ax + a^2)$. Expanding the equation, we have $\delta(a)\sigma(x) + \delta(a)\sigma(a) + \tau(a)\delta(x) + \tau(a)\delta(a) = \delta(a)\sigma(x) + \tau(a)\delta(x) + \delta(a)\sigma(a) + \tau(a)\delta(a)$. Therefore $\delta(a)\sigma(a) + \tau(a)\delta(x) = \tau(a)\delta(x) + \delta(a)\sigma(a)$; i.e. $0 = \tau(a)\delta(x) + \delta(a)\sigma(a) - \tau(a)\delta(x) - \delta(a)\sigma(a)$. But $[a, \delta(a)]_{\sigma,\tau} = 0$, it implies that $\tau(a)\delta(a) - \delta(a)\sigma(a) = 0$. Hence, we have, $0 = \tau(a)\delta(x) + \tau(a)\delta(a) - \tau(a)\delta(x) - \tau(a)\delta(a)$, which implies that $\tau(a)\delta(x, a) = 0$. Since τ is an automorphism of N, and $\tau(a)$ is not a left zero divisor, we can see that $\delta(x, a) = 0$. Hence (x, a) is constant for all $x \in N$. □

Theorem 2.6. *Let N have no non-zero divisors of zero. If N admits a non-trivial (σ, τ)-commuting (σ, τ)-derivation δ, then (N, +) is abelian.*

Proof Let c be any additive commutator. Then 2.5 implies that c is a constant. Also, for any $x \in N$, xc is also an additive commutator, and hence a constant. Thus, $0 = \delta(xc) = \delta(x)\sigma(c) + \tau(x)\delta(c)$ for all $x \in N$. This implies $\delta(x)\sigma(c) = 0$ for all $x \in N$. Since $\delta(x) \neq 0$ for some $x \in N$, we get that $\sigma(c) = 0$. Thus c = 0 for all additive commutators c. Hence, (N, +) is abelian. □

Theorem 2.7. *Let N be a prime near-ring with a nonzero (σ, τ)-derivation δ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, $a \in N$. If $\delta(N) \subseteq Z$, then (N, +) is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.*

Proof By hypothesis, $\delta(N) \subseteq Z$ and δ is non-trivial. Hence, there exists $0 \neq a \in N$ such that $z = \delta(a) \in Z - \{0\}$. It would imply that $z + z = \delta(a + a) \in Z - \{0\}$. It follows from 2.3 that $(N, +)$ is abelian. Again by hypothesis, we have $\sigma(c)\delta(ab) = \delta(ab)\sigma(c)$, for all $a, b, c \in N$. By 2.2, we have

$$\sigma(c)\delta(a)\sigma(b) + \sigma(c)\tau(a)\delta(b) = \delta(a)\sigma(b)\sigma(c) + \tau(a)\delta(b)\sigma(c)$$

for all $a, b, c \in N$. Comparing the two sides, using $\delta(N) \subseteq Z$, and the fact that $(N, +)$ is abelian, we get

$$\sigma(c)\delta(a)\sigma(b) - \delta(a)\sigma(b)\sigma(c) = \tau(a)\delta(b)\sigma(c) - \sigma(c)\tau(a)\delta(b)$$

for all $a, b, c \in N$. Thus

$$\delta(a)\sigma(c)\sigma(b) - \delta(a)\sigma(b)\sigma(c) = \tau(a)\sigma(c)\delta(b) - \sigma(c)\tau(a)\delta(b)$$

for all $a, b, c \in N$, or $\delta(a)\sigma([c, b]) = \delta(b)[\tau(a), \sigma(c)]$, for all $a, b, c \in N$. We now suppose that N is not commutative, and choose $b, c \in N$ such that $[c, b] \neq 0$, and $a = \delta(x) \in Z$. Then, we get $\delta^2(x)\sigma([c, b]) = 0$, for all $x \in N$. By 2.3 we can see that the central element $\delta^2(x)$ can not be a divisor of zero, which implies that $\delta^2(x) = 0$ for all $x \in N$. By 2.4 this cannot happen for the non-trivial δ. Thus, $\sigma([c, b]) = 0$, for all $b, c \in N$. Hence N is a commutative ring, as σ is an automorphism of N. □

Theorem 2.8. *Let N be a prime near-ring with a nonzero (σ, τ)-derivation δ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, $a \in N$. If $\delta(x)\delta(y) = \delta(y)\delta(x)$, for all $x, y \in N$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.*

Proof By hypothesis, we have $\delta(x + x)\delta(x + y) = \delta(x + y)\delta(x + x)$ for all $x, y \in N$. This implies that

$$\delta(x)\delta(x) + \delta(x)\delta(y) = \delta(x)\delta(x) + \delta(y)\delta(x)$$

for all $x, y \in N$. Hence $\delta(x)\delta(x, y) = 0$ for all $x, y \in N$, which implies that $\delta(x)\delta(c) = 0$ for all $x \in N$ and the additive commutator c. Applying 2.3 we have $\delta(c) = 0$, for all additive commutators c. Since N is a left near-ring and c is an additive commutator, we see that xc is also an additive commutator for all $x \in N$. Therefore $\delta(xc) = 0$ for all $x \in N$ and for all additive commutators c. It follows from 2.3 that $c = 0$. Hence $(N, +)$ is abelian.

