East-West J. of Mathematics: Vol. 15, No 2 (2013) pp. 135-151

ON SEMIPRIME MODULES WITH CHAIN CONDITIONS

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Abstract

Let R be an arbitrary ring, M a right R-module and $S = End_R(M)$, the endomorphism ring of M. A proper fully invariant submodule X of M is called a prime submodule of M if for any ideal I of S and any fully invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. A submodule X of M is called a semiprime submodule of M if it is an intersection of prime submodules. The module M is called a prime module if 0 is a prime submodule of M, and semiprime if 0 is a semiprime submodule of M. In this paper, we present some results on the classes of semiprime modules with chain conditions.

1. Introduction and Preliminaries

Throughout this paper, all rings are associative with identity and all modules are unitary right R-modules. Let M be a right R-module and $S = \mathrm{End}_R(M)$, its endomorphism ring. A submodule X of M is called a *fully invariant* submodule of M if for any $f \in S$, we have $f(X) \subset X$. By definition, the class of all fully invariant submodules of M is nonempty and is closed under intersections and sums. Especially, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal of R. A fully invariant proper submodule X of M is called a *prime submodule* of M if for any ideal I of S and any fully

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Key words: Full M-annihilators, maximal full M-annihilators, bimodules, bi-essential submodules, bi-uniform submodules, bi-uniform dimensions, bi-complements.
2010 AMS Subject Classification: 16D50, 16D70, 16D80.

invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. A prime submodule X of M is called *minimal* if it is minimal in the class of prime submodules of M. A fully invariant submodule X of M is called a *semiprime submodule* if it is an intersection of prime submodules of M. The module M is called a *prime module* if 0 is a prime submodule of M and *semiprime* if 0 is a semiprime submodule of M. A ring R is called a *prime ring* if R_R is a prime module and *semiprime* if R_R is a semiprime module. By symmetry, the ring R is a semiprime ring if ${}_RR$ is a semiprime module. A submodule X of M is called an *essential submodule* if for any nonzero submodule U of M, $X \cap U \neq 0$. A nonzero module M is said to be *uniform* if any two nonzero submodules of M have nonzero intersection, i.e., if each nonzero submodule of M is essential in M.

By a *complement* of X, we mean a submodule Y of M which is maximal with respect to the property $X \cap Y = 0$. A submodule K is called a *complement in* M if there exists a submodule L of M such that K is a complement of L in M. If a module M generates all its submodules, then it is called a *self-generator.*

For notations not defined here we refer the reader to [2], [5] and [10].

2. Prime submodules and semiprime submodules

Let X be a submodule of M. We denote $I_X = \{f \in S \mid f(M) \subset X\}$. Clearly I_X is a right ideal of S. If X is a fully invariant submodule of M, then I_X is an ideal of S. It had been shown in [7, Theorem 1.10] that if X is a prime submodule of M, then I_X is a prime ideal of S, where $I_X = \{f \in S \mid f(M) \subset X\}$. Conversely, if M is a self-generator and I_X is a prime ideal of S, then X is a prime submodule of M. Furthermore, we have:

Proposition 2.1. *Let M be a right R-module which is a self-generator. Then we have the following:*

(1) If X is a minimal prime submodule of M, then I_X is a minimal prime ideal of S.

(2) If P is a minimal prime ideal of S, then $X := P(M)$ is a minimal prime submodule of M and $I_X = P$.

Proof. (1) Since X is a prime submodule of M, we have I_X is a prime ideal of S. Let J be a prime ideal of S with $J \subset I_X$. Since M is a self-generator, we can write $J = I_{J(M)}$ and note that $J(M)$ is a prime submodule of M with $J(M) \subset X$. By the minimality of X, we have $J(M) = X$. Hence $J = I_X$.

(2) Since M is a self-generator, we can write $P = I_{P(M)} = I_X$. Note that $X := P(M)$ is a prime submodule of M. Let Y be a prime submodule of M

such that $Y \subset P(M)$. Then I_Y is a prime ideal of S and $I_Y \subset P = I_X$. By the minimality of P, we have $I_Y = P$, and this implies that $Y = P(M)$. $\square$

In [7] and [8] , we have proved that if M is a semiprime module, then S is a semiprime ring. Conversely, if M_R is a quasi-projective, finitely generated, self-generator module and S is a semiprime ring, then M is a semiprime module. Moreover, we have the following:

Proposition 2.2. *Let M be a right R-module.*

(1) If X is a semiprime submodule of M, then I_X is a semiprime ideal of S.

(2) If M is a self-generator and P is a semiprime ideal of S, then $X := P(M)$ is a semiprime submodule of M and $I_X = P$.

Proof. (1) Since X is a semiprime submodule of M, we can write $X = \bigcap\limits_{P\in\mathcal{F}} P$, where each $P \in \mathcal{F}$ is a prime submodule of M. So $I_X = I_{\bigcap\limits_{P\subset M, P\in\mathcal{F}} P} = \bigcap\limits_{P\subset M, P\in\mathcal{F}} I_P$. By [7, Theorem 1.10], it is easy to see that I_X is a semiprime ideal of S.

(2) Since M is a self-generator, we can write $P = I_{P(M)} = I_X$, which is a semiprime ideal of S. Hence

$$I_X = \bigcap_{K\subset S, K \text{ prime}} K = Hom(M, (\bigcap_{K\subset S, K \text{ prime}} K)(M)).$$

Let $X = P(M)$, where P is a semiprime ideal of S. Since M is a self-generator, we have $P = I_{P(M)} = I_X$ and by our assumption, $P = \bigcap\limits_{K\in\Lambda} K$, for some set Λ of prime ideals of S. Thus $I_X = Hom(M, I_X(M)) = Hom(M, (\bigcap\limits_{K\in\Lambda} K)(M))$.

On the other hand, $\bigcap\limits_{K\in\Lambda} K = \bigcap\limits_{K\in\Lambda} Hom(M, K(M)) = Hom(M, \bigcap\limits_{K\in\Lambda} K(M))$.

Thus $(\bigcap\limits_{K\in\Lambda} K)(M) = \bigcap\limits_{K\in\Lambda} K(M)$ and therefore $X = \bigcap\limits_{K\in\Lambda} K(M)$. Since K is a prime ideal of S, $K(M)$ is a prime submodule of M, proving that X is a semiprime submodule of M. $\square$

Proposition 2.3. *Let M be a right R-module which is a self-generator and X, a fully invariant submodule of M. Then X is a semiprime submodule if and only if*

($\star$) whenever $f \in S$ with $fSf(M) \subset X$, then $f(M) \subset X$.

