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THE FOGUEL ALTERNATIVE AND SWEEPING FOR AN INTERMITTENT MAP WITH MULTIPLICATIVE NOISE

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Abstract

We consider Markov operators which represent a density function of random perturbations of an intermittent map with multiplicative noise. In this paper, we give a class of intermittent maps for which the Foguel Alternative theorem holds. Actually, we prove that Markov operators are sweeping under certain conditions.

1 Introduction

Komorowski proved that if $S : [0,1] \to [0,1]$ is a piecewise convex and of class C^2 satisfying $S(0) = 0$ and $S'(0) = 1$, then S does not admit a finite invariant measure $\mu \ll \lambda$, where λ is the Lebesgue measure on $[0,1]$ in [1]. On the other hand, if we consider random perturbations of S with additive noise defined by

$$X_{n+1}^{\varepsilon} = S(X_n^{\varepsilon}) + \varepsilon Y_n \qquad (\bmod 1),$$

where $Y_0, Y_1, \cdots$ are independent random variables with values in $[0,1]$ each having the same density g and a random variable X_0 and $\{Y_n\}_{n\geq 0}$ are independent, then there always exists an invariant probability measure $\mu_\varepsilon \ll \lambda$ for each noise level $0 < \varepsilon < 1$ (see [3] for more details).

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In this paper, we consider the following random perturbation of S with multiplicative noise, that is, consider the process $\{X_n^\varepsilon\}_{n\geq 0}$ $(0<\varepsilon<1)$ defined by

$$X_{n+1}^\varepsilon = (1-\varepsilon Y_n)S(X_n^\varepsilon). \tag{1}$$

In this settings, every density function of X_n^ε is represented by n-th iterate of a Markov operator $P_\varepsilon : L^1([0,1],\lambda) \to L^1([0,1],\lambda)$ and the initial density f of X_0. We prove that if S has an infinite invariant density function $\frac{1}{x^\beta}$ $(\beta \geq 1)$ then the process $\{X_n^\varepsilon\}_{n\geq 0}$ satisfies the Foguel Alternative theorem. This implies that either P_ε has an invariant density (i.e. there exists an density function h_*^ε such that $P_\varepsilon h_*^\varepsilon = h_*^\varepsilon$) or $\{P_\varepsilon^n\}$ is sweeping, i.e.,

$$\lim_{n\to\infty} \int_{[c,1]} P_\varepsilon^n f(x)\lambda(dx) = 0 \qquad \text{for every } f \in D \text{ and } 0 < c \leq 1,$$

where $D := \{f \in L^1([0,1]) :\ f \geq 0 \text{ and } \int_{[0,1]} f(x)dx = 1\}$. Actually, we prove that $\{P_\varepsilon^n\}$ is sweeping for every $0<\varepsilon<1$ by adding certain conditions to S and the density g of $\{Y_n\}_{n\geq 0}$.

Before we introduce the Foguel Alternative theorem, we have to make the following definitions.

Let $(\Lambda, \mathcal{B}, m)$ be a σ-finite measure space and $P : L^1(\Lambda) \to L^1(\Lambda)$ be the operator

$$Pf(x) = \int_\Lambda K(x,y)f(y)m(dy),$$

where $K : \Lambda \times \Lambda \to \mathbb{R}$ is a measurable function which satisfies $K(x,y) \geq 0$ a.e. and $\int_\Lambda K(x,y)m(dx) = 1$. We call P *integral operator with stochastic kernel* $K(x,y)$.

Definition 1.1. A family $\mathcal{A} \subset \mathcal{B}$ is called admissible if $\mathcal{A}$ satisfies the following properties

1 $m(A) < \infty$ for $A \in \mathcal{A}$,

2 $A_1 \cup A_2 \in \mathcal{A}$ for $A_1, A_2 \in \mathcal{A}$,

3 There exists a sequence $\{A_n\}_{n\geq 0} \subset \mathcal{A}$ such that $\cup_{n\geq 0} A_n = \Lambda$.

Definition 1.2. Let $P : L^1(\Lambda, \mathcal{B}, m) \to L^1(\Lambda, \mathcal{B}, m)$ be an integral operator with stochastic kernel and an admissible family $\mathcal{A} \subset \mathcal{B}$ be fixed. We say that $\{P^n\}_{n\geq 0}$ is sweeping with respect to an admissible family $\mathcal{A} \subset \mathcal{B}$ if

$$\lim_{n\to\infty} \int_A P^n f dm = 0 \qquad \text{for } A \in \mathcal{A} \quad \text{and} \quad f \in D.$$

