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MATRIX MAGIC AND RECURSIVE RIDDLES

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Abstract

The Euclidean algorithm finds the greatest common divisor of a pair of positive integers recursively by forming a new pair consisting of the smaller number and the remainder of the larger divided by the smaller. Formulating this procedure using matrices and using their associativity and determinant properties gives results about continued fractions. This expository paper explores related matrix methods to locate polynomial roots, propagate waves, and design filters, fractals, and wavelets.

1 The Euclidean Algorithm and Continued Fractions

Euclid [12] recorded the ancient recursive algorithm, likely discovered by Eudoxus,

$$a = q_0 b + r_0,\ b = q_1 r_0 + r_1,\ r_0 = q_2 r_1 + r_2, \cdots, r_{n-3} = q_{n-1} r_{n-2} + r_{n-1}, r_{n-2} = q_n r_{n-1}$$

to compute r_{n-1} =gcd(a,b) of positive integers $a > b$. The following matrix formulation in the internet article en.wikipedia.org/wiki/Euclidean_ algorithm

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} q_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} q_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_{n-1} \\ 0 \end{bmatrix}.$$

gives integers $\alpha, \beta, \gamma, \delta$ with $r_{n-2} = \alpha a + \beta b$, $0 = \gamma a + \delta b$, $\alpha\delta = \beta\gamma = (-1)^{n+1}$ since

$$\begin{bmatrix} r_{n-1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & q_n \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & q_{n-1} \end{bmatrix} \cdots \begin{bmatrix} 0 & -1 \\ -1 & q_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & q_1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & q_0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

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Khinchin [16] defines continued fractions recursively

$$[a_0; a_1, a_2, ...] = a_0 + [a_1; a_2, ...]^{-1},\ [a_0; a_1, ..., a_n + x] = [a_0; a_1, ..., a_{n-1} + \frac{1}{a_n + x}]. \tag{1}$$

If a is an integer sequence then there exist integer sequences p, q, r, s such that

$$\frac{p_n + r_n x}{q_n + s_n x} = [a_0; a_1, a_2, ..., a_n + x] \tag{2}$$

for real x. For $x = 0$ and $x \to \infty$ Equations 1 and 2 give $r_n = p_{n-1},\ s_n = q_{n-1}$, and

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{bmatrix} \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix},$$

so the convergents $c_n = p_n/q_n = [a_0; a_1, a_2, ..., a_n]$ to a finite or infinite continued fraction c satisfy $c_n - c_{n-1} = \frac{(-1)^{n+1}}{q_n q_{n-1}}$ and $|c - c_n| \leq \frac{1}{q_n q_{n+1}}$. Also, Equations 1 imply that

$$[a_0; a_1, a_2, ...] = [a_0; a_1, a_2, ..., a_n + x],\ x = [a_{n+1}; a_{n+2}, ...]^{-1} \tag{3}$$

so an eventually periodic sequence a defines a quadratic continued fraction. These standard results, which are easy to derive, are obvious when formulated using matrices.

2 Schür-Cohn Stability Test

$\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, are the natural, integer, rational, real, and complex numbers. $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ is the closed unit disk. $\mathbb{D}^o = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk. $\mathbb{T}_c = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. $\mathcal{H}(\mathbb{D})$ is the space of continuous functions on $\mathbb{D}$ that are holomorphic on $\mathbb{D}^o$. For continuous $f : \mathbb{T}_c \to \mathbb{T}_c$, $W(f)$ denotes its winding number. Clearly $W(f_1 f_2) = W(f_1) + W(f_2)$.