Proof. Suppose that X is a semiprime submodule. Then $X = \bigcap\limits_{P\in\mathcal{F}} P$, for some family $\mathcal{F}$ of prime submodules of M. Let $f \in S$ with $fSf(M) \subset X$. This implies that $fSf(M) \subset P$, for all $P \in \mathcal{F}$. By the primeness of P, we have $f(M) \subset P$, for all $P \in \mathcal{F}$. Thus $f(M) \subset X$.

Conversely, suppose that $(\star)$ holds. We will show that I_X is a semiprime ideal of S. Let $f \in S$ with $fSf \subset I_X$. Then $fSf(M) \subset X$ and so $f(M) \subset X$, by $(\star)$. Thus $f \in I_X$, showing that I_X is a semiprime ideal of S. Hence X is a semiprime submodule of M. □

The following theorem gives some characterizations of semiprime submodules similar to that of semiprime ideals in associative rings and we use it as a tool for checking the semiprimeness.

Theorem 2.4. *Let M be a right R-module which is a self-generator and X, a fully invariant submodule of M. Then the following conditions are equivalent:*

(1) X is a semiprime submodule of M;

(2) If J is any ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

(3) If J is any ideal of S such that $J(M) \supsetneqq X$, then $J^2(M) \not\subset X$;

(4) If J is any right ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

(5) If J is any left ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$.

Proof. (1) ⇒ (4): Suppose that $X = \bigcap_{P \in \mathcal{F}} P$, where each $P \in \mathcal{F}$ is prime. If J is any right ideal of S with $J^2(M) \subset X$, then $J^2(M) \subset P$, for all $P \in \mathcal{F}$. By the primeness of P, we have $J(M) \subset P$, for all $P \in \mathcal{F}$. Thus $J(M) \subset X$.

(4) ⇒ (3): This part is clear.

(3) ⇒ (2): Suppose that $J(M) \not\subset X$. Then $J(M) + X \supsetneqq X$. We can write $J(M) + X = J(M) + I_X(M) = (J + I_X)(M) \supsetneqq X$. But $(J + I_X)^2(M) = (J^2 + JI_X + I_XJ + I_X^2)(M) = J^2(M) + JI_X(M) + I_XJ(M) + I_X^2(M) \subset X$, a contradiction.

(2) ⇒ (1): Let $f \in S$ with $fSf(M) \subset X$. Then $(SfS)^2(M) = (SfSfS)(M) \subset X$. So $(SfS)(M) \subset X$, and consequently, $f(M) \subset X$. Thus, by Proposition 2.3, X is a semiprime submodule of M.

(1) ⇒ (5): Suppose that $X = \bigcap_{P \in \mathcal{F}} P$ and each $P \in \mathcal{F}$ is prime. If J is any left ideal of S with $J^2(M) \subset X$, then $J^2(M) \subset P$, for all $P \in \mathcal{F}$. Then we write $J^2(M) = J(J(M)) = JS(J(M)) \subset P$. By the primeness of P, we have $J(M) \subset P$, $P \in \mathcal{F}$. Thus $J(M) \subset X$.

(5) ⇒ (3): It is clear.

(3) ⇒ (2) by the same argument as that given in (2) ⇒ (1). □

Corollary 2.5. *Let M be a right R-module which is a self-generator and let X be a semiprime submodule of M. If J is a right or left ideal of S such that $J^n(M) \subset X$ for some positive integer n, then $J(M) \subset X$.*

Proof. We prove the statement by induction on n. The case for $n = 1$ is always true. Let $n > 1$ and assume that the statement holds for lower powers. Since $n \geq 2$, we have $2n-2 \geq n$, so $(J^{n-1})^2(M) = (J^{2n-2})(M) \subset (J^n)(M) \subset X$. By Theorem 2.4, $J^{n-1}(M) \subset X$ and so $J(M) \subset X$, by the induction hypothesis. This completes the proof. □

3. Semiprime modules and chain conditions

Recall that a submodule U of a right R-module M is called an *M-annihilator* if $X = Ker(I) = \bigcap_{f \in I} Ker(f)$ for some subset I of S. Before introducing a new notion related to M- annihilators, we first prove the following results.

Lemma 3.1. *Let M be a right R-module and U, a submodule of M. If $U = Ker(I)$ for some right ideal I of S, then U is a fully invariant submodule of M.*

Proof. Take any $\varphi \in S$. Let $y \in \varphi(U)$. Then $y = \varphi(x)$, for some $x \in U$. For any $f \in I$, we have $f(y) = f(\varphi(x)) = f\varphi(x) = 0$, since $f\varphi \in I$. This shows that $y \in Ker(I) = U$, proving that U is a fully invariant submodule of M. □

Theorem 3.2. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Suppose that M is a semiprime module and X, a fully invariant submodule of M. Then X has the unique complement $Ker(I_X)$ and hence $X \oplus Ker(I_X) \subsetneq^{*} M$.*

Proof. Since M is a semiprime module, S is a semiprime ring. Put $U = Ker(I_X)$. Then $I_X(U) = 0$. Since M is a self-generator, it follows that $I_X(I_U(M)) = I_X I_U(M) = 0$. It follows that $I_X I_U = 0$. By [10, Proposition 3.13], we can see that $I_X \cap I_U = 0$. Since M is quasi-projective and finitely generated, we have $I_X = Hom(M, X)$ and $I_U = Hom(M, U) = Hom(M, Ker(I_X))$. Thus $0 = I_X \cap I_U = Hom(M, X \cap Ker(I_X))$. Since M is a self-generator, we have $X \cap Ker(I_X) = 0$. Note that I_X is a two-sided ideal of S. Let Y be any complement of X in M. Then $I_X \cap I_Y = I_{X \cap Y} = 0$, and so $I_Y I_X = 0$. Since S is a semiprime ring, we get $I_X I_Y = 0$ by [2, page 12]. This implies that $I_X I_Y(M) = 0$. So $Y = I_Y(M) \subset Ker(I_X)$. Since Y is maximal with respect to the property that $X \cap Y = 0$, we have $Y = Ker(I_X)$. Thus $Ker(I_X)$ is a unique complement of X and therefore, $X \oplus Ker(I_X) \subsetneq^{*} M$. □

Lemma 3.3. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a proper fully invariant submodule of M. If M is a semiprime module, then $Ker(I_X) \neq 0$.*

Proof. From the fact that M is a semiprime module, it implies that S is a semiprime ring. Since X is a fully invariant submodule of M, I_X is a two-sided ideal of S. Put $N = Ker(I_X)$. Then N is the unique complement of X, by Theorem 3.2. If $N = 0$, then X is essential in M. This implies that I_X is an essential ideal of S. Indeed, if J is any ideal of S with $I_X \cap J = 0$, then $0 = I_X \cap J = Hom(M, X) \cap Hom(M, J(M)) = Hom(M, X \cap J(M))$. Since M is a self-generator, $X \cap J(M) = 0$, and we have $J(M) = 0$ because X is essential in M. It follows that $J = 0$, showing that I_X is an essential ideal of S. Since $I_X \cap r(I_X) = 0$ and I_X is an essential ideal of S, we have $r(I_X) = 0$. Thus $I_X = l(r(I_X)) = l(0) = S$. It implies that $X = S(M) = M$, a contradiction. Therefore, $Ker(I_X) \neq 0$. □

In [7], we introduced the notion of minimal prime submodules and we proved that if X is a prime submodule of a module M, then X contains a minimal prime submodule of M. Moreover, we have the following proposition on minimal prime submodules of a semiprime module M. This fact can be considered as a generalization of [5, 11.40].