Definition 1.3. Let $P : L^1(\Lambda, \mathcal{B}, m) \to L^1(\Lambda, \mathcal{B}, m)$ be an integral operator with stochastic kernel and an admissible family $\mathcal{A} \subset \mathcal{B}$ be fixed. A measurable function $f : \Omega \to \mathbb{R}$ defined up to a set of measure zero is called locally integrable if

$$\int_A |f| dm < \infty \qquad \text{for } A \in \mathcal{A}$$

and $f : \Lambda \to \mathbb{R}$ is subinvariant if

$$Pf(x) \leq f(x) \qquad \text{for a.e. } x \in \Lambda.$$

The Foguel Alternative theorem was proved by Komorowski and Tyrcha [2]:

Theorem 1.4 (Foguel Alternative). *Let $P : L^1(\Lambda, \mathcal{B}, m) \to L^1(\Lambda, \mathcal{B}, m)$ be an integral operator with a stochastic kernel on a σ-finite measure space $(\Lambda, \mathcal{B}, m)$ and $\mathcal{A} \subset \mathcal{B}$ be an admissible family. If P has a locally integrable and positive ($f > 0$ a.e.) subinvariant function f with respect to $\mathcal{A}$, then either P has an invariant density or $\{P^n\}_{n \geq 0}$ is sweeping with respect to $\mathcal{A}$.*

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where $\mathcal{F}$ denotes a Borel σ-field and μ a probability measure. Let $X_0, Y_0, Y_1, \cdots$ be random variables on Ω with values in $[0, 1]$ and $S : [0, 1] \to [0, 1]$ be a non-singular measurable transformation (i.e. $\lambda(S^{-1}(A)) = 0$ for any Borel set $A \subset [0, 1]$ with $\lambda(A) = 0$, where λ is the Lebesgue measure on $[0, 1]$) and positive for λ-a.e. $x \in [0, 1]$.

Consider the following stochastic process defined by

$$X^{\varepsilon}_{n+1}(\omega) = (1 - \varepsilon Y_n) S(X^{\varepsilon}_n(\omega)) \qquad \text{for all } n \geq 0, \tag{2}$$

where $X^{\varepsilon}_0 = X_0$ for each $0 < \varepsilon < 1$.

We assume the following conditions for random perturbations $\{X^{\varepsilon}_n\}_{n \geq 0}$ generated by (2) throughout this paper :

C1 $X_0, Y_0, Y_1, Y_2, \cdots$ are independent random variables;

C2 X_0 has the density function $f_0 \in D$, i.e.

$$\mu(X_0(\omega) \in B) = \int_B f_0(x) \lambda(dx)$$

for any Borel set $B \subset [0, 1]$, where

$$D := \{f \in L^1([0, 1]) : f \geq 0 \text{ and } \int_{[0,1]} f(x) \lambda(dx) = 1\};$$

C3 each Y_n has the same density function $g \in L^\infty(\mathbb{R})$ such that $g \geq 0$,

$$\text{supp}(g) := \overline{\{x \in [0,1] :\ g(x) \neq 0\}} \subseteq [0,1] \qquad \text{with} \qquad \int_{\mathbb{R}} g(x)\lambda(dx) = 1.$$

A linear operator $P : L^1([0,1]) \to L^1([0,1])$ is said to be a Markov operator if $P(D) \subset D$. With these conditions every density function of X_n^ε is represented by n-th iterate of the Markov operator $P_\varepsilon : L^1([0,1],\lambda) \to L^1([0,1],\lambda)$ as follows:

$$\mu(\{X_n^\varepsilon \in A\}) = \int_A P_\varepsilon^n f_0(x)\lambda(dx) \qquad \text{for any Borel set } A \subset [0,1].$$

In fact, P_ε is defined by

$$\begin{aligned} P_\varepsilon f(x) \qquad &= \int_{[0,1]} f(y) g\left(\frac{1}{\varepsilon}\left(1 - \frac{x}{S(y)}\right)\right)\frac{1}{\varepsilon S(y)}\lambda(dy) \\ &= \int_{[0,1]} P_S f(y) g\left(\frac{1}{\varepsilon}\left(1 - \frac{x}{y}\right)\right)\frac{1}{\varepsilon y}\lambda(dy) \end{aligned}$$

for each $0 < \varepsilon < 1$ and $f \in L^1([0,1])$, where P_S is the Perron-Frobenius operator corresponding to S. In the following lemma, we prove these facts.

