

## ON SEMIPRIME MODULES WITH CHAIN CONDITIONS

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### Abstract

Let  $R$  be an arbitrary ring,  $M$  a right  $R$ -module and  $S = \text{End}_R(M)$ , the endomorphism ring of  $M$ . A proper fully invariant submodule  $X$  of  $M$  is called a prime submodule of  $M$  if for any ideal  $I$  of  $S$  and any fully invariant submodule  $U$  of  $M$ , if  $I(U) \subset X$ , then either  $I(M) \subset X$  or  $U \subset X$ . A submodule  $X$  of  $M$  is called a semiprime submodule of  $M$  if it is an intersection of prime submodules. The module  $M$  is called a prime module if  $0$  is a prime submodule of  $M$ , and semiprime if  $0$  is a semiprime submodule of  $M$ . In this paper, we present some results on the classes of semiprime modules with chain conditions.

## 1. Introduction and Preliminaries

Throughout this paper, all rings are associative with identity and all modules are unitary right  $R$ -modules. Let  $M$  be a right  $R$ -module and  $S = \text{End}_R(M)$ , its endomorphism ring. A submodule  $X$  of  $M$  is called a *fully invariant* submodule of  $M$  if for any  $f \in S$ , we have  $f(X) \subset X$ . By definition, the class of all fully invariant submodules of  $M$  is nonempty and is closed under intersections and sums. Especially, a right ideal of  $R$  is a fully invariant submodule of  $R_R$  if it is a two-sided ideal of  $R$ . A fully invariant proper submodule  $X$  of  $M$  is called a *prime submodule* of  $M$  if for any ideal  $I$  of  $S$  and any fully

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invariant submodule  $U$  of  $M$ , if  $I(U) \subset X$ , then either  $I(M) \subset X$  or  $U \subset X$ . A prime submodule  $X$  of  $M$  is called *minimal* if it is minimal in the class of prime submodules of  $M$ . A fully invariant submodule  $X$  of  $M$  is called a *semiprime submodule* if it is an intersection of prime submodules of  $M$ . The module  $M$  is called a *prime module* if  $0$  is a prime submodule of  $M$  and *semiprime* if  $0$  is a semiprime submodule of  $M$ . A ring  $R$  is called a *prime ring* if  $R_R$  is a prime module and *semiprime* if  $R_R$  is a semiprime module. By symmetry, the ring  $R$  is a semiprime ring if  ${}_R R$  is a semiprime module. A submodule  $X$  of  $M$  is called an *essential submodule* if for any nonzero submodule  $U$  of  $M$ ,  $X \cap U \neq 0$ . A nonzero module  $M$  is said to be *uniform* if any two nonzero submodules of  $M$  have nonzero intersection, i.e., if each nonzero submodule of  $M$  is essential in  $M$ .

By a *complement* of  $X$ , we mean a submodule  $Y$  of  $M$  which is maximal with respect to the property  $X \cap Y = 0$ . A submodule  $K$  is called a *complement in  $M$*  if there exists a submodule  $L$  of  $M$  such that  $K$  is a complement of  $L$  in  $M$ . If a module  $M$  generates all its submodules, then it is called a *self-generator*.

For notations not defined here we refer the reader to [2], [5] and [10].

## 2. Prime submodules and semiprime submodules

Let  $X$  be a submodule of  $M$ . We denote  $I_X = \{f \in S \mid f(M) \subset X\}$ . Clearly  $I_X$  is a right ideal of  $S$ . If  $X$  is a fully invariant submodule of  $M$ , then  $I_X$  is an ideal of  $S$ . It had been shown in [7, Theorem 1.10] that if  $X$  is a prime submodule of  $M$ , then  $I_X$  is a prime ideal of  $S$ , where  $I_X = \{f \in S \mid f(M) \subset X\}$ . Conversely, if  $M$  is a self-generator and  $I_X$  is a prime ideal of  $S$ , then  $X$  is a prime submodule of  $M$ . Furthermore, we have:

**Proposition 2.1.** *Let  $M$  be a right  $R$ -module which is a self-generator. Then we have the following:*

- (1) *If  $X$  is a minimal prime submodule of  $M$ , then  $I_X$  is a minimal prime ideal of  $S$ .*
- (2) *If  $P$  is a minimal prime ideal of  $S$ , then  $X := P(M)$  is a minimal prime submodule of  $M$  and  $I_X = P$ .*

**Proof.** (1) Since  $X$  is a prime submodule of  $M$ , we have  $I_X$  is a prime ideal of  $S$ . Let  $J$  be a prime ideal of  $S$  with  $J \subset I_X$ . Since  $M$  is a self-generator, we can write  $J = I_{J(M)}$  and note that  $J(M)$  is a prime submodule of  $M$  with  $J(M) \subset X$ . By the minimality of  $X$ , we have  $J(M) = X$ . Hence  $J = I_X$ .

(2) Since  $M$  is a self-generator, we can write  $P = I_{P(M)} = I_X$ . Note that  $X := P(M)$  is a prime submodule of  $M$ . Let  $Y$  be a prime submodule of  $M$

such that  $Y \subset P(M)$ . Then  $I_Y$  is a prime ideal of  $S$  and  $I_Y \subset P = I_X$ . By the minimality of  $P$ , we have  $I_Y = P$ , and this implies that  $Y = P(M)$ .  $\square$

In [7] and [8], we have proved that if  $M$  is a semiprime module, then  $S$  is a semiprime ring. Conversely, if  $M_R$  is a quasi-projective, finitely generated, self-generator module and  $S$  is a semiprime ring, then  $M$  is a semiprime module. Moreover, we have the following:

**Proposition 2.2.** *Let  $M$  be a right  $R$ -module.*

- (1) *If  $X$  is a semiprime submodule of  $M$ , then  $I_X$  is a semiprime ideal of  $S$ .*
- (2) *If  $M$  is a self-generator and  $P$  is a semiprime ideal of  $S$ , then  $X := P(M)$  is a semiprime submodule of  $M$  and  $I_X = P$ .*

**Proof.** (1) Since  $X$  is a semiprime submodule of  $M$ , we can write  $X = \bigcap_{P \in \mathcal{F}} P$ , where each  $P \in \mathcal{F}$  is a prime submodule of  $M$ . So  $I_X = I_{\bigcap_{P \in \mathcal{F}} P} = \bigcap_{P \in \mathcal{F}} I_P$ . By [7, Theorem 1.10], it is easy to see that  $I_X$  is a semiprime ideal of  $S$ .

(2) Since  $M$  is a self-generator, we can write  $P = I_{P(M)} = I_X$ , which is a semiprime ideal of  $S$ . Hence

$$I_X = \bigcap_{K \subset S, K \text{ prime}} K = \text{Hom}(M, (\bigcap_{K \subset S, K \text{ prime}} K)(M)).$$

Let  $X = P(M)$ , where  $P$  is a semiprime ideal of  $S$ . Since  $M$  is a self-generator, we have  $P = I_{P(M)} = I_X$  and by our assumption,  $P = \bigcap_{K \in \Lambda} K$ , for some set  $\Lambda$  of prime ideals of  $S$ . Thus  $I_X = \text{Hom}(M, I_X(M)) = \text{Hom}(M, (\bigcap_{K \in \Lambda} K)(M))$ .

On the other hand,  $\bigcap_{K \in \Lambda} K = \bigcap_{K \in \Lambda} \text{Hom}(M, K(M)) = \text{Hom}(M, \bigcap_{K \in \Lambda} K(M))$ .