Schür [35] and Cohn [5] developed algorithms to test if a degree N polynomial

$$A_N(z) = 1 + a_1 z + \cdots + a_N z^N$$

has no roots in the closed disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, a condition that ensures the stability of the inverse filter $1/A_N(z)$. We give a matrix formulation of the version of this test described in [30], slightly modified by replacing z^{-1} with z and incorporating complex valued coefficients, and use this formulation to validate the test. Construct the reciprocal polynomial $B_N(z) = z^N \overline{A_N}(z^{-1})$ and construct K_m and degree m polynomials $A_m(z)$ and $B_m(z)$ for $1 \leq m \leq N$ by following recursive algorithm:

For $m = N$ to 2

$$K_m = \text{coefficient of } z^m \text{ in } A_m(z),$$

$$\begin{bmatrix} A_{m-1}(z) \\ B_{m-1}(z) \end{bmatrix} = \left(1 - |K_m|^2\right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & -K_m \\ -\overline{K_m} & 1 \end{bmatrix} \begin{bmatrix} A_m(z) \\ B_m(z) \end{bmatrix}. \tag{4}$$

Therefore 1 is the constant term in each A_m, $A_1(z) = 1 + K_1 z$, $B_1(z) = \overline{K_1} + z$,

$$\begin{bmatrix} A_m(z) \\ B_m(z) \end{bmatrix} = \begin{bmatrix} 1 & K_m \\ \overline{K_m} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} A_{m-1}(z) \\ B_{m-1}(z) \end{bmatrix}, \tag{5}$$

and the rational functions $R_m = \frac{B_m}{A_m}$ satisfy

$$R_m(z) = \frac{\overline{K_m} + zR_{m-1}(z)}{1 + K_m z R_{m-1}(z)}, \quad 2 \leq m \leq N. \tag{6}$$

Lemma 1. *If A_m has no zeros in $\mathbb{D}$ then $R_m \in \mathcal{H}(\mathbb{D})$, $R_m(\mathbb{T}_c) = \mathbb{T}_c$, $W(R_m) = m$, and $R_m(\mathbb{D}) = \mathbb{D}$.*

Proof Since A_m has no zeros in $\mathbb{D}$ and constant term 1 there exist $\lambda_1, ..., \lambda_m \in \mathbb{D}^o$ such that $A_m(z) = \prod_{j=1}^m (1 - \lambda_j\, z)$, so $B_m(z) = \prod_{j=1}^m (z - \overline{\lambda_j})$ and $R_m(z) = \prod_{j=1}^m F_j(z)$, where the Blaschke factor $F_j(z) = \frac{z - \overline{\lambda_j}}{1 - \lambda_j\, z}$. Then $F_j(\mathbb{T}_c) = \mathbb{T}_c$ and $|\lambda_j| < 1$ implies $W(F_j) = 1$ so $R_m(\mathbb{T}_c) = \mathbb{T}_c$ and $W(R_m) = m$. Since $R_m \in \mathcal{H}(\mathbb{D})$ the maximum modulus principle gives $R_m(\mathbb{D}) \subseteq \mathbb{D}$. Since $W(R_m) = m$ a homotopy argument gives $R_m(\mathbb{D}) = \mathbb{D}$. □

Theorem 1. *$A_N(z)$ has no zeros in $\mathbb{D}$ if and only if $|K_m| < 1, \ 1 \leq m \leq N$.*

Proof $A_1(z) = 1 + K_1 z$ so the theorem holds for $N = 1$. We assume that $N \geq 2$ and proceed by induction. Assume that $|K_m| < 1$ for $1 \leq m \leq N$. By induction R_{N-1} has no zeros in $\mathbb{D}$. Then Equation 6 and Lemma 1 give $R_N(\mathbb{T}_c) = \mathbb{T}_c$ and $R_N(\mathbb{D}) = \mathbb{D}$ so A_N has no zeros in $\mathbb{D}$. To prove the converse assume that A_N has no zeros in $\mathbb{D}$. Then since A_N has constant term 1, $|K_N| < 1$. Solving Equation 6 for R_{N-1} gives

$$R_{N-1}(z) = z^{-1} \frac{R_N - \overline{K_N}}{1 - K_N R_N(z)}. \tag{7}$$

Therefore $R_{N-1}(\mathbb{T}_c) = \mathbb{T}_c$ and $R_{N-1}(\mathbb{D}) = \mathbb{D}$ so A_{N-1} has no zeros in $\mathbb{D}$. Then $|K_{N-1}| < 1$. Proceeding recursively gives $|K_j| < 1$ for $1 \leq j \leq N$. □

Since every closed disk in $\mathbb{C}$ is homeomorphic to $\mathbb{D}$ by a linear transformation, and every closed half-plane in $\mathbb{C}$ is homeomorphic to $\mathbb{D} \backslash \{0\}$ by a linear fractional transformation, the Schür-Cohn test can be easily extended to test for roots in these sets.