Lemma 2.1. *Let $S : [0,1] \to [0,1]$ be a non-singular positive a.e. measurable transformation and $\{X_n^\varepsilon\}_{n\geq 0}$ be a random perturbation defined by (2). If Conditions C1-C3 are valid for $\{X_n^\varepsilon\}_{n\geq 0}$, then each density function of X_n^ε is represented by n-th iterate of the Markov operator $P_\varepsilon : L^1([0,1]) \to L^1([0,1])$ define by*

$$P_\varepsilon f(x) = \int_{[0,1]} f(y) g\left(\frac{1}{\varepsilon}\left(1 - \frac{x}{S(y)}\right)\right)\frac{1}{\varepsilon S(y)}\lambda(dy) \tag{3}$$

for each $0 < \varepsilon < 1$.

Proof Fix $0 < \varepsilon < 1$ arbitrarily. We assume that there exists the density function f_n^ε of X_n^ε.

Let $g_\varepsilon(x) = \frac{1}{\varepsilon} g(\frac{x}{\varepsilon})$ and $1 - A := \{1 - x :\ x \in A\}$ for $A \subset [0,1]$. Since

$$\int_A h(1-x)\lambda(dx) = \int_{1-A} h(x)\lambda(dx),$$

for any Borel set $A \subset [0,1]$ and $h \in L^1(\mathbb{R})$ with respect to the one-dimensional Lebesgue integration, we have

$$\mu\left(1 - \varepsilon Y_n \in A\right) \qquad = \int_{1-\varepsilon x \in A} g(x)dx = \int_{1-x\in A} g_\varepsilon(x)dx = \int_{x\in A} g_\varepsilon(1-x)dx$$

for all $n \geq 0$. This implies that the sequence $\{1-\varepsilon Y_n\}_{n\geq 0}$ is the i.i.d. sequence and has the same density function $g_\varepsilon(1-x)$. Thus we have

$$\begin{aligned}\mu\left(X^\varepsilon_{n+1} \in A\right) &= \mu\left((1-\varepsilon Y_n)S(X^\varepsilon_n) \in A\right)\\ &= \int\int_{xS(y)\in A} f^\varepsilon_n(y) g_\varepsilon(1-x)\lambda(dy)\lambda(dx).\end{aligned}$$

We remark that the set $S^{-1}(\{0\})$ is λ-null set by the assumption about S. By Condition $C3$, we have $g_\varepsilon(1-x) = \frac{1}{\varepsilon}g(\frac{1}{\varepsilon}(1-x)) = 0$ for any $x > 1$ and $x < 0$ because $\frac{1}{\varepsilon}(1-x) < 0$ and $\frac{1}{\varepsilon}(1-x) > \frac{1}{\varepsilon} > 1$ respectively. Thus setting $a = xS(y)$ and $b = y$, we obtain

$$\begin{aligned}\mu\left(X^\varepsilon_{n+1} \in A\right) &= \int_{a\in A}\int_{\{b\in[0,1]:\ \frac{a}{S(b)}\in[0,1],\ S(b)\neq 0\}} f^\varepsilon_n(b) g_\varepsilon\Big(1-\frac{a}{S(b)}\Big)\frac{1}{S(b)}\lambda(db)\lambda(da)\\ &= \int_{a\in A}\int_{\{b\in[0,1]:\ S(b)\neq 0\}} f^\varepsilon_n(b) g_\varepsilon\left(1-\frac{a}{S(b)}\right)\frac{1}{S(b)}\lambda(db)\lambda(da)\\ &= \int_{a\in A}\int_{b\in[0,1]} f^\varepsilon_n(b) g\left(\frac{1}{\varepsilon}\left(1-\frac{a}{S(b)}\right)\right)\frac{1}{\varepsilon S(b)}\lambda(db)\lambda(da)\\ &= \int_A P_\varepsilon f^\varepsilon_n(a)\lambda(da).\end{aligned}$$

This equation implies that if f^ε_n exits then the density function f^ε_{n+1} of X^ε_{n+1} also exists and given by

$$f^\varepsilon_{n+1}(x) = \int_{y\in[0,1]} f^\varepsilon_n(y) g\left(\frac{1}{\varepsilon}\left(1-\frac{x}{S(y)}\right)\right)\frac{1}{\varepsilon S(y)}\lambda(dy) =: P_\varepsilon f^\varepsilon_n(x) \qquad \text{a.e.}$$