Thus  $(\bigcap_{K \in \Lambda} K)(M) = \bigcap_{K \in \Lambda} K(M)$  and therefore  $X = \bigcap_{K \in \Lambda} K(M)$ . Since  $K$  is a prime ideal of  $S$ ,  $K(M)$  is a prime submodule of  $M$ , proving that  $X$  is a semiprime submodule of  $M$ .  $\square$

**Proposition 2.3.** *Let  $M$  be a right  $R$ -module which is a self-generator and  $X$ , a fully invariant submodule of  $M$ . Then  $X$  is a semiprime submodule if and only if*

$$(\star) \quad \text{whenever } f \in S \text{ with } fSf(M) \subset X, \text{ then } f(M) \subset X.$$

**Proof.** Suppose that  $X$  is a semiprime submodule. Then  $X = \bigcap_{P \in \mathcal{F}} P$ , for some family  $\mathcal{F}$  of prime submodules of  $M$ . Let  $f \in S$  with  $fSf(M) \subset X$ . This

implies that  $fSf(M) \subset P$ , for all  $P \in \mathcal{F}$ . By the primeness of  $P$ , we have  $f(M) \subset P$ , for all  $P \in \mathcal{F}$ . Thus  $f(M) \subset X$ .

Conversely, suppose that  $(\star)$  holds. We will show that  $I_X$  is a semiprime ideal of  $S$ . Let  $f \in S$  with  $fSf \subset I_X$ . Then  $fSf(M) \subset X$  and so  $f(M) \subset X$ , by  $(\star)$ . Thus  $f \in I_X$ , showing that  $I_X$  is a semiprime ideal of  $S$ . Hence  $X$  is a semiprime submodule of  $M$ .  $\square$

The following theorem gives some characterizations of semiprime submodules similar to that of semiprime ideals in associative rings and we use it as a tool for checking the semiprimeness.

**Theorem 2.4.** *Let  $M$  be a right  $R$ -module which is a self-generator and  $X$ , a fully invariant submodule of  $M$ . Then the following conditions are equivalent:*

- (1)  $X$  is a semiprime submodule of  $M$ ;
- (2) If  $J$  is any ideal of  $S$  such that  $J^2(M) \subset X$ , then  $J(M) \subset X$ ;
- (3) If  $J$  is any ideal of  $S$  such that  $J(M) \not\subseteq X$ , then  $J^2(M) \not\subset X$ ;
- (4) If  $J$  is any right ideal of  $S$  such that  $J^2(M) \subset X$ , then  $J(M) \subset X$ ;
- (5) If  $J$  is any left ideal of  $S$  such that  $J^2(M) \subset X$ , then  $J(M) \subset X$ .

**Proof.** (1)  $\Rightarrow$  (4): Suppose that  $X = \bigcap_{P \in \mathcal{F}} P$ , where each  $P \in \mathcal{F}$  is prime. If  $J$  is any right ideal of  $S$  with  $J^2(M) \subset X$ , then  $J^2(M) \subset P$ , for all  $P \in \mathcal{F}$ . By the primeness of  $P$ , we have  $J(M) \subset P$ , for all  $P \in \mathcal{F}$ . Thus  $J(M) \subset X$ .

(4)  $\Rightarrow$  (3): This part is clear.

(3)  $\Rightarrow$  (2): Suppose that  $J(M) \not\subset X$ . Then  $J(M) + X \not\subseteq X$ . We can write  $J(M) + X = J(M) + I_X(M) = (J + I_X)(M) \not\subseteq X$ . But  $(J + I_X)^2(M) = (J^2 + JI_X + I_XJ + I_X^2)(M) = J^2(M) + JI_X(M) + I_XJ(M) + I_X^2(M) \subset X$ , a contradiction.

(2)  $\Rightarrow$  (1): Let  $f \in S$  with  $fSf(M) \subset X$ . Then  $(SfS)^2(M) = (SfSfS)(M) \subset X$ . So  $(SfS)(M) \subset X$ , and consequently,  $f(M) \subset X$ . Thus, by Proposition 2.3,  $X$  is a semiprime submodule of  $M$ .

(1)  $\Rightarrow$  (5): Suppose that  $X = \bigcap_{P \in \mathcal{F}} P$  and each  $P \in \mathcal{F}$  is prime. If  $J$  is any left ideal of  $S$  with  $J^2(M) \subset X$ , then  $J^2(M) \subset P$ , for all  $P \in \mathcal{F}$ . Then we write  $J^2(M) = J(J(M)) = JS(J(M)) \subset P$ . By the primeness of  $P$ , we have  $J(M) \subset P$ ,  $P \in \mathcal{F}$ . Thus  $J(M) \subset X$ .

(5)  $\Rightarrow$  (3): It is clear.

(3)  $\Rightarrow$  (2) by the same argument as that given in (2)  $\Rightarrow$  (1).  $\square$

**Corollary 2.5.** *Let  $M$  be a right  $R$ -module which is a self-generator and let  $X$  be a semiprime submodule of  $M$ . If  $J$  is a right or left ideal of  $S$  such that  $J^n(M) \subset X$  for some positive integer  $n$ , then  $J(M) \subset X$ .*

**Proof.** We prove the statement by induction on  $n$ . The case for  $n = 1$  is always true. Let  $n > 1$  and assume that the statement holds for lower powers. Since  $n \geq 2$ , we have  $2n - 2 \geq n$ , so  $(J^{n-1})^2(M) = (J^{2n-2})(M) \subset (J^n)(M) \subset X$ . By Theorem 2.4,  $J^{n-1}(M) \subset X$  and so  $J(M) \subset X$ , by the induction hypothesis. This completes the proof.  $\square$

### 3. Semiprime modules and chain conditions

Recall that a submodule  $U$  of a right  $R$ -module  $M$  is called an  $M$ -annihilator if  $X = \text{Ker}(I) = \bigcap_{f \in I} \text{Ker}(f)$  for some subset  $I$  of  $S$ . Before introducing a new notion related to  $M$ -annihilators, we first prove the following results.

**Lemma 3.1.** *Let  $M$  be a right  $R$ -module and  $U$ , a submodule of  $M$ . If  $U = \text{Ker}(I)$  for some right ideal  $I$  of  $S$ , then  $U$  is a fully invariant submodule of  $M$ .*

**Proof.** Take any  $\varphi \in S$ . Let  $y \in \varphi(U)$ . Then  $y = \varphi(x)$ , for some  $x \in U$ . For any  $f \in I$ , we have  $f(y) = f(\varphi(x)) = f\varphi(x) = 0$ , since  $f\varphi \in I$ . This shows that  $y \in \text{Ker}(I) = U$ , proving that  $U$  is a fully invariant submodule of  $M$ .  $\square$

**Theorem 3.2.** *Let  $M$  be a quasi-projective, finitely generated right  $R$ -module which is a self-generator. Suppose that  $M$  is a semiprime module and  $X$ , a fully invariant submodule of  $M$ . Then  $X$  has the unique complement  $\text{Ker}(I_X)$  and hence  $X \oplus \text{Ker}(I_X) \lesssim M$ .*

**Proof.** Since  $M$  is a semiprime module,  $S$  is a semiprime ring. Put  $U = \text{Ker}(I_X)$ . Then  $I_X(U) = 0$ . Since  $M$  is a self-generator, it follows that  $I_X(I_U(M)) = I_X I_U(M) = 0$ . It follows that  $I_X I_U = 0$ . By [10, Proposition 3.13], we can see that  $I_X \cap I_U = 0$ . Since  $M$  is quasi-projective and finitely generated, we have  $I_X = \text{Hom}(M, X)$  and  $I_U = \text{Hom}(M, U) = \text{Hom}(M, \text{Ker}(I_X))$ . Thus  $0 = I_X \cap I_U = \text{Hom}(M, X \cap \text{Ker}(I_X))$ . Since  $M$  is a self-generator, we have  $X \cap \text{Ker}(I_X) = 0$ . Note that  $I_X$  is a two-sided ideal of  $S$ . Let  $Y$  be any complement of  $X$  in  $M$ . Then  $I_X \cap I_Y = I_{X \cap Y} = 0$ , and so  $I_Y I_X = 0$ . Since  $S$  is a semiprime ring, we get  $I_X I_Y = 0$  by [2, page 12]. This implies that  $I_X I_Y(M) = 0$ . So  $Y = I_Y(M) \subset \text{Ker}(I_X)$ . Since  $Y$  is maximal with respect to the property that  $X \cap Y = 0$ , we have  $Y = \text{Ker}(I_X)$ . Thus  $\text{Ker}(I_X)$  is a unique complement of  $X$  and therefore,  $X \oplus \text{Ker}(I_X) \lesssim M$ .  $\square$