Proposition 3.4. *Let M be a semiprime right R-module which is a self-generator. Let X be a fully invariant submodule of M and Ω, the set of minimal prime submodules of M which do not contain X. Then $Ker(I_X) = \cap\{P : P \in \Omega\}$.*

Proof. Let $B = \cap\{P : P \in \Omega\}$. Then any element in $X \cap B$ is in the intersection of all minimal prime submodules of M and this intersection is 0 because M is a semiprime module. Thus $X \cap B = 0$. Note that I_X and I_B are two-sided ideals of S. We have $I_X \cap I_B = I_{X\cap B} = 0$. Since S is a semiprime ring, by [10, 3.13], we get $I_X I_B = 0$, and consequently, $I_X I_B(M) = 0$. It follows that $I_X(B) = 0$. So $B \subset Ker(I_X)$. On the other hand, for any $P \in \Omega$, we have $I_X(Ker(I_X)) = 0 \subset P$. Since P is a prime submodule of M and $X = I_X(M) \not\subset P$, we must have $Ker(I_X) \subset P$. This implies that $Ker(I_X) \subset B$. Therefore, $Ker(I_X) = B$, proving our proposition. □

Definition 3.5. A fully invariant submodule X of a right R-module M is called a *full M-annihilator* if $X = Ker(I)$ for some ideal I of S. A full M-annihilator X of a right R-module M is said to be *maximal* if $X \neq M$ and there are no full M-annihilators strictly between X and M.

As an immediate consequence of Proposition 3.3 is the following characterization of full M-annihilators in a semiprime module M.

Lemma 3.6. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and U, a full M-annihilator of M. If M is a semiprime module and $N = Ker(I_U)$, then $U = Ker(I_N)$.*

Proof. Since U is a full M-annihilator of M, we can write $U = Ker(K)$ for some ideal K of S. Then $U = Ker(K) = Ker(I_{K(M)})$ is a unique complement of $X = K(M)$, by Theorem 3.2. Since $N = Ker(I_U)$ is a unique complement of U, we have $X \subset N$. This implies that $I_X \subset I_N$. So $U = Ker(I_X) \supset Ker(I_N)$. Since $Ker(I_N)$ is a unique complement of N and $U \cap N = 0$, we have $U = Ker(I_N)$, proving our lemma. □

Proposition 3.7. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a fully invariant submodule of M. If M is a semiprime module, then X is a full M-annihilator if and only if X is a complement of Y for some fully invariant submodule Y of M.*

Proof. Suppose that X is a full M-annihilator. Then $X = Ker(K)$ for some ideal K of S. Since M is a self-generator, we can write $K = I_{K(M)}$. It follows that $X = Ker(I_{K(M)})$ is a unique complement of $Y = K(M)$, by Theorem 3.2.

Conversely, suppose that X is a complement of Y, where Y is a fully invariant submodule of M. Since Y has a unique complement $Ker(I_Y)$, we have $X = Ker(I_Y)$, proving that X is a full M-annihilator. □

Note that, if we consider the bimodule ${}_SM_R$, then X is a fully invariant submodule of a right R-module M if and only if X is a bi-submodule of ${}_SM_R$. For convenience, we use the terminology "bi-submodule". To avoid some confusions, we will use roughly the terminology bi-essential submodules, bi-uniform submodules, bi-uniform dimensions, and bi-complements in the following definitions.

Definition 3.8. Consider the bimodule ${}_SM_R$ and a bi-submodule X of ${}_SM_R$. We say that X is a *bi-essential submodule of* ${}_SM_R$ (or X *is bi-essential in* ${}_SM_R$) if for any bi-submodule U of ${}_SM_R$, $X \cap U = 0$ implies $U = 0$. If X is bi-essential in ${}_SM_R$, we denote $X \subsetneq^{*} {}_SM_R$. A bi-submodule X of ${}_SM_R$ is called a *bi-uniform submodule* if every nonzero bi-submodule of X is bi-essential in X. The bimodule ${}_SM_R$ is called a *uniform bimodule* if every nonzero bi-submodule of ${}_SM_R$ is bi-essential in ${}_SM_R$.

Note that, by Theorem 3.2, for a semiprime right R-module M, and for any bi-submodule X of ${}_SM_R$, $Ker(I_X)$ is the unique complement of X and it is fully invariant too. Therefore, in the context of semiprime modules, bi-essential is essenatial and bi-complements are complements as the usual definitions. From the above definitions, we can prove the following results.

Proposition 3.9. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a fully invariant submodule of M. Suppose that M is a semiprime module. Then we have the following:*

(1) X is a bi-essential submodule of ${}_SM_R$ if and only if $Ker(I_X) = 0$;

(2) If X is not contained in any minimal prime submodule of M, then X is

a bi-essential submodule of ${}_SM_R$.

Proof. (1) It is clear because $Ker(I_X)$ is the unique complement of X, by Theorem 3.2.

(2) Since X is not contained in any minimal prime submodule of M, $Ker(I_X)$ is the intersection of all minimal prime submodules of M, by Proposition 3.4. Therefore, $Ker(I_X) = 0$ because M is semiprime and the result follows from (1). □

Lemma 3.10.

(1) Let A, B, X, N *be bi-submodules of* ${}_SM_R$ *with* $A \subset B \subset X \subset N$. *If* A *is bi-essential in* ${}_SN_R$, *then* B *is bi-essential in* ${}_SX_R$*;*

(2) If $A_1, \ldots, A_n$ *are bi-essential submodules of* ${}_SM_R$, *then* $\bigcap\limits_{i=1}^{n} A_i$ *is bi-essential in* ${}_SM_R$*;*

(3) If A, B *are bi-submodules of* ${}_SM_R$ *such that* A *is bi-essential in* ${}_SB_R$ *and* B *is bi-essential in* ${}_SM_R$, *then* A *is bi-essential in* ${}_SM_R$.

Proof. The proof of this lemma is a routine and we present here for the sake of completeness.

(1) Let U be a bi-submodule of ${}_SX_R$ and $B \cap U = 0$. Then U is a bi-submodule of ${}_SN_R$ and $A \cap U = 0$. Since A is bi-essential in ${}_SN_R$, we have $U = 0$, and consequently, B is bi-essential in ${}_SX_R$.