From the linearity of integral, the operator P_ε is linear and $P_\varepsilon f \geq 0$ for any $f \geq 0$ because of $g \geq 0$. Moreover, since $supp(g) \subset [0,1] \subset [\frac{1}{\varepsilon} - \frac{1}{\varepsilon S(y)}, \frac{1}{\varepsilon}]$ for each $0 < \varepsilon < 1$, we have

$$\begin{aligned}\|P_\varepsilon f\|_{L^1([0,1])} &= \int_{[0,1]} P_\varepsilon f(x)\lambda(dx)\\ &= \int_{[0,1]} f(y)\left\{\int_{[0,1]} g_\varepsilon\left(1-\frac{x}{S(y)}\right)\frac{1}{S(y)}\lambda(dx)\right\}\lambda(dy) \text{ (by Fubini's theorem)}\\ &= \int_{[0,1]} f(y)\left\{\int_{[0,\frac{1}{S(y)}]} g_\varepsilon(1-x)\lambda(dx)\right\}\lambda(dy)\\ &= \int_{[0,1]} f(y)\left\{\int_{[1-\frac{1}{S(y)},1]} g_\varepsilon(x)\lambda(dx)\right\}\lambda(dy)\\ &= \int_{[0,1]} f(y)\left\{\int_{[\frac{1}{\varepsilon}-\frac{1}{\varepsilon S(y)},\frac{1}{\varepsilon}]\cap[0,1]} g(x)\lambda(dx)\right\}\lambda(dy) = \int_{[0,1]} f(y)\lambda(dy)\\ &= \|f\|_{L^1([0,1])}.\end{aligned}$$

for any $f \geq 0$. Therefore P_ε is the Markov operator. $\square$

Remark 2.2. It is obviously that the Markov operator defined by (3) is the integral operator with stochastic kernel $K(x,y) := g\left(\frac{1}{\varepsilon}\left(1 - \frac{x}{S(y)}\right)\right)\frac{1}{\varepsilon S(y)}$ because

$$\int_{[0,1]} g\left(\frac{1}{\varepsilon}\left(1 - \frac{x}{S(y)}\right)\right)\frac{1}{\varepsilon S(y)}\lambda(dx) = \int_{[0,1]} g\left(\frac{1}{\varepsilon}(1-x)\right)\frac{1}{\varepsilon}\lambda(dx) = 1.$$

Remark 2.3. The Perron-Frobenius operator P_S corresponding to S exists because S is non-singular transformation. Hence we can write the Markov operator P_ε defined by (3) as

$$P_\varepsilon f(x) = \int_{[0,1]\setminus\{0\}} P_S f(y) g\left(\frac{1}{\varepsilon}\left(1 - \frac{x}{y}\right)\right)\frac{1}{\varepsilon y}\lambda(dy) \tag{4}$$

and by the change of variables with respect to the one-dimensional Lebesgue integral and Condition C3, we also have

$$P_\varepsilon f(x) = \int_{[0,\frac{1}{\varepsilon}(1-x)]} P_S f\left(\frac{x}{1-\varepsilon y}\right)\frac{g(y)}{1-\varepsilon y}\lambda(dy). \tag{5}$$

3 Main Results

We prove that the Foguel Alternative theorem holds for the Markov operator $\{P_\varepsilon^n\}$ defined by (3).

Let $\mathcal{A} := \{\{0\} \cup [c,1] : \ 0 < c \leq 1\}$. It is easy to see that $\mathcal{A}$ satisfies (1)-(3) in Definition 1.1, so that $\mathcal{A}$ is an admissible subfamily of Borel σ-algebra on $[0,1]$. Consequently, we have one of our main theorem.

Theorem 3.1. *Let $S : [0,1] \to [0,1]$ be a non-singular positive a.e. transformation and P_ε be the Markov operator defined by (3) for each $0 < \varepsilon < 1$. Suppose that there exists an invariant infinite density function $h_\beta : (x) = \frac{1}{x^\beta}$ ($\beta \geq 1$) such that $P_S h_\beta(x) = h_\beta(x)$ a.e. x, where P_S is the Perron-Frobenius operator corresponding to S. Then h_β is a locally integrable, positive and subinvariant function with respect to $\mathcal{A}$ and P_ε. Consequently, the Foguel alternative theorem holds for P_ε, that is, either P_ε has an invariant density or sweeping with respect to $\mathcal{A}$.*

Proof Obviously, $\int_A h_\beta(x)dx < \infty$ for all $A \in \mathcal{A}$ and $h_\beta(x) > 0$ a.e. $x \in [0,1]$. Hence h_β is locally integrable positive function with respect to $\mathcal{A}$.