3 Wave Propagation

We discuss matrix formulations of some one dimensional wave propagation models that arise in classical and quantum mechanics.

3.1 Scattering from Layered Media

$\mathbb{U} = \{x + iy \in \mathbb{C} : y \geq 0\}$ is the closed upper half space. $\mathbb{U}^o = \{x + iy \in \mathbb{C} : y > 0\}$ is the open upper half plane. $\mathcal{H}(\mathbb{U})$ is the space of functions that are continuous on $\mathbb{U}$ and holomorphic on $\mathbb{U}^o$.

We follow the derivation by Robinson and Treitel in [34], based in part on Robinson's PhD Thesis research [33], of the reflection from a layered media with m-interfaces that separate $m+1$ homogeneous regions each having constant acoustic impedance. We extend the derivations by allowing different travel times between consecutive interfaces. Assume that the interfaces form horizontal planes with the m-th interface on the top. Let τ_j, $j = 2, ..., m$ be the round trip time delays between the interface j and interface $j-1$. A downward traveling wave whose amplitude at the m-th interface is $e^{i\omega t}$ gives:

1. An upward traveling wave whose amplitude at the m-th interface equals $R_m(\omega)\, e^{i\omega t}$, where $R_m(\omega)$ is the reflection transfer function at frequency $\omega \in \mathbb{R}$.

2. A downward travelling wave whose amplitude at the 1-st interface equals $T_m(\omega)\, e^{i\omega t}$, where $T_m(\omega)$ is the transmission transfer function at frequency $\omega \in \mathbb{R}$.

If only the m-th interface existed then:

1. A downward traveling wave whose amplitude at the m-th interface equals $e^{i\omega t}$ would result in an upward traveling wave whose amplitude at the m-th interface equals $r_m\, e^{i\omega t}$ and a downward traveling wave whose amplitude at the m-th interface equals $(1 + r_m)\, e^{i\omega t}$ where the m-th reflection coefficient $r_m \in (-1, 1)$.

2. An upward traveling wave whose amplitude at the m-th interface equals $e^{i\omega t}$ would result in a downward traveling wave whose amplitude at the m-th interface equals $-r_m\, e^{i\omega t}$ and an upward traveling wave whose amplitude at the m-th interface equals $(1 - r_m)\, e^{i\omega t}$.

Clearly $R_1(\omega) = r_1$ and $T_1(\omega) = 1 + r_1$. The derivation in ([34], p. 450), which is based on summing geometric series, modified by replacing z by $e^{i\omega\tau_m}$, gives

$$R_m(\omega) = \frac{r_m + R_{m-1}(\omega)\, e^{i\omega\tau_m}}{1 + r_m R_{m-1}(\omega)\, e^{i\omega\tau_m}}, \quad 2 \leq m, \tag{8}$$

and

$$T_m(\omega) = \frac{(1+r_m)T_{m-1}(\omega)}{1 + r_m R_{m-1}(\omega)\, e^{i\omega\tau_m}}, \quad 2 \leq m. \tag{9}$$