Assume now that N is 2-torsion free, $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, $a \in N$. Then by 2.1 and 2.2 we have

$$\delta(\delta(x)y))\delta(z) = (\delta^2(x)\tau(y) + \sigma(\delta(x))\delta(y))\delta(z) = \delta^2(x)\tau(y)\delta(z) + \sigma(\delta(x))\delta(y)\delta(z)$$

for all $x, y, z \in N$. This implies that

$$\delta^2(x)\tau(y)\delta(z) = \delta(\delta(x)y))\delta(z) - \sigma(\delta(x))\delta(y)\delta(z)$$

for all $x, y, z \in N$.

Moreover, since $\delta(x)\delta(y) = \delta(y)\delta(x)$, for all $x, y \in N$, we have $\delta(\delta(x)y))\delta(z) = \delta(z)\delta(\delta(x)y)) = \delta(z)\delta^2(x)\tau(y) + \sigma(\delta(x))\delta(y) = \delta(z)\delta^2(x)\tau(y) + \delta(z)\sigma(\delta(x))\delta(y) = \delta^2(x)\delta(z)\tau(y) + \sigma(\delta(x))\delta(y)\delta(z)$, for all $x, y, z \in N$.

Combining the results, we get

$$\delta^2(x)\tau(y)\delta(z) - \delta^2(x)\delta(z)\tau(y) = 0$$

for all $x, y, z \in N$; i.e.

$$\delta^2(x)(\tau(y)\delta(z) - \delta(z)\tau(y)) = 0$$

for all $x, y, z \in N$. Replacing y by ya we have

$$\delta^2(x)(\tau(ya)\delta(z) - \delta(z)\tau(ya)) = 0$$

for all $a, x, y, z \in N$, i.e.,

$$\delta^2(x)\tau(y)(\tau(a)\delta(z) - \delta(z)\tau(a)) = 0$$

for all $a, x, y, z \in N$. Thus, $\delta^2(x)N(\tau(a)\delta(z) - \delta(z)\tau(a)) = 0$ for all $a, x, y, z \in N$. Since N is prime and τ is an automorphism, $\delta^2(x) = 0$, or $a\delta(z) - \delta(z)a = 0$, for all $a, x, z \in N$. Referring to 2.4 $\delta^2(x) = 0$ is not possible. Hence $a\delta(z) - \delta(z)a = 0$, for all $a, z \in N$. Therefore, $\delta(N) \subseteq Z$ and it follows from 2.7 that N is commutative. □

Theorem 2.9. *Let N be a 2-torsion free prime near-ring N, δ_1 be a (σ, τ)-derivation of N and δ_2 be a derivation of N. If $\delta_1\delta_2(N) = 0$, then $\delta_1 = 0$, or $\delta_2 = 0$.*

Proof By hypothesis $\delta_1\delta_2(ab) = 0$ for all $a, b \in N$. Therefore, we have $0 = \delta_1(\delta_2(a)b + a\delta_2(b)) = \delta_1(\delta_2(a)b) + \delta_1(a\delta_2(b)) = \delta_1(\delta_2(a))\sigma(b) + \tau(\delta_2(a))\delta_1(b) + \delta_1(a)\sigma(\delta_2(b)) + \tau(a)\delta_1(\delta_2(b))$. Thus, we have $\tau(\delta_2(a))\delta_1(b) + \delta_1(a)\sigma(\delta_2(b)) = 0$ for all $a, b \in N$. Replacing a by $\delta_2(a)$, we get $\tau(\delta_2^2(a))\delta_1(b) = 0$ for all $a, b \in N$. By 2.3, it implies that $\delta_1 = 0$, or $\delta_2^2 = 0$. If $\delta_2^2 = 0$, then by 2.4, $\delta_2 = 0$. □

Theorem 2.10. *Let N be a 2-torsion free prime near-ring N, δ_1 be a derivation of N and δ_2 be a (σ, τ)- derivation of N such that $\tau\delta_1 = \delta_1\tau$ and $\tau\delta_2 = \delta_2\tau$. If $\delta_1\delta_2(N) = 0$, then $\delta_1 = 0$, or $\delta_2 = 0$.*

Proof By hypothesis $\delta_1\delta_2(ab) = 0$, for all $a, b \in N$. Therefore, we have $0 = \delta_1(\delta_2(a)\sigma(b) + \tau(a)\delta_2(b)) = \delta_1(\delta_2(a)\sigma(b)) + \delta_1(\tau(a)\delta_2(b)) = \delta_1(\delta_2(a))\sigma(b) + \delta_2(a)\delta_1(\sigma(b)) + \delta_1(\tau(a))\delta_2(b) + \tau(a)\delta_1\delta_2(b))$. This implies that

$$\delta_2(a)\delta_1(\sigma(b)) + \delta_1(\tau(a))\delta_2(b) = 0$$

for all $a, b \in N$. Replacing a by $\delta_2(a)$, and using the fact that $\tau\delta_1 = \delta_1\tau$ and $\tau\delta_2 = \delta_2\tau$, we have $\delta_2^2(a)\delta_1(\sigma(b)) = 0$, for all $a, b \in N$. Applying 2.3, we have $\delta_1 = 0$, or $\delta_2^2 = 0$. If $\delta_2^2 = 0$, then by 2.4, $\delta_2 = 0$, proving our Theorem. □

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