Fix $0 < x \le 1$ arbitrarily. Hence there exists $0 < c < x$ and we denote $\frac{1}{x^\beta}\mathbf{1}_{[c,1]}(x)$ by $f_*(x)$. Since $f_*(x) \le \frac{1}{x^\beta} =: h_\beta(x)$, we have

$$\begin{aligned} P_\varepsilon h_\beta(x) = P_\varepsilon f_*(x) &= \int_{[0,1]} P_S f_*(y) \frac{1}{\varepsilon y} g\left(\frac{1}{\varepsilon}\left(1 - \frac{x}{y}\right)\right)\lambda(dy) \\ &= \int_{[0,\frac{1}{\varepsilon}(1-x)]} P_S f_*\left(\frac{x}{1-\varepsilon y}\right)\frac{g(y)}{1-\varepsilon y}\lambda(dy) \\ &\le \int_{[0,\frac{1}{\varepsilon}(1-x)]} P_S h_\beta\left(\frac{x}{1-\varepsilon y}\right)\frac{g(y)}{1-\varepsilon y}\lambda(dy) \\ &\le \frac{1}{x^\beta}\int_{[0,\frac{1}{\varepsilon}(1-x)]} g(y)\lambda(dy) \le \frac{1}{x^\beta} = h_\beta(x). \end{aligned}$$

This yields $P_\varepsilon h_\beta(x) \le h_\beta(x)$ for a.e. $x \in (0,1]$. Therefore h_β is a locally integrable, positive and subinvariant function with respect to $\mathcal{A}$. □

Actually, the Markov operators defined by (3) with respect to some intermittent maps are sweeping for all noise level $0 < \varepsilon < 1$.

From now, we add assumptions to a non-singular positive a.e. transformation $S : [0,1] \to [0,1]$:

S1 There exists a partition $0 = a_0 < a_1 < \cdots < a_m = 1$ such that for each integer j, the restriction S_j of S to the interval $[a_j, a_{j+1})$ is C^1 monotonic function for $j = 1, \cdots, m-1$ and $S(0) = 0$ and $S'(0) = 1$.

S2 $a_1 \ge \frac{1}{2}$.

S3 $S(a_1) \ne 1$ if $m \ge 2$.

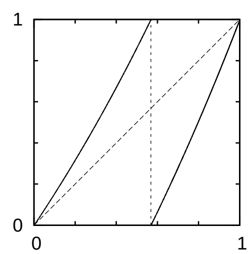

Figure 1: example of an intermittent map S satisfying S1-S3.

Lemma 3.2. *We denote $\varepsilon S(x)$ by $S_\varepsilon(x)$ for $x \in [0,1]$. Let P_{S_ε} be the Perron-Frobenius operator corresponding to S_ε for which conditions (S1)-(S3) are satisfied. Let $\mu_n^\varepsilon(dx) := P_{S_\varepsilon}^n f(x)dx$ for an arbitrarily $f \in D$. Then we have*

$$\mu_n^\varepsilon \Longrightarrow \delta_0 \qquad \textit{in weakly} \qquad \textit{as } n \to \infty$$

for each $0 < \varepsilon \le \frac{1}{2}$.

Proof We have $\lim_{n\to\infty} S_\varepsilon^n(x) = 0$ for all $x \in [0,1]$ because $S_\varepsilon([0,\frac{1}{2}])$ is included in $[0,\frac{1}{2})$ and $S_\varepsilon([0,1]) \subset [0,\frac{1}{2}]$ by the assumptions about S. This implies that for any bounded continuous function $r(x)$ on $[0,1]$,

$$\lim_{n\to\infty}\int_{[0,1]} r(x)\mu_n^\varepsilon(dx) = \lim_{n\to\infty}\int_{[0,1]} r\left(S_\varepsilon^n(x)\right) f(x)\lambda(dx) = \int_{[0,1]} r(0)f(x)\lambda(dx) = r(0)$$

by the dominated convergence theorem. Therefore μ_n^ε converges to the Dirac measurer supported by $\{0\}$. □

Remark 3.3. By the piecewise monotonicity of S from condition S1, we can see that S_ε is also the non-singular transformation for each $0 < \varepsilon \le \frac{1}{2}$.

The following theorem is our main result.