(2) We prove the statement by induction on n. The case $n = 1$ is trivial by assumption. Suppose that $A = \bigcap\limits_{i=1}^{n-1} A_i$ is bi-essential in ${}_SM_R$. Let U be a bi-submodule of ${}_SM_R$ and $A \cap A_n \cap U = 0$. Then $A_n \cap U = 0$ because A is bi-essential in ${}_SM_R$. Since A_n is bi-essential in ${}_SM_R$, we have $U = 0$. Thus the result follows.

(3) Let X be a bi-submodule of ${}_SM_R$ and $X \cap A = 0$. Then $(X \cap B) \cap A = 0$ and since A is bi-essential in ${}_SB_R$, we have $X \cap B = 0$. Thus $X = 0$ because B is bi-essential in ${}_SM_R$. □

Lemma 3.11. *Let* A *be a bi-submodule of* ${}_SM_R$. *Then* A *is bi-essential in* ${}_SM_R$ *if and only if for any* $m \in M$ *with* $m \neq 0$, *there exist* $f_1, \ldots, f_n \in S$ *and* $r_1, \ldots, r_n \in R$ *such that* $\sum\limits_{i=1}^{n} f_i(mr_i) \neq 0$ *and* $\sum\limits_{i=1}^{n} f_i(mr_i) \in A$.

Proof. Suppose that A is bi-essential in ${}_SM_R$. Since $m \neq 0$, we have $SmR \neq 0$ and so $A \cap SmR \neq 0$. We can find $f_1, \ldots, f_n \in S$ and $r_1, \ldots, r_n \in R$ such that $\sum\limits_{i=1}^{n} f_i(mr_i) \neq 0$ and $\sum\limits_{i=1}^{n} f_i(mr_i) \in A$.

Conversely, suppose that for any $m \in M$ with $m \neq 0$, there exist $f_1, \ldots, f_n \in S$ and $r_1, \ldots, r_n \in R$ such that $\sum_{i=1}^{n} f_i(mr_i) \neq 0$ and $\sum_{i=1}^{n} f_i(mr_i) \in A$. Let B be a nonzero bi-submodule of ${}_SM_R$. Then there exists $0 \neq m \in B$ such that $\sum_{i=1}^{n} f_i(mr_i) \in B$. Hence $A \cap B \neq 0$. This completes the proof. $\square$

Lemma 3.12. *Let A, B, C, U be bi-submodules of ${}_SM_R$ with $A \subsetneq^*_S B_R$ and $C \subsetneq^*_S U_R$. If the sums $A + C$ and $B + U$ are direct, then $A \oplus C$ is bi-essential in $B \oplus U$ as a bi-submodule.*

Proof. Let $m = b + u \in B \oplus U$ with $b \in B, u \in U$ and $m \neq 0$. If $b = 0$, then $u \neq 0$. So $SuR \neq 0$ and $SuR \cap C \neq 0$ because $C \subsetneq^*_S U_R$. This implies that $SmR \cap (A + C) \neq 0$, so we are done. Let therefore $b \neq 0$. Since $A \subsetneq^*_S B_R$, there exist $f_1, \ldots, f_n \in S$ and $r_1, \ldots, r_n \in R$ such that $0 \neq \sum_{i=1}^{n} f_i(br_i) \in A$. If $\sum_{i=1}^{n} f_i(ur_i) = 0$, then $\sum_{i=1}^{n} f_i(mr_i) = \sum_{i=1}^{n} f_i((b+u)r_i) = \sum_{i=1}^{n} f_i(br_i) \neq 0$ and $\sum_{i=1}^{n} f_i(mr_i) \in A + C$. Then we are done. If $\sum_{i=1}^{n} f_i(ur_i) \neq 0$, then there exist $g_1, \ldots, g_k \in S$ and $t_1, \ldots, t_k \in R$ such that $0 \neq \sum_{j=1}^{k} g_j(\sum_{i=1}^{n} f_i(ur_i))t_j \in C$ since $C \subsetneq^*_S U_R$. Consider the element $\sum_{i=\overline{1,n}; j=\overline{1,k}} g_j f_i(mr_it_j) = \sum_{i=\overline{1,n}; j=\overline{1,k}} g_j f_i(br_it_j) + \sum_{i=\overline{1,n}; j=\overline{1,k}} g_j f_i(ur_it_j) = \sum_{j=1}^{k} g_j(\sum_{i=1}^{n} f_i(br_i))t_j + \sum_{j=1}^{k} g_j(\sum_{i=1}^{n} f_i(ur_i))t_j \in A + C$. If $\sum_{i=\overline{1,n}; j=\overline{1,k}} g_j f_i(mr_it_j) = 0$, then $\sum_{j=1}^{k} g_j(\sum_{i=1}^{n} f_i(ur_i))t_j \in A \cap C = 0$, a contradiction. Thus $\sum_{i=\overline{1,n}; j=\overline{1,k}} g_j f_i(mr_it_j) \neq 0$. It follows that $A \oplus C$ is bi-essential in $B \oplus U$ as a bi-submodule, by Lemma 3.11. $\square$

Since Lemma 3.11 and Lemma 3.12 hold, so by induction, we can conclude that the following also holds.

Proposition 3.13. *Let A_i, B_i for $i = 1, \ldots, n$, be bi-submodules of ${}_SM_R$. If A_i is bi-essential in ${}_S{B_i}_R$, for all $i = 1, \ldots, n$ and the sums $\sum_{i=1}^{n} A_i$, $\sum_{i=1}^{n} B_i$ are direct, then $\bigoplus_{i=1}^{n} A_i$ is bi-essential in $\bigoplus_{i=1}^{n} B_i$ as a bi-submodule.*

The following theorem gives some characterizations of maximal full M-annihilators in a semiprime module M similar to that of maximal annihilators in a semiprime ring R.

Theorem 3.14. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a proper fully invariant submodule of M. If M is a semiprime module, then the following conditions are equivalent:*

(1) X is a maximal full M-annihilator;

(2) X is a minimal prime submodule and a full M-annihilator;

(3) X is a prime submodule and a full M-annihilator;

(4) $X = Ker(I_U)$ for some bi-uniform submodule U of ${}_SM_R$.

Moreover, if M has only finitely many minimal prime submodules, then the above conditions are equivalent to:

(5) X is a minimal prime submodule.

Proof. Since M is a semiprime module, we see that S is a semiprime ring.