**Lemma 3.3.** *Let  $M$  be a quasi-projective, finitely generated right  $R$ -module which is a self-generator and  $X$ , a proper fully invariant submodule of  $M$ . If  $M$  is a semiprime module, then  $\text{Ker}(I_X) \neq 0$ .*

**Proof.** From the fact that  $M$  is a semiprime module, it implies that  $S$  is a

semiprime ring. Since  $X$  is a fully invariant submodule of  $M$ ,  $I_X$  is a two-sided ideal of  $S$ . Put  $N = \text{Ker}(I_X)$ . Then  $N$  is the unique complement of  $X$ , by Theorem 3.2. If  $N = 0$ , then  $X$  is essential in  $M$ . This implies that  $I_X$  is an essential ideal of  $S$ . Indeed, if  $J$  is any ideal of  $S$  with  $I_X \cap J = 0$ , then  $0 = I_X \cap J = \text{Hom}(M, X) \cap \text{Hom}(M, J(M)) = \text{Hom}(M, X \cap J(M))$ . Since  $M$  is a self-generator,  $X \cap J(M) = 0$ , and we have  $J(M) = 0$  because  $X$  is essential in  $M$ . It follows that  $J = 0$ , showing that  $I_X$  is an essential ideal of  $S$ . Since  $I_X \cap r(I_X) = 0$  and  $I_X$  is an essential ideal of  $S$ , we have  $r(I_X) = 0$ . Thus  $I_X = l(r(I_X)) = l(0) = S$ . It implies that  $X = S(M) = M$ , a contradiction. Therefore,  $\text{Ker}(I_X) \neq 0$ .  $\square$

In [7], we introduced the notion of minimal prime submodules and we proved that if  $X$  is a prime submodule of a module  $M$ , then  $X$  contains a minimal prime submodule of  $M$ . Moreover, we have the following proposition on minimal prime submodules of a semiprime module  $M$ . This fact can be considered as a generalization of [5, 11.40].

**Proposition 3.4.** *Let  $M$  be a semiprime right  $R$ -module which is a self-generator. Let  $X$  be a fully invariant submodule of  $M$  and  $\Omega$ , the set of minimal prime submodules of  $M$  which do not contain  $X$ . Then  $\text{Ker}(I_X) = \cap\{P : P \in \Omega\}$ .*

**Proof.** Let  $B = \cap\{P : P \in \Omega\}$ . Then any element in  $X \cap B$  is in the intersection of all minimal prime submodules of  $M$  and this intersection is 0 because  $M$  is a semiprime module. Thus  $X \cap B = 0$ . Note that  $I_X$  and  $I_B$  are two-sided ideals of  $S$ . We have  $I_X \cap I_B = I_{X \cap B} = 0$ . Since  $S$  is a semiprime ring, by [10, 3.13], we get  $I_X I_B = 0$ , and consequently,  $I_X I_B(M) = 0$ . It follows that  $I_X(B) = 0$ . So  $B \subset \text{Ker}(I_X)$ . On the other hand, for any  $P \in \Omega$ , we have  $I_X(\text{Ker}(I_X)) = 0 \subset P$ . Since  $P$  is a prime submodule of  $M$  and  $X = I_X(M) \not\subset P$ , we must have  $\text{Ker}(I_X) \subset P$ . This implies that  $\text{Ker}(I_X) \subset B$ . Therefore,  $\text{Ker}(I_X) = B$ , proving our proposition.  $\square$

**Definition 3.5.** A fully invariant submodule  $X$  of a right  $R$ -module  $M$  is called a *full  $M$ -annihilator* if  $X = \text{Ker}(I)$  for some ideal  $I$  of  $S$ . A full  $M$ -annihilator  $X$  of a right  $R$ -module  $M$  is said to be *maximal* if  $X \neq M$  and there are no full  $M$ -annihilators strictly between  $X$  and  $M$ .

As an immediate consequence of Proposition 3.3 is the following characterization of full  $M$ -annihilators in a semiprime module  $M$ .

**Lemma 3.6.** *Let  $M$  be a quasi-projective, finitely generated right  $R$ -module which is a self-generator and  $U$ , a full  $M$ -annihilator of  $M$ . If  $M$  is a semiprime module and  $N = \text{Ker}(I_U)$ , then  $U = \text{Ker}(I_N)$ .*

**Proof.** Since  $U$  is a full  $M$ -annihilator of  $M$ , we can write  $U = \text{Ker}(K)$  for

some ideal  $K$  of  $S$ . Then  $U = Ker(K) = Ker(I_{K(M)})$  is a unique complement of  $X = K(M)$ , by Theorem 3.2. Since  $N = Ker(I_U)$  is a unique complement of  $U$ , we have  $X \subset N$ . This implies that  $I_X \subset I_N$ . So  $U = Ker(I_X) \supset Ker(I_N)$ . Since  $Ker(I_N)$  is a unique complement of  $N$  and  $U \cap N = 0$ , we have  $U = Ker(I_N)$ , proving our lemma.  $\square$

**Proposition 3.7.** *Let  $M$  be a quasi-projective, finitely generated right  $R$ -module which is a self-generator and  $X$ , a fully invariant submodule of  $M$ . If  $M$  is a semiprime module, then  $X$  is a full  $M$ -annihilator if and only if  $X$  is a complement of  $Y$  for some fully invariant submodule  $Y$  of  $M$ .*

**Proof.** Suppose that  $X$  is a full  $M$ -annihilator. Then  $X = Ker(K)$  for some ideal  $K$  of  $S$ . Since  $M$  is a self-generator, we can write  $K = I_{K(M)}$ . It follows that  $X = Ker(I_{K(M)})$  is a unique complement of  $Y = K(M)$ , by Theorem 3.2.

Conversely, suppose that  $X$  is a complement of  $Y$ , where  $Y$  is a fully invariant submodule of  $M$ . Since  $Y$  has a unique complement  $Ker(I_Y)$ , we have  $X = Ker(I_Y)$ , proving that  $X$  is a full  $M$ -annihilator.  $\square$

Note that, if we consider the bimodule  ${}_S M_R$ , then  $X$  is a fully invariant submodule of a right  $R$ -module  $M$  if and only if  $X$  is a bi-submodule of  ${}_S M_R$ . For convenience, we use the terminology "bi-submodule". To avoid some confusions, we will use roughly the terminology bi-essential submodules, bi-uniform submodules, bi-uniform dimensions, and bi-complements in the following definitions.

**Definition 3.8.** Consider the bimodule  ${}_S M_R$  and a bi-submodule  $X$  of  ${}_S M_R$ . We say that  $X$  is a *bi-essential submodule* of  ${}_S M_R$  (or  $X$  is *bi-essential in*  ${}_S M_R$ ) if for any bi-submodule  $U$  of  ${}_S M_R$ ,  $X \cap U = 0$  implies  $U = 0$ . If  $X$  is bi-essential in  ${}_S M_R$ , we denote  $X \lesssim {}_S M_R$ . A bi-submodule  $X$  of  ${}_S M_R$  is called a *bi-uniform submodule* if every nonzero bi-submodule of  $X$  is bi-essential in  $X$ . The bimodule  ${}_S M_R$  is called a *uniform bimodule* if every nonzero bi-submodule of  ${}_S M_R$  is bi-essential in  ${}_S M_R$ .

Note that, by Theorem 3.2, for a semiprime right  $R$ -module  $M$ , and for any bi-submodule  $X$  of  ${}_S M_R$ ,  $Ker(I_X)$  is the unique complement of  $X$  and it is fully invariant too. Therefore, in the context of semiprime modules, bi-essential is essential and bi-complements are complements as the usual definitions. From the above definitions, we can prove the following results.