Theorem 3.4. *Let $S : [0,1] \to [0,1]$ be a transformation satisfying S1-S3 and P_ε be the Markov operator defined by (3) with respect to S and $0 < \varepsilon < 1$. Suppose that $(P_{S_\sigma}^n \mathbf{1}_{[0,1]}(x))' \le 0$ holds for all $n \ge 1$ and $\sigma \le \frac{1}{2}$.*

1 *For any $0 < \varepsilon \le \frac{1}{2}$, if the density function g satisfies*

$$-\|g\|_{L^\infty} \log(1-\varepsilon) \le 1, \tag{6}$$

then $\{P_\varepsilon^n\}$ is sweeping with respect to $\mathcal{A}$.

2 *For any $\frac{1}{2} < \varepsilon < 1$, if the density function g satisfies that*

$$\frac{-\|g\|_{L^\infty}}{\varepsilon}(1-\varepsilon)\log(1-\varepsilon) \le 1, \tag{7}$$

then $\{P_\varepsilon^n\}$ is sweeping with respect to $\mathcal{A}$.

Proof Fix $x \in (0,1]$ arbitrarily. Firstly, we consider the case **1**. For $0 < \varepsilon \le \frac{1}{2}$,

$$\begin{aligned}
P_\varepsilon \mathbf{1}_{[0,1]}(x) &= \int_{[0,1]} \mathbf{1}_{[0,1]}(y)\frac{1}{\varepsilon S(y)} g\left(\frac{1}{\varepsilon}\left(1-\frac{x}{S(y)}\right)\right)\lambda(dy)\\
&= \int_{[0,1]} P_{S_\varepsilon}\mathbf{1}_{[0,1]}(y)\frac{1}{y} g\left(\frac{1}{\varepsilon}-\frac{x}{y}\right)\lambda(dy)\\
&= \int_{[\varepsilon x, \frac{\varepsilon x}{1-\varepsilon}]} P_{S_\varepsilon}\mathbf{1}_{[0,1]}(y)\frac{1}{y} g\left(\frac{1}{\varepsilon}-\frac{x}{y}\right)\lambda(dy)
\end{aligned}$$

since support of g is included in $[0,1]$, the support of $g\left(\frac{1}{\varepsilon}-\frac{x}{y}\right)$ is included in

$\left[\varepsilon x, \frac{\varepsilon x}{1-\varepsilon}\right]$. Because of $(P_{S_\varepsilon}\mathbf{1}_{[0,1]}(x))' \le 0$ and Condition (6), we have

$$\begin{aligned} P_\varepsilon \mathbf{1}_{[0,1]}(x) &\le P_{S_\varepsilon}\mathbf{1}_{[0,1]}(\varepsilon x)\lambda(dy) \cdot \|g\|_{L^\infty} \int_{[\varepsilon x, \frac{\varepsilon x}{1-\varepsilon}]} \frac{1}{y}\lambda(dy) \\ &= P_{S_\varepsilon}\mathbf{1}_{[0,1]}(\varepsilon x) \cdot \|g\|_{L^\infty} \log\left(\frac{1}{1-\varepsilon}\right) \\ &\le P_{S_\varepsilon}\mathbf{1}_{[0,1]}(x). \end{aligned}$$

If $P_\varepsilon^n \mathbf{1}_{[0,1]}(x) \le P_{S_\varepsilon}^n \mathbf{1}_{[0,1]}(x)$ holds for some $n \ge 2$, then we have

$$\begin{aligned} P_\varepsilon^{n+1}\mathbf{1}_{[0,1]}(x) &\le P_\varepsilon(P_{S_\varepsilon}^n f(x)) \\ &= \int_{[\varepsilon x, \frac{\varepsilon x}{1-\varepsilon}]} P_{S_\varepsilon}^{n+1}\mathbf{1}_{[0,1]}(y)\frac{1}{y} g\left(\frac{1}{\varepsilon} - \frac{x}{y}\right)\lambda(dy) \\ &\le P_{S_\varepsilon}^{n+1}\mathbf{1}_{[0,1]}(\varepsilon x) = P_{S_\varepsilon}^{n+1}\mathbf{1}_{[0,1]}(x) \end{aligned}$$

whence by induction, it follows that

$$P_\varepsilon^n \mathbf{1}_{[0,1]}(x) \le P_{S_\varepsilon}^n \mathbf{1}_{[0,1]}(x) \qquad \text{for all } n \ge 0.$$

Therefore we have

$$\begin{aligned} &\int_{[c,1]} P_\varepsilon^n \mathbf{1}_{[0,1]}(x)\lambda(dx) \\ &\le \int_{[c,1]} P_{S_\varepsilon}^n \mathbf{1}_{[0,1]}(x)\lambda(dx) \to 0 \quad \text{as } n \to \infty \quad \text{for } 0 < c \le 1 \end{aligned}$$

by Lemma 3.2.