Lemma 2. *Let $R_m = \frac{B_m}{A_m}$, where B_m and A_m are (possibly anharmonic) trigonometric polynomials. Then*

$$\left[\begin{array}{c} A_m(\omega) \\ B_m(\omega) \end{array}\right] = \left[\begin{array}{cc} 1 & r_m \\ r_m & 1 \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ 0 & e^{i\omega\tau_m} \end{array}\right] \left[\begin{array}{c} A_{m-1}(\omega) \\ B_{m-1}(\omega) \end{array}\right], \quad 2 \leq m, \tag{10}$$

and

$$T_m(\omega) = \frac{(1+r_1)\cdots(1+r_m)}{A_m(\omega)}. \tag{11}$$

Proof Equation 10 is obtained by substituting $R_m = B_m/A_m$ in Equation 3. Equation 11 holds for $m = 1$ since $T_1(\omega) = 1 + r_1$ and $A_1(\omega) = 1$. It holds for $m > 1$ by substituting $R_m = B_m/A_m$ and Equation 10 in Equation 9 and using the induction hypothesis that $T_{m-1}(\omega) = \frac{(1+r_1)\cdots(1+r_{m-1})}{A_{m-1}(\omega)}$. □

Lemma 3. *If $r_1, ..., r_m \in (-1, 1)$ and $\omega \in \mathbb{U}$ then $R_j(\omega) \in \mathbb{D}^0, , j = 1, ..., m$.*

Proof The assertion holds for $m = 1$ since $R_1(\omega) = r_1$. We proceed by induction and assume that $R_{m-1}(\omega) \in \mathbb{D}^0$. If $\omega \in \mathbb{U}$ then $\lambda = R_{m-1}(\omega)\, e^{i\omega\tau_m} \in \mathbb{D}^0$. Then $|R_m(\omega)|^2 = \frac{r_m^2 + |\lambda|^2 + 2r_m \Re\lambda}{1 + r_m^2|\lambda|^2 + 2r_m\Re\lambda} < 1$ and hence $R_m(\omega) \in \mathbb{D}^o$. □

Theorem 2. *If $R_m = \frac{B_m}{A_m}$, where B_m and A_m are (possibly anharmonic) trigonometric polynomials then*

$$1 - |R_m(\omega)|^2 = \frac{(1-r_1)\cdots(1-r_m)}{(1+r_1)\cdots(1+r_m)}\ |T_m(\omega)|^2, \quad \omega \in \mathbb{R}. \tag{12}$$

Proof Equation 10 gives

$$|A_m(\omega)|^2 - |B_m(\omega)|^2 = (1-r_m^2)(|A_{m-1}(\omega)|^2 - |B_{m-1}(\omega)|^2) = \prod_{j=1}^{m}(1-r_j^2), \quad 2 \leq m,\ \omega \in \mathbb{R},$$

so Equation 12 follows from Equation 11. In practice the layers are chosen so that all τ_m equal a constant $\tau > 0$ and R_m is expressed as a rational function of $z = e^{i\omega\tau_m}$. If a unit spike is transmitted then the autocorrelation sequence $\psi_j, j = 0, ..., m$ of the reflected time series is computed and the normal equations

$$\left[\begin{array}{cccc} 1-\psi_0 & -\psi_1 & \cdots & -\psi_m \\ -\psi_1 & 1-\psi_0 & \cdots & -\psi_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\psi_m & -\psi_{m-1} & \cdots & 1-\psi_0 \end{array}\right] \left[\begin{array}{c} 1 \\ a_1 \\ \vdots \\ a_m \end{array}\right] = \left[\begin{array}{c} \sigma_m^2 \\ 0 \\ \vdots \\ 0 \end{array}\right],$$

to obtain $A_m(z) = 1 + a_1 z + \cdots + a_m z^m$ as described in ([34], p. 457) where the connections with prediction theory is discussed. This connection is related to spectral factorization of the positive trigonometric polynomial $|A_m(e^{i\tau\omega})|^2$ to obtain the polynomial $A_m(z)$ that has no roots in $\mathbb{D}$. Our generalization allows $A_m(\omega)$ and $B_m(\omega)$ to be anharmonic trigonometric polynomials and thus almost periodic functions of the type pioneered by Bohr [8]. The $|A_m(\omega)|^2$ is extends to be an entire function of exponential type and the relevant spectral factorization theorem is that of Ahiezer [2], ([7], Theorem 7.5.1). In [22] we explored some aspects of spectral factorization of trigonometric polynomials of two variables. In [23] we used Bohr almost periodic functions and their extensions discussed by Besicovitch [6] to develop a theory that relates Helson and Lowdenslager's [13] spectral factorization of multidimensional trigonometric polynomials to Ahiezer's spectral factorization.