(1) $\Rightarrow$ (2): Assume that X is a full M-annihilator. So we can write $X = Ker(K)$ for some nonzero ideal K of S. Let I be an ideal of S and U, a fully invariant submodule of M such that $I(U) \subset X$ and $I(M) \not\subset X$. We must show that $U \subset X$. Since $I(M) \not\subset X$, then $KI(M) \neq 0$ and so $0 \neq KI \subset K$. Therefore, $M \neq Ker(KI) \supset Ker(K) = X$. By the maximality of X, we have $Ker(KI) = Ker(K)$. Now $I(U) \subset X$ implies $K(I(U)) = KI(U) = 0$. It follows that $U \subset Ker(KI) = Ker(K)$. Thus, X is a prime submodule of M.

We now suppose that P is a prime submodule of M and $P \subsetneqq X$. Then $K(X) = 0 \subset P$. Since P is prime and $X \not\subset P$, we must have $K(M) \subset P \subsetneqq X$. So $0 = K(K(M)) = K^2(M)$ implies that $K^2 = 0$, a contradiction to S being a semiprime ring.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (4): Put $U = Ker(I_X)$. Then $X = Ker(I_U)$, by Lemma 3.6. We now show that U is a bi-uniform submodule of ${}_SM_R$. Suppose U is not a bi-uniform submodule of ${}_SM_R$. Then there are nonzero fully invariant submodules X_1, X_2 with $X_1 \oplus X_2 \subset U$. Since all minimal prime submodules intersect at 0, so $X_1 \not\subset P$ for some minimal prime submodule P. Then $I_{X_1}(Ker(I_{X_1})) = 0 \subset P$. Since P is prime and $X_1 = I_{X_1}(M) \not\subset P$, we must have $Ker(I_{X_1}) \subset P$. Since $Ker(I_{X_1})$ is the unique complement of X_1, we have $I_{X_1}(X_2) = 0$ and $I_U(X_2) \neq 0$. This implies that $Ker(I_{X_1}) \supsetneqq Ker(I_U) = X$. So $P \supsetneqq X$, a contradiction to the fact that P is a minimal prime submodule of M.

(4) $\Rightarrow$ (1): Suppose that $X = Ker(I_U)$ for some bi-uniform submodule U of ${}_SM_R$ and that $X \subsetneqq B$, where B is a full M-annihilator. Since $X = Ker(I_U)$ is the unique complement of U, we have $B \cap U \neq 0$. This implies that $B \cap U$ is a nonzero bi-submodule of ${}_SM_R$ and so $B \cap U$ is an essential submodule of U. Therefore, $(B \cap U) \oplus Ker(I_U)$ is essential in $U \oplus Ker(I_U)$ and $U \oplus Ker(I_U)$ is essential in ${}_SM_R$, i.e., $(B \cap U) \oplus Ker(I_U)$ is essential in ${}_SM_R$. Since $(B \cap U) \oplus Ker(I_U)$ is contained in B, we see that so B is essential in M. It follows that I_B is an essential ideal of S. Since $I_B \cap r(I_B) = 0$, we have $r(I_B) = 0$. Thus $I_B = l(r(I_B)) = S$ and so $B = M$. Therefore, $X \subsetneqq B = M$.

Finally, suppose that M has only finitely many minimal prime submodules $P_1, \ldots, P_t$. Using $P_1 \cap \cdots \cap P_t = 0$, we see by Proposition 3.4 that $P_i = Ker(I_{Q_i})$, where $Q_i = \bigcap_{k \neq i} P_k$. In this case, (2) $\Leftrightarrow$ (5). $\square$

Proposition 3.15. *Let $U = U_1 \oplus \cdots \oplus U_m$ and $V = V_1 \oplus \cdots \oplus V_n$ be bi-essential submodules of ${}_SM_R$, where the U_i's and V_j's are bi-uniform submodules. Then $m = n$.*

Proof. We may assume that $n \geq m$. We claim that $\overline{U} := U_2 \oplus \cdots \oplus U_m$ intersects trivially with some V_j. If otherwise, $\overline{U} \cap V_j \neq 0$, for all $j = 1, \ldots, n$; then we would have $\overline{U} \cap V_j \subsetneq^* V_j$, since V_j is bi-uniform, and we get $(\overline{U} \cap V_1) \oplus \cdots \oplus (\overline{U} \cap V_n)$ is bi-essential in $V_1 \oplus \cdots \oplus V_n = V$ and hence also $\overline{U} \cap V \subsetneq^* {}_SV_R$ as a bi-submodule and $V \subsetneq^* {}_SM_R$. This implies that $\overline{U} \cap V \subsetneq^* {}_SM_R$, and consequently, $\overline{U}$ is bi-essential in ${}_SM_R$, a contradiction. Therefore, $\overline{U} \cap V_j = 0$ for some j. After relabelling the V_j's, we may assume that $\overline{U} \cap V_1 = 0$. Let $U' = \overline{U} \oplus V_1$. We must then have $U' \cap U_1 \neq 0$. If otherwise, $U_1 + \cdots + U_m + V_1$ would be a direct sum, a contradiction to the fact that $U \subsetneq^* {}_SM_R$. So $(U' \cap U_1) \oplus U_2 \oplus \cdots \oplus U_m$ is bi-essential in $U_1 \oplus U_2 \oplus \cdots \oplus U_m = U$ as a bi-submodule and $U \subsetneq^* {}_SM_R$. Since the left hand side is contained in U', it follows that $U' \subsetneq^* {}_SM_R$. We have thus replaced the summand U_1 by V_1 when going from U to U'. Repeating the process, we can pass from U' to some bi-essential submodule $U'' = V_1 \oplus V_2 \oplus U_3 \oplus \cdots \oplus U_m$. After m steps, we have a bi-essential submodule $U^{(m)} = V_1 \oplus \cdots \oplus V_m$. But $V = V_1 \oplus \cdots \oplus V_n \subsetneq^* {}_SM_R$, so we must have $m = n$. $\square$

Definition 3.16. We say that the bimodule ${}_SM_R$ has bi-uniform dimension n (denote $dim({}_SM_R) = n$) if there is a bi-essential submodule V of ${}_SM_R$ such that V is a direct sum of n bi-uniform submodules. By Proposition 3.15, $dim({}_SM_R)$ is well-defined. If no such integer exists, we write $dim({}_SM_R) = \infty$. We can check that $dim({}_SM_R) = 0$ if and only if ${}_SM_R = 0$, and $dim({}_SM_R) = 1$ if and only if ${}_SM_R$ is a uniform bimodule.

Theorem 3.17. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Suppose that $dim({}_SM_R) = n < \infty$. Then $dim({}_SS_S) = n$.*

Proof. We first show that if U is a bi-uniform submodule of ${}_SM_R$, then I_U is a uniform ideal of ${}_SS_S$. Let J, K be nonzero ideals of S and $J, K \subset I_U$. Then $0 \neq J(M), K(M) \subset U$. So $J(M) \cap K(M) \neq 0$ since U is a bi-uniform submodule of ${}_SM_R$. But $J \cap K = Hom(M, J(M)) \cap Hom(M, K(M) = Hom(M, J(M) \cap K(M))$. Therefore, $J \cap K \neq 0$. It follows that I_U is a uniform ideal of S.