**Proposition 3.9.** *Let  $M$  be a quasi-projective, finitely generated right  $R$ -module which is a self-generator and  $X$ , a fully invariant submodule of  $M$ . Suppose that  $M$  is a semiprime module. Then we have the following:*

- (1)  $X$  is a bi-essential submodule of  ${}_S M_R$  if and only if  $Ker(I_X) = 0$ ;
- (2) If  $X$  is not contained in any minimal prime submodule of  $M$ , then  $X$  is

a bi-essential submodule of  ${}_S M_R$ .

**Proof.** (1) It is clear because  $\text{Ker}(I_X)$  is the unique complement of  $X$ , by Theorem 3.2.

(2) Since  $X$  is not contained in any minimal prime submodule of  $M$ ,  $\text{Ker}(I_X)$  is the intersection of all minimal prime submodules of  $M$ , by Proposition 3.4. Therefore,  $\text{Ker}(I_X) = 0$  because  $M$  is semiprime and the result follows from (1).  $\square$

**Lemma 3.10.**

- (1) Let  $A, B, X, N$  be bi-submodules of  ${}_S M_R$  with  $A \subset B \subset X \subset N$ . If  $A$  is bi-essential in  ${}_S N_R$ , then  $B$  is bi-essential in  ${}_S X_R$ ;
- (2) If  $A_1, \dots, A_n$  are bi-essential submodules of  ${}_S M_R$ , then  $\bigcap_{i=1}^n A_i$  is bi-essential in  ${}_S M_R$ ;
- (3) If  $A, B$  are bi-submodules of  ${}_S M_R$  such that  $A$  is bi-essential in  ${}_S B_R$  and  $B$  is bi-essential in  ${}_S M_R$ , then  $A$  is bi-essential in  ${}_S M_R$ .

**Proof.** The proof of this lemma is a routine and we present here for the sake of completeness.

(1) Let  $U$  be a bi-submodule of  ${}_S X_R$  and  $B \cap U = 0$ . Then  $U$  is a bi-submodule of  ${}_S N_R$  and  $A \cap U = 0$ . Since  $A$  is bi-essential in  ${}_S N_R$ , we have  $U = 0$ , and consequently,  $B$  is bi-essential in  ${}_S X_R$ .

(2) We prove the statement by induction on  $n$ . The case  $n = 1$  is trivial by assumption. Suppose that  $A = \bigcap_{i=1}^{n-1} A_i$  is bi-essential in  ${}_S M_R$ . Let  $U$  be a bi-submodule of  ${}_S M_R$  and  $A \cap A_n \cap U = 0$ . Then  $A_n \cap U = 0$  because  $A$  is bi-essential in  ${}_S M_R$ . Since  $A_n$  is bi-essential in  ${}_S M_R$ , we have  $U = 0$ . Thus the result follows.

(3) Let  $X$  be a bi-submodule of  ${}_S M_R$  and  $X \cap A = 0$ . Then  $(X \cap B) \cap A = 0$  and since  $A$  is bi-essential in  ${}_S B_R$ , we have  $X \cap B = 0$ . Thus  $X = 0$  because  $B$  is bi-essential in  ${}_S M_R$ .  $\square$

**Lemma 3.11.** Let  $A$  be a bi-submodule of  ${}_S M_R$ . Then  $A$  is bi-essential in  ${}_S M_R$  if and only if for any  $m \in M$  with  $m \neq 0$ , there exist  $f_1, \dots, f_n \in S$  and  $r_1, \dots, r_n \in R$  such that  $\sum_{i=1}^n f_i(mr_i) \neq 0$  and  $\sum_{i=1}^n f_i(mr_i) \in A$ .

**Proof.** Suppose that  $A$  is bi-essential in  ${}_S M_R$ . Since  $m \neq 0$ , we have  $SmR \neq 0$  and so  $A \cap SmR \neq 0$ . We can find  $f_1, \dots, f_n \in S$  and  $r_1, \dots, r_n \in R$  such that  $\sum_{i=1}^n f_i(mr_i) \neq 0$  and  $\sum_{i=1}^n f_i(mr_i) \in A$ .

Conversely, suppose that for any  $m \in M$  with  $m \neq 0$ , there exist  $f_1, \dots, f_n \in S$  and  $r_1, \dots, r_n \in R$  such that  $\sum_{i=1}^n f_i(mr_i) \neq 0$  and  $\sum_{i=1}^n f_i(mr_i) \in A$ . Let  $B$  be a nonzero bi-submodule of  ${}_S M_R$ . Then there exists  $0 \neq m \in B$  such that  $\sum_{i=1}^n f_i(mr_i) \in B$ . Hence  $A \cap B \neq 0$ . This completes the proof.  $\square$

**Lemma 3.12.** *Let  $A, B, C, U$  be bi-submodules of  ${}_S M_R$  with  $A \subseteq_S^* B_R$  and  $C \subseteq_S^* U_R$ . If the sums  $A + C$  and  $B + U$  are direct, then  $A \oplus C$  is bi-essential in  $B \oplus U$  as a bi-submodule.*

**Proof.** Let  $m = b + u \in B \oplus U$  with  $b \in B, u \in U$  and  $m \neq 0$ . If  $b = 0$ , then  $u \neq 0$ . So  $SuR \neq 0$  and  $SuR \cap C \neq 0$  because  $C \subseteq_S^* U_R$ . This implies that  $SmR \cap (A + C) \neq 0$ , so we are done. Let therefore  $b \neq 0$ . Since  $A \subseteq_S^* B_R$ , there exist  $f_1, \dots, f_n \in S$  and  $r_1, \dots, r_n \in R$  such that  $0 \neq \sum_{i=1}^n f_i(br_i) \in A$ . If  $\sum_{i=1}^n f_i(ur_i) = 0$ , then  $\sum_{i=1}^n f_i(mr_i) = \sum_{i=1}^n f_i((b + u)r_i) = \sum_{i=1}^n f_i(br_i) \neq 0$  and  $\sum_{i=1}^n f_i(mr_i) \in A + C$ . Then we are done. If  $\sum_{i=1}^n f_i(ur_i) \neq 0$ , then there exist  $g_1, \dots, g_k \in S$  and  $t_1, \dots, t_k \in R$  such that  $0 \neq \sum_{j=1}^k g_j(\sum_{i=1}^n f_i(ur_i))t_j \in C$  since  $C \subseteq_S^* U_R$ . Consider the element  $\sum_{i=1, n; j=1, k} g_j f_i(mr_i t_j) = \sum_{i=1, n; j=1, k} g_j f_i(br_i t_j) + \sum_{i=1, n; j=1, k} g_j f_i(ur_i t_j) = \sum_{j=1}^k g_j(\sum_{i=1}^n f_i(br_i))t_j + \sum_{j=1}^k g_j(\sum_{i=1}^n f_i(ur_i))t_j \in A + C$ . If  $\sum_{i=1, n; j=1, k} g_j f_i(mr_i t_j) = 0$ , then  $\sum_{j=1}^k g_j(\sum_{i=1}^n f_i(ur_i))t_j \in A \cap C = 0$ , a contradiction. Thus  $\sum_{i=1, n; j=1, k} g_j f_i(mr_i t_j) \neq 0$ . It follows that  $A \oplus C$  is bi-essential in  $B \oplus U$  as a bi-submodule, by Lemma 3.11.  $\square$

Since Lemma 3.11 and Lemma 3.12 hold, so by induction, we can conclude that the following also holds.

**Proposition 3.13.** *Let  $A_i, B_i$  for  $i = 1, \dots, n$ , be bi-submodules of  ${}_S M_R$ . If  $A_i$  is bi-essential in  ${}_S B_{iR}$ , for all  $i = 1, \dots, n$  and the sums  $\sum_{i=1}^n A_i, \sum_{i=1}^n B_i$  are direct, then  $\bigoplus_{i=1}^n A_i$  is bi-essential in  $\bigoplus_{i=1}^n B_i$  as a bi-submodule.*

The following theorem gives some characterizations of maximal full  $M$ -annihilators in a semiprime module  $M$  similar to that of maximal annihilators in a semiprime ring  $R$ .