One consequence of allowing variable τ_m is that we can consider the limit as the distance between the m-th and $(m-1)$-th interface approaches 0. Then

$$\frac{R_m(\omega) - R_{m-1}(\omega)}{\tau_m} \approx \frac{r_m}{\tau_m} + i\omega R_{m-1}(\omega) - \frac{r_m}{\tau_m} R_{m-1}^2(\omega).$$

Replacing m by a time variable τ, setting $F_\omega(\tau) = R_\tau(\omega)$, and assuming that the impedance $Z(\tau)$ is differentiable gives the Riccati [32] differential equation

$$\frac{F_\omega(\tau)}{d\tau} = \frac{d\log Z(\tau)}{d\tau} + i\omega F_\omega(\tau) - \frac{d\log Z(\tau)}{d\tau} F_\omega(\tau)^2. \tag{13}$$

Compare this simple derivation of the Riccati scattering equation with Chen's [4].

3.2 Vibrations of Random Media

As Lin explained in her honors thesis [25], the displacement amplitudes $v_{1,j}, \ldots, v_{n,j}$ for the eigenmodes with eigenvalue λ_j for a circular chain on harmonic oscillators with unit stiffness and masses $m_1, \ldots, m_n$ satisfies the generalized eigenvector equation

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 2 & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{bmatrix} \begin{bmatrix} v_{1,j} \\ v_{2,j} \\ \vdots \\ v_{n-1,j} \\ v_{n,j} \end{bmatrix} = \lambda_j \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & m_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & m_n \end{bmatrix} \begin{bmatrix} v_{1,j} \\ v_{2,j} \\ \vdots \\ v_{n-1,j} \\ v_{n,j} \end{bmatrix},$$

$$j = 1, \ldots, n$$

which can be formulated using matrices. Using a slightly different notation than used by Lin we define a family of transfer matrices

$$T(k, \mu) = \begin{bmatrix} 0 & 1 \\ -1 & 2 - \mu\, m_k \end{bmatrix}, \quad \mu \in \mathbb{R}, \tag{14}$$

and their product

$$P(\mu) = T(n,\mu)T(n-1,\mu)\cdots T(1,\mu), \quad \mu \in \mathbb{R}. \tag{15}$$

Clearly $\det P(\mu) = \prod_{i=1}^{n} \det T(i,\mu) = 1$.

Theorem 3. *For an eigenvalue λ_j of the generalized eigenvector equation, the displacement amplitudes satisfy*

$$\begin{bmatrix} v_{k,j} \\ v_{k+1,j} \end{bmatrix} = T(n,\lambda_j) \begin{bmatrix} v_{k-1,j} \\ v_{k,j} \end{bmatrix}, \tag{16}$$

where we define $v_{n+1,j} = v_{1,j}$, and consequently

$$\begin{bmatrix} v_{n,j} \\ v_{1,j} \end{bmatrix} = P(\lambda_j) \begin{bmatrix} v_{n,j} \\ v_{1,j} \end{bmatrix}, \tag{17}$$

and hence 1 is an eigenvalue of $P(\lambda_j) = 1$. Conversely, if 1 is an eigenvalue of $P(\lambda)$ then $\lambda = \lambda_j$ for some $j = 1, ..., n$.