Since $dim({}_SM_R) = n < \infty$, there exist n bi-uniform submodules $U_1, \ldots, U_n$ such that the sum $U_1 + \cdots + U_n$ is direct and is bi-essential in ${}_SM_R$. Then $I_{U_1}, \ldots, I_{U_n}$ are uniform ideals of S. It is easy to check that $I_{U_i} \cap \sum_{j \neq i} I_{U_j} = 0$, so the sum $I_{U_1} + \cdots + I_{U_n}$ is direct. We will show that $I_{U_1} + \cdots + I_{U_n}$ is essential in S. Let K be an ideal of S with $(I_{U_1} + \cdots + I_{U_n}) \cap K = 0$. Then $K(M)$ is a bi-submodule of ${}_SM_R$ and we can write $0 = (I_{U_1} + \cdots + I_{U_n}) \cap K = Hom(M, (I_{U_1} + \cdots + I_{U_n})(M)) \cap Hom(M, K(M)) = Hom(M, I_{U_1}(M) + \cdots + I_{U_n}(M)) \cap Hom(M, K(M)) = Hom(M, U_1 + \cdots + U_n) \cap Hom(M, K(M)) = Hom(M, (U_1 + \cdots + U_n) \cap K(M))$. Since M is a self-generator, $(U_1 + \cdots + U_n) \cap K(M) = 0$, and because $U_1 + \cdots + U_n$ is bi-essential in ${}_SM_R$, we have $K(M) = 0$. Thus $K = 0$, proving that $I_{U_1} + \cdots + I_{U_n}$ is an essential ideal of S. Thus $dim({}_SS_S) = n$. □

Theorem 3.18. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Suppose that $dim({}_SS_S) = n < \infty$. Then $dim({}_SM_R) = n$.*

Proof. We first show that if K is a uniform ideal of S, then $K(M)$ is a bi-uniform submodule of ${}_SM_R$ and $K = I_{K(M)}$. Suppose that X_1, X_2 are nonzero bi-submodules of ${}_SM_R$ such that $X_1, X_2 \subset K(M)$. Then $0 \neq I_{X_1}, I_{X_2} \subset I_{K(M)} = K$. We can write $I_{X_1 \cap X_2} = I_{X_1} \cap I_{X_2} \neq 0$ since K is a uniform ideal of S. So $X_1 \cap X_2 \neq 0$.

From $dim({}_SS_S) = n < \infty$, there exist n uniform ideals $K_1, \ldots, K_n$ of S such that the sum $K_1 + \cdots + K_n$ is direct and is essential in S. Then $K_1(M)$, $\ldots$, $K_n(M)$ are bi-uniform submodules of ${}_SM_R$ and $K_i = I_{K_i(M)}$. Now, we show that the sum $K_1(M) + \cdots + K_n(M)$ is direct and is bi-essential in ${}_SM_R$. We have

$$\begin{aligned} 0 = K_i \cap \sum_{j\neq i} K_j &= Hom(M, K_i(M)) \cap Hom(M, (\sum_{j\neq i} K_j)(M)) \\ &= Hom(M, K_i(M)) \cap Hom(M, \sum_{j\neq i} K_j(M)) \\ &= Hom(M, K_i(M) \cap \sum_{j\neq i} K_j(M)). \end{aligned}$$

So $K_i(M) \cap \sum_{j\neq i} K_j(M) = 0$ since M is a self-generator. It shows that the sum $K_1(M) + \cdots + K_n(M)$ is direct. Let X be a nonzero bi-submodule of ${}_SM_R$ such that $(K_1(M) + \cdots + K_n(M)) \cap X = 0$. Then we can write $(K_1 + \cdots + K_n) \cap I_X = Hom(M, (K_1 + \cdots + K_n)(M)) \cap Hom(M, I_X(M)) = Hom(M, (K_1(M) + \cdots + K_n(M)) \cap X) = 0$. So $I_X = 0$ since $K_1 + \cdots + K_n$ is essential in S. It follows that $X = 0$, proving that $K_1(M) + \cdots + K_n(M)$ is a bi-essential submodule of ${}_SM_R$. Thus $dim({}_SM_R) = n$. □

Proposition 3.19. *Consider the bimodule ${}_SM_R$. Suppose that $dim({}_SM_R) = n < \infty$ and $V_1, \ldots, V_n$ are bi-uniform submodules of ${}_SM_R$ such that the sum $V = V_1 + \cdots + V_n$ is direct and is bi-essential in ${}_SM_R$. Then any direct sum of nonzero bi-submodules $N = N_1 \oplus \cdots \oplus N_k \subset {}_SM_R$ has $k \leq n$ summands.*

Proof. We prove the statement by induction on n. First consider the case

$n = 1$. Then we see that ${}_SM_R$ is a uniform bimodule. So the direct sum of nonzero bi-submodules of ${}_SM_R$ has only $k = 1$ summand. Suppose that the statement holds for $n-1$ summands. We check for the case $dim({}_SM_R) = n$. Since V is bi-essential in ${}_SM_R$, put $N_i^{'} := N_i \cap V \neq 0$ and $V \supset N_1^{'} \oplus \cdots \oplus N_k^{'}$. Thus we may assume that $M = V$, say $M = V_1 \oplus \cdots \oplus V_n$, where all the V_i's are bi-uniform. Let $\overline{N} = N_2 \oplus \cdots \oplus N_k$. If $\overline{N} \cap V_i \neq 0$ for all $i = 1, \ldots, n$, then $(\overline{N} \cap V_1) \oplus \cdots \oplus (\overline{N} \cap V_n) \subsetneq^* V_1 \oplus \cdots \oplus V_n = M$. This implies that $\overline{N} \subsetneq^* {}_SM_R$, a contradiction. Therefore, $\overline{N} \cap V_i = 0$ for some i. After relabelling the V_i's, we may assume that $\overline{N} \cap V_1 = 0$. Projecting M modulo V_1 onto $V_2 \oplus \cdots \oplus V_n$, we have then an embedding of $\overline{N}$ into $V_2 \oplus \cdots \oplus V_n$. By assumption, we get $k - 1 \leq n - 1$ and so $k \leq n$. □

Lemma 3.20. *Consider the bimodule ${}_SM_R$. If ${}_SM_R$ does not contain a direct sum of an infinite number of nonzero bi-submodules, then any nonzero bi-submodule $N \subset {}_SM_R$ contains a bi-uniform submodule.*

Proof. If ${}_SN_R$ does not contain any bi-uniform submodule, then ${}_SN_R$ itself is not bi-uniform, so ${}_SN_R$ contains some $A_1 \oplus B_1$, where A_1, B_1 are nonzero bi-submodules. Then B_1 is also not bi-uniform, so B_1 contains some $A_2 \oplus B_2$, where A_2, B_2 are nonzero bi-submodules. Continuing the process, we will get an infinite direct sum $A_1 \oplus A_2 \oplus A_3 \oplus \cdots \subset {}_SM_R$, a contradiction. Thus the result follows. □

Proposition 3.21. *$dim({}_SM_R) = \infty$ if and only if ${}_SM_R$ contains an infinite direct sum of nonzero bi-submodules.*

Proof. If ${}_SM_R$ contains an infinite direct sum of nonzero bi-submodules, then $dim({}_SM_R) = \infty$, by Lemma 3.20.