**Theorem 3.14.** *Let  $M$  be a quasi-projective, finitely generated right  $R$ -module which is a self-generator and  $X$ , a proper fully invariant submodule of  $M$ . If  $M$  is a semiprime module, then the following conditions are equivalent:*

- (1)  $X$  is a maximal full  $M$ -annihilator;
- (2)  $X$  is a minimal prime submodule and a full  $M$ -annihilator;
- (3)  $X$  is a prime submodule and a full  $M$ -annihilator;
- (4)  $X = \text{Ker}(I_U)$  for some bi-uniform submodule  $U$  of  ${}_S M_R$ .

Moreover, if  $M$  has only finitely many minimal prime submodules, then the above conditions are equivalent to:

- (5)  $X$  is a minimal prime submodule.

**Proof.** Since  $M$  is a semiprime module, we see that  $S$  is a semiprime ring.

(1)  $\Rightarrow$  (2): Assume that  $X$  is a full  $M$ -annihilator. So we can write  $X = \text{Ker}(K)$  for some nonzero ideal  $K$  of  $S$ . Let  $I$  be an ideal of  $S$  and  $U$ , a fully invariant submodule of  $M$  such that  $I(U) \subset X$  and  $I(M) \not\subset X$ . We must show that  $U \subset X$ . Since  $I(M) \not\subset X$ , then  $KI(M) \neq 0$  and so  $0 \neq KI \subset K$ . Therefore,  $M \neq \text{Ker}(KI) \supset \text{Ker}(K) = X$ . By the maximality of  $X$ , we have  $\text{Ker}(KI) = \text{Ker}(K)$ . Now  $I(U) \subset X$  implies  $K(I(U)) = KI(U) = 0$ . It follows that  $U \subset \text{Ker}(KI) = \text{Ker}(K)$ . Thus,  $X$  is a prime submodule of  $M$ .

We now suppose that  $P$  is a prime submodule of  $M$  and  $P \subsetneq X$ . Then  $K(X) = 0 \subset P$ . Since  $P$  is prime and  $X \not\subset P$ , we must have  $K(M) \subset P \subsetneq X$ . So  $0 = K(K(M)) = K^2(M)$  implies that  $K^2 = 0$ , a contradiction to  $S$  being a semiprime ring.

(2)  $\Rightarrow$  (3): Obvious.

(3)  $\Rightarrow$  (4): Put  $U = \text{Ker}(I_X)$ . Then  $X = \text{Ker}(I_U)$ , by Lemma 3.6. We now show that  $U$  is a bi-uniform submodule of  ${}_S M_R$ . Suppose  $U$  is not a bi-uniform submodule of  ${}_S M_R$ . Then there are nonzero fully invariant submodules  $X_1, X_2$  with  $X_1 \oplus X_2 \subset U$ . Since all minimal prime submodules intersect at 0, so  $X_1 \not\subset P$  for some minimal prime submodule  $P$ . Then  $I_{X_1}(\text{Ker}(I_{X_1})) = 0 \subset P$ . Since  $P$  is prime and  $X_1 = I_{X_1}(M) \not\subset P$ , we must have  $\text{Ker}(I_{X_1}) \subset P$ . Since  $\text{Ker}(I_{X_1})$  is the unique complement of  $X_1$ , we have  $I_{X_1}(X_2) = 0$  and  $I_U(X_2) \neq 0$ . This implies that  $\text{Ker}(I_{X_1}) \supsetneq \text{Ker}(I_U) = X$ . So  $P \supsetneq X$ , a contradiction to the fact that  $P$  is a minimal prime submodule of  $M$ .

(4)  $\Rightarrow$  (1): Suppose that  $X = \text{Ker}(I_U)$  for some bi-uniform submodule  $U$  of  ${}_S M_R$  and that  $X \subsetneq B$ , where  $B$  is a full  $M$ -annihilator. Since  $X = \text{Ker}(I_U)$  is the unique complement of  $U$ , we have  $B \cap U \neq 0$ . This implies that  $B \cap U$  is a nonzero bi-submodule of  ${}_S M_R$  and so  $B \cap U$  is an essential submodule of  $U$ . Therefore,  $(B \cap U) \oplus \text{Ker}(I_U)$  is essential in  $U \oplus \text{Ker}(I_U)$  and  $U \oplus \text{Ker}(I_U)$  is essential in  ${}_S M_R$ , i.e.,  $(B \cap U) \oplus \text{Ker}(I_U)$  is essential in  ${}_S M_R$ . Since  $(B \cap U) \oplus \text{Ker}(I_U)$  is contained in  $B$ , we see that so  $B$  is essential in  $M$ . It follows that  $I_B$  is an essential ideal of  $S$ . Since  $I_B \cap r(I_B) = 0$ , we have  $r(I_B) = 0$ . Thus  $I_B = l(r(I_B)) = S$  and so  $B = M$ . Therefore,  $X \subsetneq B = M$ .

Finally, suppose that  $M$  has only finitely many minimal prime submodules  $P_1, \dots, P_t$ . Using  $P_1 \cap \dots \cap P_t = 0$ , we see by Proposition 3.4 that  $P_i = \text{Ker}(I_{Q_i})$ , where  $Q_i = \bigcap_{k \neq i} P_k$ . In this case, (2)  $\Leftrightarrow$  (5).  $\square$

**Proposition 3.15.** *Let  $U = U_1 \oplus \dots \oplus U_m$  and  $V = V_1 \oplus \dots \oplus V_n$  be bi-essential submodules of  ${}_S M_R$ , where the  $U_i$ 's and  $V_j$ 's are bi-uniform submodules. Then  $m = n$ .*

**Proof.** We may assume that  $n \geq m$ . We claim that  $\overline{U} := U_1 \oplus \dots \oplus U_m$  intersects trivially with some  $V_j$ . If otherwise,  $\overline{U} \cap V_j \neq 0$ , for all  $j = 1, \dots, n$ ; then we would have  $\overline{U} \cap V_j \lesssim^* V_j$ , since  $V_j$  is bi-uniform, and we get  $(\overline{U} \cap V_1) \oplus \dots \oplus (\overline{U} \cap V_n)$  is bi-essential in  $V_1 \oplus \dots \oplus V_n = V$  and hence also  $\overline{U} \cap V \lesssim^* V$  as a bi-submodule and  $V \lesssim^* {}_S M_R$ . This implies that  $\overline{U} \cap V \lesssim^* {}_S M_R$ , and consequently,  $\overline{U}$  is bi-essential in  ${}_S M_R$ , a contradiction. Therefore,  $\overline{U} \cap V_j = 0$  for some  $j$ . After relabelling the  $V_j$ 's, we may assume that  $\overline{U} \cap V_1 = 0$ . Let  $U' = \overline{U} \oplus V_1$ . We must then have  $U' \cap U_1 \neq 0$ . If otherwise,  $U_1 + \dots + U_m + V_1$  would be a direct sum, a contradiction to the fact that  $U \lesssim^* {}_S M_R$ . So  $(U' \cap U_1) \oplus U_2 \oplus \dots \oplus U_m$  is bi-essential in  $U_1 \oplus U_2 \oplus \dots \oplus U_m = U$  as a bi-submodule and  $U \lesssim^* {}_S M_R$ . Since the left hand side is contained in  $U'$ , it follows that  $U' \lesssim^* {}_S M_R$ . We have thus replaced the summand  $U_1$  by  $V_1$  when going from  $U$  to  $U'$ . Repeating the process, we can pass from  $U'$  to some bi-essential submodule  $U'' = V_1 \oplus V_2 \oplus U_3 \oplus \dots \oplus U_m$ . After  $m$  steps, we have a bi-essential submodule  $U^{(m)} = V_1 \oplus \dots \oplus V_m$ . But  $V = V_1 \oplus \dots \oplus V_n \lesssim^* {}_S M_R$ , so we must have  $m = n$ .  $\square$

**Definition 3.16.** We say that the bimodule  ${}_S M_R$  has bi-uniform dimension  $n$  (denote  $\dim({}_S M_R) = n$ ) if there is a bi-essential submodule  $V$  of  ${}_S M_R$  such that  $V$  is a direct sum of  $n$  bi-uniform submodules. By Proposition 3.15,  $\dim({}_S M_R)$  is well-defined. If no such integer exists, we write  $\dim({}_S M_R) = \infty$ . We can check that  $\dim({}_S M_R) = 0$  if and only if  ${}_S M_R = 0$ , and  $\dim({}_S M_R) = 1$  if and only if  ${}_S M_R$  is a uniform bimodule.