Consider the case **2**. With analogous considerations we have

$$\begin{aligned} P_\varepsilon \mathbf{1}_{[0,1]}(x) &= \int_{[0,1]} \mathbf{1}_{[0,1]}(y)\tfrac{1-\varepsilon}{\varepsilon} \cdot \tfrac{1}{(1-\varepsilon)S(y)} \cdot g\left(\tfrac{1}{\varepsilon} - \tfrac{1-\varepsilon}{\varepsilon} \cdot \tfrac{x}{(1-\varepsilon)S(y)}\right)\lambda(dy) \\ &= \int_{[0,1]} P_{S_{(1-\varepsilon)}}\mathbf{1}_{[0,1]}(y)\tfrac{1-\varepsilon}{\varepsilon y} \cdot g\left(\tfrac{1}{\varepsilon} - \tfrac{1-\varepsilon}{\varepsilon} \cdot \tfrac{x}{y}\right)\lambda(dy) \\ &= \int_{[(1-\varepsilon)x, x]} P_{S_{(1-\varepsilon)}}\mathbf{1}_{[0,1]}(y)\tfrac{1-\varepsilon}{\varepsilon y} g\left(\tfrac{1}{\varepsilon} - \tfrac{1-\varepsilon}{\varepsilon}\tfrac{x}{y}\right)\lambda(dy) \end{aligned}$$

since support of g is included in $[0,1]$, the support of $g\left(\frac{1}{\varepsilon} - \frac{x}{y}\right)$ is included in $[(1-\varepsilon)x, x]$. Because of $(P_{S_(1-\varepsilon)}\mathbf{1}_{[0,1]}(x))' \le 0$ and Condition (7), we have

$$\begin{aligned} P_\varepsilon \mathbf{1}_{[0,1]}(x) &\le P_{S_{(1-\varepsilon)}}\mathbf{1}_{[0,1]}((1-\varepsilon)x) \cdot \|g\|_{L^\infty} \int_{[(1-\varepsilon)x, x]} \frac{1-\varepsilon}{\varepsilon y}\lambda(dy) \\ &= P_{S_{(1-\varepsilon)}}\mathbf{1}_{[0,1]}(x) \cdot \|g\|_{L^\infty} \frac{1-\varepsilon}{\varepsilon} \log\left(\frac{1}{1-\varepsilon}\right) \\ &\le P_{S_{(1-\varepsilon)}}\mathbf{1}_{[0,1]}(x). \end{aligned}$$

Therefore by induction, it follows that

$$P_\varepsilon^n \mathbf{1}_{[0,1]}(x) \le P_{S_{(1-\varepsilon)}}^n \mathbf{1}_{[0,1]}(x) \qquad \text{for } x \in (0,1].$$

Therefore we have

$$\begin{aligned}&\int_{[c,1]} P_\varepsilon^n \mathbf{1}_{[0,1]}(x)\lambda(dx)\\ &\quad \le \int_{[c,1]} P_{S_{(1-\varepsilon)}}^n \mathbf{1}_{[0,1]}(x)\lambda(dx) \to 0 \quad \text{as } n\to\infty \quad \text{for } 0 < c \le 1\end{aligned}$$

by Lemma 3.2.

Give an arbitrary density function $f \in D$. Since for any $\delta > 0$, there exists a constant $M > 0$ such that

$$\int_{[0,1]} (f-M)^+ \lambda(dx) \le \delta,$$

where $(f)^+ = \max\{0, f-M\}$, we have that

$$\int_c^1 P_\varepsilon^n f(x)\lambda(dx) \le M \int_c^1 P_\varepsilon^n \mathbf{1}_{[0,1]}(x)\lambda(dx) + \delta.$$

Since $\{P_\varepsilon^n \mathbf{1}_{[0,1]}\}$ converges uniformly to zero on $[c,1]$ for each $0 < \varepsilon \le 1$, we have

$$\lim_{n\to\infty} \int_c^1 P_\varepsilon^n f(x)\lambda(dx) = 0 \qquad \text{for } 0 < c \le 1.$$

Then the proof is now completed. □

4 Examples

In this section, we give two examples which satisfy the sufficient conditions of Theorem 3.1 and 3.4.