Conversely, suppose that ${}_SM_R$ does not contain an infinite direct sum of nonzero bi-submodules. Pick a bi-uniform submodule $V_1 \subset {}_SM_R$. If V_1 is not bi-essential in ${}_SM_R$, then ${}_SM_R$ contains $V_1 \oplus V_2$, for some nonzero bi-submodule V_2, and we may assume that V_2 is bi-uniform. If $V_1 \oplus V_2$ is not bi-essential in ${}_SM_R$, then $V_1 \oplus V_2 \oplus V_3 \subset {}_SM_R$ where V_3 is a bi-uniform submodule. By our assumption, this process must stop, and we arrive at some bi-essential submodule $V_1 \oplus \cdots \oplus V_n$ where each V_i is bi-uniform. By definition, we have $dim({}_SM_R) = n$. □

We now explore the meaning of bi-uniform dimensions and we need more concepts of bi-complements in the bimodule ${}_SM_R$.

Definition 3.22. Let X be a bi-submodule of the bimodule ${}_SM_R$. We say that X is a *bi-complement* in ${}_SM_R$ (denote $X \subset_c {}_SM_R$) if there exists a bi-submodule $Y \subset {}_SM_R$ such that X is a bi-complement of Y in ${}_SM_R$.

Proposition 3.23. *Consider the bimodule ${}_SM_R$. Suppose that $X \subset_c {}_SM_R$ and*

T is a bi-submodule of ${}_SM_R$ such that $X \cap T = 0$. Then X is a bi-complement of T if and only if $X \oplus T \subsetneq^* {}_SM_R$.

Proof. If X is a bi-complement of T, then $X \oplus T \subsetneq^* {}_SM_R$. Conversely, assume that $X \oplus T \subsetneq^* {}_SM_R$. Since $X \subset_c {}_SM_R$, there is a bi-submodule $U \subset {}_SM_R$ such that X is a bi-complement of U. We show that X is maximal with respect to the property that $X \cap T = 0$. Let D be a bi-submodule of ${}_SM_R$ such that $X \subset D$ and $D \cap T = 0$. We have $(X+T) \cap (D \cap U) = ((X+T) \cap D) \cap U = X \cap U = 0$. Since $X \oplus T \subsetneq^* {}_SM_R$, we have $D \cap U = 0$. This implies that $D = X$. Thus X is a bi-complement of T. $\square$

Corollary 3.24. *Suppose that $X \subset_c {}_SM_R$. Let T be a bi-complement of X in M. Then X is a bi-complement of T.*

Proof. Since T is a bi-complement of X, we have $T \oplus X \subsetneq^* {}_SM_R$. But then by Proposition 3.23, we can conclude that X is a bi-complement of T. $\square$

The next result describes some basic properties of bi-complements in the bimodule ${}_SM_R$.

Proposition 3.25. *Let X, N be bi-submodules of the bimodule ${}_SM_R$ such that $X \subset N \subset {}_SM_R$. Then we have the following:*

(1) If $X \subset_c {}_SM_R$, then $X \subset_c {}_SN_R$.

(2) If $X \subset_c {}_SN_R$ and $N \subset_c {}_SM_R$, then $X \subset_c {}_SM_R$.

Proof. (1) If $X \subset_c {}_SM_R$, then there exists a bi-submodule $Y \subset {}_SM_R$ such that X is a bi-complement of Y in ${}_SM_R$. This implies that X is a bi-complement of $Y \cap N$ in ${}_SN_R$. Thus $X \subset_c {}_SN_R$.

(2) Suppose that X is a bi-complement of U in ${}_SN_R$ and N is a bi-complement of T in ${}_SM_R$. We will show that X is a bi-complement of $U \oplus T$ in ${}_SM_R$. It is clear that $X \cap (U+T) = 0$. Let D be a bi-submodule of ${}_SM_R$ such that $D \supsetneqq X$. We need to show that $D \cap (U+T) \neq 0$. If $D \cap N \supsetneqq X$, then $(D \cap N) \cap U \neq 0$. So $D \cap U \neq 0$, and consequently, $D \cap (U+T) \neq 0$. Then we are done. Now, consider the case $D \cap N = X$. Then there exists $d \in D \setminus N$ such that $(N + SdR) \cap T \neq 0$. Therefore, there exist $n \in N, t \in T \setminus \{0\}, f_1, \ldots, f_k \in S, r_1, \ldots, r_k \in R$ such that

$$n + \sum_{i=1}^{k} f_i(dr_i) = t \tag{1}$$

If $n \in X$, then $n + \sum_{i=1}^{k} f_i(dr_i) \in D$ and so $D \cap T \neq 0$. It follows that $D \cap (U+T) \neq 0$ and we are done. If $n \notin X$, then $(X + SnR) \cap U \neq 0$ and there exist $x \in X, u \in U \setminus \{0\}, g_1, \dots, g_m \in S, t_1, \dots, t_m \in R$ such that

$$x + \sum_{j=1}^{m} g_j(nt_j) = u \tag{2}$$

From (1), we have

$$\sum_{j=1}^{m} g_j(nt_j) + \sum_{j=1}^{m} g_j(\sum_{i=1}^{k} f_i(dr_i))t_j = \sum_{j=1}^{m} g_j(tt_j) \tag{3}$$

Subtracting (3) from (2), we get

$$x - \sum_{j=1}^{m} g_j(\sum_{i=1}^{k} f_i(dr_i))t_j = u - \sum_{j=1}^{m} g_j(tt_j) \in (D \cap (U \oplus T)) \setminus \{0\}.$$

□

In the next few results, we shall explore the relationship between bi-uniform dimensions and bi-complements. The first result about bimodules of finite bi-uniform dimensions is an analogue of Proposition 3.19.