**Theorem 3.17.** *Let  $M$  be a quasi-projective, finitely generated right  $R$ -module which is a self-generator. Suppose that  $\dim({}_S M_R) = n < \infty$ . Then  $\dim({}_S S_S) = n$ .*

**Proof.** We first show that if  $U$  is a bi-uniform submodule of  ${}_S M_R$ , then  $I_U$  is a uniform ideal of  ${}_S S_S$ . Let  $J, K$  be nonzero ideals of  $S$  and  $J, K \subset I_U$ . Then  $0 \neq J(M), K(M) \subset U$ . So  $J(M) \cap K(M) \neq 0$  since  $U$  is a bi-uniform submodule of  ${}_S M_R$ . But  $J \cap K = \text{Hom}(M, J(M)) \cap \text{Hom}(M, K(M)) = \text{Hom}(M, J(M) \cap K(M))$ . Therefore,  $J \cap K \neq 0$ . It follows that  $I_U$  is a uniform ideal of  $S$ .

Since  $\dim({}_S M_R) = n < \infty$ , there exist  $n$  bi-uniform submodules  $U_1, \dots, U_n$  such that the sum  $U_1 + \dots + U_n$  is direct and is bi-essential in  ${}_S M_R$ . Then  $I_{U_1}, \dots, I_{U_n}$  are uniform ideals of  $S$ . It is easy to check that  $I_{U_i} \cap \sum_{j \neq i} I_{U_j} = 0$ ,

so the sum  $I_{U_1} + \cdots + I_{U_n}$  is direct. We will show that  $I_{U_1} + \cdots + I_{U_n}$  is essential in  $S$ . Let  $K$  be an ideal of  $S$  with  $(I_{U_1} + \cdots + I_{U_n}) \cap K = 0$ . Then  $K(M)$  is a bi-submodule of  ${}_S M_R$  and we can write  $0 = (I_{U_1} + \cdots + I_{U_n}) \cap K = \text{Hom}(M, (I_{U_1} + \cdots + I_{U_n})(M)) \cap \text{Hom}(M, K(M)) = \text{Hom}(M, I_{U_1}(M) + \cdots + I_{U_n}(M)) \cap \text{Hom}(M, K(M)) = \text{Hom}(M, U_1 + \cdots + U_n) \cap \text{Hom}(M, K(M)) = \text{Hom}(M, (U_1 + \cdots + U_n) \cap K(M))$ . Since  $M$  is a self-generator,  $(U_1 + \cdots + U_n) \cap K(M) = 0$ , and because  $U_1 + \cdots + U_n$  is bi-essential in  ${}_S M_R$ , we have  $K(M) = 0$ . Thus  $K = 0$ , proving that  $I_{U_1} + \cdots + I_{U_n}$  is an essential ideal of  $S$ . Thus  $\dim({}_S S_S) = n$ .  $\square$

**Theorem 3.18.** *Let  $M$  be a quasi-projective, finitely generated right  $R$ -module which is a self-generator. Suppose that  $\dim({}_S S_S) = n < \infty$ . Then  $\dim({}_S M_R) = n$ .*

**Proof.** We first show that if  $K$  is a uniform ideal of  $S$ , then  $K(M)$  is a bi-uniform submodule of  ${}_S M_R$  and  $K = I_{K(M)}$ . Suppose that  $X_1, X_2$  are nonzero bi-submodules of  ${}_S M_R$  such that  $X_1, X_2 \subset K(M)$ . Then  $0 \neq I_{X_1}, I_{X_2} \subset I_{K(M)} = K$ . We can write  $I_{X_1 \cap X_2} = I_{X_1} \cap I_{X_2} \neq 0$  since  $K$  is a uniform ideal of  $S$ . So  $X_1 \cap X_2 \neq 0$ .

From  $\dim({}_S S_S) = n < \infty$ , there exist  $n$  uniform ideals  $K_1, \dots, K_n$  of  $S$  such that the sum  $K_1 + \cdots + K_n$  is direct and is essential in  $S$ . Then  $K_1(M), \dots, K_n(M)$  are bi-uniform submodules of  ${}_S M_R$  and  $K_i = I_{K_i(M)}$ . Now, we show that the sum  $K_1(M) + \cdots + K_n(M)$  is direct and is bi-essential in  ${}_S M_R$ . We have

$$\begin{aligned} 0 &= K_i \cap \sum_{j \neq i} K_j = \text{Hom}(M, K_i(M)) \cap \text{Hom}(M, (\sum_{j \neq i} K_j)(M)) \\ &= \text{Hom}(M, K_i(M)) \cap \text{Hom}(M, \sum_{j \neq i} K_j(M)) \\ &= \text{Hom}(M, K_i(M) \cap \sum_{j \neq i} K_j(M)). \end{aligned}$$

So  $K_i(M) \cap \sum_{j \neq i} K_j(M) = 0$  since  $M$  is a self-generator. It shows that the sum  $K_1(M) + \cdots + K_n(M)$  is direct. Let  $X$  be a nonzero bi-submodule of  ${}_S M_R$  such that  $(K_1(M) + \cdots + K_n(M)) \cap X = 0$ . Then we can write  $(K_1 + \cdots + K_n) \cap I_X = \text{Hom}(M, (K_1 + \cdots + K_n)(M)) \cap \text{Hom}(M, I_X(M)) = \text{Hom}(M, (K_1(M) + \cdots + K_n(M)) \cap X) = 0$ . So  $I_X = 0$  since  $K_1 + \cdots + K_n$  is essential in  $S$ . It follows that  $X = 0$ , proving that  $K_1(M) + \cdots + K_n(M)$  is a bi-essential submodule of  ${}_S M_R$ . Thus  $\dim({}_S M_R) = n$ .  $\square$

**Proposition 3.19.** *Consider the bimodule  ${}_S M_R$ . Suppose that  $\dim({}_S M_R) = n < \infty$  and  $V_1, \dots, V_n$  are bi-uniform submodules of  ${}_S M_R$  such that the sum  $V = V_1 + \cdots + V_n$  is direct and is bi-essential in  ${}_S M_R$ . Then any direct sum of nonzero bi-submodules  $N = N_1 \oplus \cdots \oplus N_k \subset {}_S M_R$  has  $k \leq n$  summands.*

**Proof.** We prove the statement by induction on  $n$ . First consider the case

$n = 1$ . Then we see that  ${}_S M_R$  is a uniform bimodule. So the direct sum of nonzero bi-submodules of  ${}_S M_R$  has only  $k = 1$  summand. Suppose that the statement holds for  $n - 1$  summands. We check for the case  $\dim({}_S M_R) = n$ . Since  $V$  is bi-essential in  ${}_S M_R$ , put  $N'_i := N_i \cap V \neq 0$  and  $V \supset N'_1 \oplus \cdots \oplus N'_k$ . Thus we may assume that  $M = V$ , say  $M = V_1 \oplus \cdots \oplus V_n$ , where all the  $V_i$ 's are bi-uniform. Let  $\overline{N} = N_2 \oplus \cdots \oplus N_k$ . If  $\overline{N} \cap V_i \neq 0$  for all  $i = 1, \dots, n$ , then  $(\overline{N} \cap V_1) \oplus \cdots \oplus (\overline{N} \cap V_n) \subseteq^* V_1 \oplus \cdots \oplus V_n = M$ . This implies that  $\overline{N} \subseteq^* {}_S M_R$ , a contradiction. Therefore,  $\overline{N} \cap V_i = 0$  for some  $i$ . After relabelling the  $V_i$ 's, we may assume that  $\overline{N} \cap V_1 = 0$ . Projecting  $M$  modulo  $V_1$  onto  $V_2 \oplus \cdots \oplus V_n$ , we have then an embedding of  $\overline{N}$  into  $V_2 \oplus \cdots \oplus V_n$ . By assumption, we get  $k - 1 \leq n - 1$  and so  $k \leq n$ .  $\square$