Example 1

Let $S : [0,1] \to [0,1]$ be a map defined by

$$S(x) = \begin{cases} \dfrac{x}{1-x} & x \in \left[0, \dfrac{1}{2}\right) \\ 2x-1 & x \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

Indeed, S has the invariant density $\frac{1}{x}$ (cf. [5]). Therefore the Foguel Alternative theorem holds for the Markov operator defined by (3) with respect to $\mathcal{A} =$

$\{\{0\}\cup[c,1]:\ 0<c\leq 1\}$. Moreover this transformation satisfied the assumption of Theorem 3.4. Fix $0<\varepsilon\leq\frac{1}{2}$ arbitrarily. Since

$$S_\varepsilon(x)=\varepsilon S(x)\begin{cases}\dfrac{\varepsilon x}{1-x} & x\in\left[0,\dfrac{1}{2}\right)\\ (2x-1)\varepsilon & x\in\left[\dfrac{1}{2},1\right],\end{cases}$$

we have

$$P_{S_\varepsilon}f(x)=\frac{\varepsilon}{(\varepsilon+x)^2}f\left(\frac{x}{\varepsilon+x}\right)\cdot\mathbf{1}_{[0,\varepsilon]}(x)+\frac{1}{2\varepsilon}f\left(\frac{1}{2}+\frac{x}{2\varepsilon}\right)\cdot\mathbf{1}_{[0,\varepsilon]}(x).$$

First of all, we have

$$(P_{S_\varepsilon}\mathbf{1}_{[0,1]}(x))'=-\frac{2\varepsilon}{(\varepsilon+x)^3}\mathbf{1}_{[0,\varepsilon]}(x)\leq 0.$$

Furthermore, if we assume $(P^k_{S_\varepsilon}\mathbf{1}_{[0,1]}(x))'\leq 0$ for some $k\geq 2$ then we have

$$\begin{aligned}(P^{k+1}_{S_\varepsilon}\mathbf{1}_{[0,1]}(x))' &= \Bigg(\frac{\varepsilon}{(\varepsilon+x)^2}P^k_{S_\varepsilon}\mathbf{1}_{[0,1]}\left(\frac{x}{\varepsilon+x}\right)\cdot\mathbf{1}_{[0,\varepsilon]}(x)\\ &\qquad +\frac{1}{2\varepsilon}P^k_{S_\varepsilon}\mathbf{1}_{[0,1]}\left(\frac{1}{2}+\frac{x}{2\varepsilon}\right)\cdot\mathbf{1}_{[0,\varepsilon]}(x)\Bigg)'\\ &= \frac{-2\varepsilon x}{(\varepsilon+x)^3}P^k_{S_\varepsilon}\mathbf{1}_{[0,1]}\left(\frac{x}{\varepsilon+x}\right)\cdot\mathbf{1}_{[0,\varepsilon]}(x)\\ &\qquad +\frac{\varepsilon^2}{(\varepsilon+x)^4}\left(P^k_{S_\varepsilon}\mathbf{1}_{[0,1]}\left(\frac{x}{\varepsilon+x}\right)\right)'\cdot\mathbf{1}_{[0,\varepsilon]}(x)\\ &\qquad +\frac{1}{4\varepsilon^2}\left(P^k_{S_\varepsilon}\mathbf{1}_{[0,1]}\left(\frac{1}{2}+\frac{x}{2\varepsilon}\right)\right)'\cdot\mathbf{1}_{[0,\varepsilon]}(x)\\ &\leq 0 \qquad \text{for all } x\in[0,1].\end{aligned}$$

Therefore by induction, we have $(P^n_{S_\varepsilon}\mathbf{1}_{[0,1]}(x))'\leq 0$ for all $n\geq 1$. Therefore the intermittent map S satisfies the sufficient conditions of Theorem 3.4.

Example 2
Let $S:[0,1]\to[0,1]$ be a map defined by $S(x)=x$. Since $P_Sf(x)=f(x)$, it is obviously that $\frac{1}{x}$ is a positive subinvariant function with respect to $\mathcal{A}=\{\{0\}\cup[c,1]:\ 0<c\leq 1\}$ and S satisfies (S1)-(S3). Since $P^n_{S_\varepsilon}\mathbf{1}_{[0,1]}(x)=\frac{1}{\varepsilon^n}\mathbf{1}_{[0,\varepsilon^n]}(x)$ for $\varepsilon\leq\frac{1}{2}$, we have $P^n_{S_\varepsilon}\mathbf{1}_{[0,1]}(x)'=0$ for all $x\in[0,1]$ and $n\geq 0$. Therefore S satisfies the sufficient conditions of Theorem 3.4.

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