Proposition 3.26. *Suppose that $dim({}_SM_R) = n < \infty$. Then any chain of bi-complements in ${}_SM_R$ has length $\leq n$. More precisely, if $C_0 \subsetneqq C_1 \subsetneqq \cdots \subsetneqq C_k$ where the C_i's are bi-complements in ${}_SM_R$, then $k \leq n$.*

Proof. By Proposition 3.25(1), we have $C_{i-1} \subset_c C_i$, say, C_{i-1} is a bi-complement of U_i in C_i for $1 \leq i \leq k$. Since $C_{i-1} \neq C_i$, then $U_i \neq 0$. Now we have $U_1 \oplus \cdots \oplus U_k \subset {}_SM_R$, so $k \leq n$, by Proposition 3.19. □

Next we present the analogue of Proposition 3.21.

Proposition 3.27. *For the bimodule ${}_SM_R$, the following are equivalent:*

(1) $dim({}_SM_R) = \infty$;

(2) There exists an infinite strictly ascending chain of bi-complements in ${}_SM_R$;

(3) There exists an infinite strictly descending chain of bi-complements in ${}_SM_R$.

Proof. (1) ⇒ (2): By Proposition 3.21, ${}_SM_R$ contains $U_1 \oplus U_2 \oplus \cdots$, where each U_i is a nonzero bi-submodule of ${}_SM_R$. Enlarge U_1 into a bi-complement to $U_2 \oplus U_3 \oplus \cdots$, say C_1. Then we enlarge $C_1 \oplus U_2$ into a bi-complement to $U_3 \oplus U_4 \oplus \cdots$, say C_2. In this way, we get an ascending chain $C_1 \subset C_2 \subset \cdots$, where each C_i is a bi-complement in ${}_SM_R$. Since $C_i \supset U_i$ and $C_{i-1} \cap U_i = 0$, we have $C_{i-1} \neq C_i$ for each i.

$(2) \Rightarrow (3)$: Suppose that we have a strictly ascending chain of bi-complements in ${}_SM_R$, say $C_0 \subsetneqq C_1 \subsetneqq \cdots$. Then C_{i-1} is a bi-complement to some nonzero U_i in C_i. Enlarge $U_1 \oplus U_2 \oplus \cdots$ into a bi-complement to C_0, say Y_1. Working in Y_1, enlarge $U_2 \oplus U_3 \oplus \cdots$ into a bi-complement to U_1 in Y_1, say Y_2. By Proposition 3.25(2), $Y_2 \subset_c Y_1 \subset_c {}_SM_R$ implies that $Y_2 \subset_c {}_SM_R$. We have $Y_1 \neq Y_2$ because $Y_1 \supset U_1$ and $Y_2 \cap U_1 = 0$. Continuing this process, we get a strictly descending chain of bi-complements $Y_1 \supsetneqq Y_2 \supsetneqq \cdots$ in M.

$(3) \Rightarrow (1)$: Follows from Proposition 3.26. □

Negating the three statements in Proposition 3.27, we get the following equivalent result.

Proposition 3.28. *For the bimodule ${}_SM_R$, the following are equivalent:*

(1) $dim({}_SM_R) < \infty$;

(2) The bi-complements in ${}_SM_R$ satisfy the ACC;

(3) The bi-complements in ${}_SM_R$ satisfy the DCC.

Finally, we get the following theorem which offers various criteria for a semiprime module to have only finitely many minimal prime submodules.

Theorem 3.29. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a semiprime module, then the following conditions are equivalent:*

(1) $n := dim({}_SM_R) < \infty$;

(2) The number t of minimal prime submodules of M is finite;

(3) The number m of full M-annihilators of M is finite;

(4) M_R has the ACC on full M-annihilators;

(4′) M_R has the DCC on full M-annihilators;

(5) ${}_SM_R$ has the ACC on bi-complements;

(5′) ${}_SM_R$ has the DCC on bi-complements.

If these conditions hold, then $n = t$ and $m = 2^t$. Finally, $n = t = 1$ if and only if M is a prime module.

Proof. $(1) \Rightarrow (2)$: Let U_i $(1 \leq i \leq n)$ be bi-uniform submodules of the bimodule ${}_SM_R$ such that the direct sum $U_1 \oplus \cdots \oplus U_n$ is a bi-essential submodule of ${}_SM_R$. Put $P_i = Ker(I_{U_i})$. Then P_i is a minimal prime submodule of M, by Theorem 3.14. Let P be a minimal prime submodule of M. Then for each

$i = 1, \ldots, n$, we have $I_{U_i}(P_i) = 0 \subset P$. By the primeness of P, we have either $I_{U_i}(M) \subset P$ or $P_i \subset P$, i.e., either $U_i \subset P$ or $P_i \subset P$. If $U_i \subset P$ for all $i = 1, \ldots, n$, then $U_1 \oplus \cdots \oplus U_n \subset P$. So $Ker(I_{U_1 \oplus \cdots \oplus U_n}) \supset Ker(I_P) \neq 0$. This implies that $Ker(I_{U_1 \oplus \cdots \oplus U_n}) \neq 0$. On the other hand, $Ker(I_{U_1 \oplus \cdots \oplus U_n}) \cap (U_1 \oplus \cdots \oplus U_n) = 0$ and $U_1 \oplus \cdots \oplus U_n$ is a bi-essential submodule of ${}_SM_R$, implying that $Ker(I_{U_1 \oplus \cdots \oplus U_n}) = 0$, a contradiction. Thus, $P_i \subset P$ for some i. By the minimality of P, we have $P_i = P$. This shows (2), and we see that $t = n$.

(2) $\Rightarrow$ (3): From (2), we see that t is finite. By Proposition 3.4, we have $m \leq 2^t < \infty$. By Theorem 3.14, each minimal prime submodule of M is a full M-annihilator. This implies that the finite intersection of full M-annihilators is also a full M-annihilator. So $2^t \leq m$. Thus $m = 2^t$.

(3) $\Rightarrow$ (4): Clear from (3).

(4) $\Leftrightarrow$ (5) and (4$'$) $\Leftrightarrow$ (5$'$) follow from Proposition 3.7 and Proposition 3.28.

(4) $\Rightarrow$ (4$'$): Suppose that we have a descending chain of full M-annihilators, say $U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$. Then $Ker(I_{U_1}) \subset Ker(I_{U_2}) \subset \cdots \subset Ker(I_{U_n} \subset \cdots$ is an ascending chain of full M-annihilators. By (4), there is an integer k such that $Ker(I_{U_k}) = Ker(I_{U_j})$ for all $j > k$. Put $N_i = Ker(I_{U_i})$, so we have $U_k = Ker(I_{N_k}) = Ker(I_{N_j}) = U_j$ for all $j > k$, by Lemma 3.6. Thus M_R has the DCC for full M-annihilators.

(4$'$) $\Rightarrow$ (4): Similar to (4) $\Rightarrow$ (4$'$).

(5) $\Rightarrow$ (1): Follows from Proposition 3.28.

The last statement in this proposition is clear. □

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