**Lemma 3.20.** *Consider the bimodule  ${}_S M_R$ . If  ${}_S M_R$  does not contain a direct sum of an infinite number of nonzero bi-submodules, then any nonzero bi-submodule  $N \subset {}_S M_R$  contains a bi-uniform submodule.*

**Proof.** If  ${}_S N_R$  does not contain any bi-uniform submodule, then  ${}_S N_R$  itself is not bi-uniform, so  ${}_S N_R$  contains some  $A_1 \oplus B_1$ , where  $A_1, B_1$  are nonzero bi-submodules. Then  $B_1$  is also not bi-uniform, so  $B_1$  contains some  $A_2 \oplus B_2$ , where  $A_2, B_2$  are nonzero bi-submodules. Continuing the process, we will get an infinite direct sum  $A_1 \oplus A_2 \oplus A_3 \oplus \cdots \subset {}_S M_R$ , a contradiction. Thus the result follows.  $\square$

**Proposition 3.21.**  *$\dim({}_S M_R) = \infty$  if and only if  ${}_S M_R$  contains an infinite direct sum of nonzero bi-submodules.*

**Proof.** If  ${}_S M_R$  contains an infinite direct sum of nonzero bi-submodules, then  $\dim({}_S M_R) = \infty$ , by Lemma 3.20.

Conversely, suppose that  ${}_S M_R$  does not contain an infinite direct sum of nonzero bi-submodules. Pick a bi-uniform submodule  $V_1 \subset {}_S M_R$ . If  $V_1$  is not bi-essential in  ${}_S M_R$ , then  ${}_S M_R$  contains  $V_1 \oplus V_2$ , for some nonzero bi-submodule  $V_2$ , and we may assume that  $V_2$  is bi-uniform. If  $V_1 \oplus V_2$  is not bi-essential in  ${}_S M_R$ , then  $V_1 \oplus V_2 \oplus V_3 \subset {}_S M_R$  where  $V_3$  is a bi-uniform submodule. By our assumption, this process must stop, and we arrive at some bi-essential submodule  $V_1 \oplus \cdots \oplus V_n$  where each  $V_i$  is bi-uniform. By definition, we have  $\dim({}_S M_R) = n$ .  $\square$

We now explore the meaning of bi-uniform dimensions and we need more concepts of bi-complements in the bimodule  ${}_S M_R$ .

**Definition 3.22.** Let  $X$  be a bi-submodule of the bimodule  ${}_S M_R$ . We say that  $X$  is a *bi-complement* in  ${}_S M_R$  (denote  $X \subset_c {}_S M_R$ ) if there exists a bi-submodule  $Y \subset {}_S M_R$  such that  $X$  is a bi-complement of  $Y$  in  ${}_S M_R$ .

**Proposition 3.23.** *Consider the bimodule  ${}_S M_R$ . Suppose that  $X \subset_c {}_S M_R$  and*

$T$  is a bi-submodule of  ${}_S M_R$  such that  $X \cap T = 0$ . Then  $X$  is a bi-complement of  $T$  if and only if  $X \oplus T \lesssim_S M_R$ .

**Proof.** If  $X$  is a bi-complement of  $T$ , then  $X \oplus T \lesssim_S M_R$ . Conversely, assume that  $X \oplus T \lesssim_S M_R$ . Since  $X \subset_c {}_S M_R$ , there is a bi-submodule  $U \subset {}_S M_R$  such that  $X$  is a bi-complement of  $U$ . We show that  $X$  is maximal with respect to the property that  $X \cap T = 0$ . Let  $D$  be a bi-submodule of  ${}_S M_R$  such that  $X \subset D$  and  $D \cap T = 0$ . We have  $(X + T) \cap (D \cap U) = ((X + T) \cap D) \cap U = X \cap U = 0$ . Since  $X \oplus T \lesssim_S M_R$ , we have  $D \cap U = 0$ . This implies that  $D = X$ . Thus  $X$  is a bi-complement of  $T$ .  $\square$

**Corollary 3.24.** *Suppose that  $X \subset_c {}_S M_R$ . Let  $T$  be a bi-complement of  $X$  in  $M$ . Then  $X$  is a bi-complement of  $T$ .*

**Proof.** Since  $T$  is a bi-complement of  $X$ , we have  $T \oplus X \lesssim_S M_R$ . But then by Proposition 3.23, we can conclude that  $X$  is a bi-complement of  $T$ .  $\square$

The next result describes some basic properties of bi-complements in the bimodule  ${}_S M_R$ .

**Proposition 3.25.** *Let  $X, N$  be bi-submodules of the bimodule  ${}_S M_R$  such that  $X \subset N \subset {}_S M_R$ . Then we have the following:*

- (1) *If  $X \subset_c {}_S M_R$ , then  $X \subset_c {}_S N_R$ .*
- (2) *If  $X \subset_c {}_S N_R$  and  $N \subset_c {}_S M_R$ , then  $X \subset_c {}_S M_R$ .*

**Proof.** (1) If  $X \subset_c {}_S M_R$ , then there exists a bi-submodule  $Y \subset {}_S M_R$  such that  $X$  is a bi-complement of  $Y$  in  ${}_S M_R$ . This implies that  $X$  is a bi-complement of  $Y \cap N$  in  ${}_S N_R$ . Thus  $X \subset_c {}_S N_R$ .

(2) Suppose that  $X$  is a bi-complement of  $U$  in  ${}_S N_R$  and  $N$  is a bi-complement of  $T$  in  ${}_S M_R$ . We will show that  $X$  is a bi-complement of  $U \oplus T$  in  ${}_S M_R$ . It is clear that  $X \cap (U + T) = 0$ . Let  $D$  be a bi-submodule of  ${}_S M_R$  such that  $D \supseteq X$ . We need to show that  $D \cap (U + T) \neq 0$ . If  $D \cap N \supseteq X$ , then  $(D \cap N) \cap U \neq 0$ . So  $D \cap U \neq 0$ , and consequently,  $D \cap (U + T) \neq 0$ . Then we are done. Now, consider the case  $D \cap N = X$ . Then there exists  $d \in D \setminus N$  such that  $(N + SdR) \cap T \neq 0$ . Therefore, there exist  $n \in N, t \in T \setminus \{0\}, f_1, \dots, f_k \in S, r_1, \dots, r_k \in R$  such that

$$n + \sum_{i=1}^k f_i(dr_i) = t \quad (1)$$

If  $n \in X$ , then  $n + \sum_{i=1}^k f_i(dr_i) \in D$  and so  $D \cap T \neq 0$ . It follows that  $D \cap (U + T) \neq 0$  and we are done. If  $n \notin X$ , then  $(X + SnR) \cap U \neq 0$  and

there exist  $x \in X, u \in U \setminus \{0\}, g_1, \dots, g_m \in S, t_1, \dots, t_m \in R$  such that

$$x + \sum_{j=1}^m g_j(nt_j) = u \tag{2}$$

From (1), we have

$$\sum_{j=1}^m g_j(nt_j) + \sum_{j=1}^m g_j\left(\sum_{i=1}^k f_i(dr_i)\right)t_j = \sum_{j=1}^m g_j(tt_j) \tag{3}$$

Subtracting (3) from (2), we get

$$x - \sum_{j=1}^m g_j\left(\sum_{i=1}^k f_i(dr_i)\right)t_j = u - \sum_{j=1}^m g_j(tt_j) \in (D \cap (U \oplus T)) \setminus \{0\}.$$

□

In the next few results, we shall explore the relationship between bi-uniform dimensions and bi-complements. The first result about bimodules of finite bi-uniform dimensions is an analogue of Proposition 3.19.

**Proposition 3.26.** *Suppose that  $\dim({}_S M_R) = n < \infty$ . Then any chain of bi-complements in  ${}_S M_R$  has length  $\leq n$ . More precisely, if  $C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_k$  where the  $C_i$ 's are bi-complements in  ${}_S M_R$ , then  $k \leq n$ .*

**Proof.** By Proposition 3.25(1), we have  $C_{i-1} \subset_c C_i$ , say,  $C_{i-1}$  is a bi-complement of  $U_i$  in  $C_i$  for  $1 \leq i \leq k$ . Since  $C_{i-1} \neq C_i$ , then  $U_i \neq 0$ . Now we have  $U_1 \oplus \dots \oplus U_k \subset {}_S M_R$ , so  $k \leq n$ , by Proposition 3.19. □

Next we present the analogue of Proposition 3.21.

**Proposition 3.27.** *For the bimodule  ${}_S M_R$ , the following are equivalent:*

- (1)  $\dim({}_S M_R) = \infty$ ;
- (2) *There exists an infinite strictly ascending chain of bi-complements in  ${}_S M_R$ ;*
- (3) *There exists an infinite strictly descending chain of bi-complements in  ${}_S M_R$ .*

**Proof.** (1)  $\Rightarrow$  (2): By Proposition 3.21,  ${}_S M_R$  contains  $U_1 \oplus U_2 \oplus \dots$ , where each  $U_i$  is a nonzero bi-submodule of  ${}_S M_R$ . Enlarge  $U_1$  into a bi-complement to  $U_2 \oplus U_3 \oplus \dots$ , say  $C_1$ . Then we enlarge  $C_1 \oplus U_2$  into a bi-complement to  $U_3 \oplus U_4 \oplus \dots$ , say  $C_2$ . In this way, we get an ascending chain  $C_1 \subset C_2 \subset \dots$ , where each  $C_i$  is a bi-complement in  ${}_S M_R$ . Since  $C_i \supset U_i$  and  $C_{i-1} \cap U_i = 0$ , we have  $C_{i-1} \neq C_i$  for each  $i$ .

(2)  $\Rightarrow$  (3): Suppose that we have a strictly ascending chain of bi-complements in  ${}_S M_R$ , say  $C_0 \subsetneq C_1 \subsetneq \cdots$ . Then  $C_{i-1}$  is a bi-complement to some nonzero  $U_i$  in  $C_i$ . Enlarge  $U_1 \oplus U_2 \oplus \cdots$  into a bi-complement to  $C_0$ , say  $Y_1$ . Working in  $Y_1$ , enlarge  $U_2 \oplus U_3 \oplus \cdots$  into a bi-complement to  $U_1$  in  $Y_1$ , say  $Y_2$ . By Proposition 3.25(2),  $Y_2 \subset_c Y_1 \subset_c {}_S M_R$  implies that  $Y_2 \subset_c {}_S M_R$ . We have  $Y_1 \neq Y_2$  because  $Y_1 \supset U_1$  and  $Y_2 \cap U_1 = 0$ . Continuing this process, we get a strictly descending chain of bi-complements  $Y_1 \supsetneq Y_2 \supsetneq \cdots$  in  $M$ .

(3)  $\Rightarrow$  (1): Follows from Proposition 3.26.  $\square$

Negating the three statements in Proposition 3.27, we get the following equivalent result.

**Proposition 3.28.** *For the bimodule  ${}_S M_R$ , the following are equivalent:*

- (1)  $\dim({}_S M_R) < \infty$ ;
- (2) The bi-complements in  ${}_S M_R$  satisfy the ACC;
- (3) The bi-complements in  ${}_S M_R$  satisfy the DCC.

Finally, we get the following theorem which offers various criteria for a semiprime module to have only finitely many minimal prime submodules.

**Theorem 3.29.** *Let  $M$  be a quasi-projective, finitely generated right  $R$ -module which is a self-generator. If  $M$  is a semiprime module, then the following conditions are equivalent:*

- (1)  $n := \dim({}_S M_R) < \infty$ ;
- (2) The number  $t$  of minimal prime submodules of  $M$  is finite;
- (3) The number  $m$  of full  $M$ -annihilators of  $M$  is finite;
- (4)  $M_R$  has the ACC on full  $M$ -annihilators;
- (4')  $M_R$  has the DCC on full  $M$ -annihilators;
- (5)  ${}_S M_R$  has the ACC on bi-complements;
- (5')  ${}_S M_R$  has the DCC on bi-complements.

*If these conditions hold, then  $n = t$  and  $m = 2^t$ . Finally,  $n = t = 1$  if and only if  $M$  is a prime module.*

**Proof.** (1)  $\Rightarrow$  (2): Let  $U_i$  ( $1 \leq i \leq n$ ) be bi-uniform submodules of the bimodule  ${}_S M_R$  such that the direct sum  $U_1 \oplus \cdots \oplus U_n$  is a bi-essential submodule of  ${}_S M_R$ . Put  $P_i = \text{Ker}(I_{U_i})$ . Then  $P_i$  is a minimal prime submodule of  $M$ , by Theorem 3.14. Let  $P$  be a minimal prime submodule of  $M$ . Then for each

$i = 1, \dots, n$ , we have  $I_{U_i}(P_i) = 0 \subset P$ . By the primeness of  $P$ , we have either  $I_{U_i}(M) \subset P$  or  $P_i \subset P$ , i.e., either  $U_i \subset P$  or  $P_i \subset P$ . If  $U_i \subset P$  for all  $i = 1, \dots, n$ , then  $U_1 \oplus \dots \oplus U_n \subset P$ . So  $\text{Ker}(I_{U_1 \oplus \dots \oplus U_n}) \supset \text{Ker}(I_P) \neq 0$ . This implies that  $\text{Ker}(I_{U_1 \oplus \dots \oplus U_n}) \neq 0$ . On the other hand,  $\text{Ker}(I_{U_1 \oplus \dots \oplus U_n}) \cap (U_1 \oplus \dots \oplus U_n) = 0$  and  $U_1 \oplus \dots \oplus U_n$  is a bi-essential submodule of  ${}_S M_R$ , implying that  $\text{Ker}(I_{U_1 \oplus \dots \oplus U_n}) = 0$ , a contradiction. Thus,  $P_i \subset P$  for some  $i$ . By the minimality of  $P$ , we have  $P_i = P$ . This shows (2), and we see that  $t = n$ .

(2)  $\Rightarrow$  (3): From (2), we see that  $t$  is finite. By Proposition 3.4, we have  $m \leq 2^t < \infty$ . By Theorem 3.14, each minimal prime submodule of  $M$  is a full  $M$ -annihilator. This implies that the finite intersection of full  $M$ -annihilators is also a full  $M$ -annihilator. So  $2^t \leq m$ . Thus  $m = 2^t$ .

(3)  $\Rightarrow$  (4): Clear from (3).

(4)  $\Leftrightarrow$  (5) and (4')  $\Leftrightarrow$  (5') follow from Proposition 3.7 and Proposition 3.28.

(4)  $\Rightarrow$  (4'): Suppose that we have a descending chain of full  $M$ -annihilators, say  $U_1 \supset U_2 \supset \dots \supset U_n \supset \dots$ . Then  $\text{Ker}(I_{U_1}) \subset \text{Ker}(I_{U_2}) \subset \dots \subset \text{Ker}(I_{U_n}) \subset \dots$  is an ascending chain of full  $M$ -annihilators. By (4), there is an integer  $k$  such that  $\text{Ker}(I_{U_k}) = \text{Ker}(I_{U_j})$  for all  $j > k$ . Put  $N_i = \text{Ker}(I_{U_i})$ , so we have  $U_k = \text{Ker}(I_{N_k}) = \text{Ker}(I_{N_j}) = U_j$  for all  $j > k$ , by Lemma 3.6. Thus  $M_R$  has the DCC for full  $M$ -annihilators.

(4')  $\Rightarrow$  (4): Similar to (4)  $\Rightarrow$  (4').

(5)  $\Rightarrow$  (1): Follows from Proposition 3.28.

The last statement in this proposition is clear.  $\square$

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