Proof The first assertion is obvious. If 1 is an eigenvalue of $P(\lambda)$ then there exist $v_n, v_1 \in \mathbb{R}$ such that

$$\begin{bmatrix} v_n \\ v_1 \end{bmatrix} = P(\lambda) \begin{bmatrix} v_n \\ v_1 \end{bmatrix}. \tag{18}$$

Recursively define for $k = 1$ to $k = n-1$

$$\begin{bmatrix} v_k \\ v_{k+1} \end{bmatrix} = T(k,\lambda) \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}. \tag{19}$$

Equations 16 and 19 give

$$\begin{bmatrix} v_n \\ v_1 \end{bmatrix} = P(\lambda) \begin{bmatrix} v_n \\ v_1 \end{bmatrix} = T(n,\lambda) \begin{bmatrix} v_{n-1} \\ v_n \end{bmatrix},$$

and hence $v_1, ..., v_n$ are amplitudes that satisfy the generalized eigenvector equation with eigenvalue λ so $\lambda = \lambda_j$ for some $j = 1, ..., n$. □

A theorem of Furstenberg [11] implies that if μ is fixed, then as $n \to \infty$ if m_j are independent random variables such that the probability measure of the transfer matrices $T(j,\mu)$ are not supported in a compact subgroup of $SL(2,\mathbb{R})$, then trace$(P(\mu)) \to \infty$. Ishii [15] has argued that this result explains the localization of amplitude sequences $v_{1,j}, ..., v_{n,j}$ for large values for $j \geq \sqrt{n}$ for large n and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Lin ([25], §3.3.3) raises the fact that the matrices $P(\lambda_j)$ are not products of independent random variables and thus do not satisfy the hypotheses of Furstenberg's theorem. She performed extensive computational experiments using MATLAB to show how the eigenvalues make the transfer matrices statistically dependent and also to explore the degree of eigenvalue localization. A rigorous formulation of eigenvector localization as j increases is required to devise a proof of localization and such a formulation appears to be lacking.

3.3 The Ten Martini Problem

Avila and Jitomirskaya [1] solved this famous problem by proving that the spectrum of the almost Mathiew operator on $\ell^2(\mathbb{Z})$

$$(H_{\lambda,\alpha,\theta}\, u)_n = u_{n-1} + u_{n+1} + 2\lambda \cos(2\pi(\theta + n\alpha))\, u_n,$$

which models the Quantum Hall Effect, is a Cantor set if α is irrational, $\omega \in \mathbb{R}$, and $\lambda > 0$. $u \in \ell^2(\mathbb{Z}$ satisfies $H_{\lambda,\alpha,\theta}u = Eu$ if and only if

$$\begin{bmatrix} u_k \\ u_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & E - 2\lambda\cos(2\pi(\theta + n\alpha)) \end{bmatrix} \begin{bmatrix} u_{k-1} \\ u_k \end{bmatrix}.$$

Compare the structure of this equation with Equations 14 and 16. Their transfer matrices are deterministic but associated with the minimal dynamical system $x \to x + \alpha$ on the circle group $\mathbb{R}/\mathbb{Z}$. Their proof uses deep properties of the dynamics of the products of transfer matrices and involves extraordinarily detailed computations and hard as well as functional analysis. A more detailed discussion of it is far beyond the scope of this paper.

4 Filters, Fractals and Wavelets

Resnikoff and Wells [31] make extensive use of matrices to discuss the topics in this section. Given a bisequence A_k with values in the algebra of $m \times m$ matrices over a subfield $\mathbb{F} \subseteq \mathbb{C}$, they associate a formal Laurent series ([31], Equation 4.4)

$$A(z) = \sum_{k\in\mathbb{Z}} A_k\, z^k$$

and they define its adjoint

$$\widetilde{A}(z) = \sum_{k\in\mathbb{Z}} A_k^*\, z^{-k}.$$

$A(z)$ is a paraunitary m-channel filter bank if

$$A(z)\, \widetilde{A}(z) = m\, I$$

where I is the $m \times m$ identity matrix, and an m-channel wavelet matrix if in addition

$$A(1)\,[1\,1\,...\,1]^* = [m\,0\,...\,0]^*.$$

They use wavelet matrices to design orthogonal wavelets. This involves factorization wavelet matrices identical to the factorization in Equation 5 in the Schür-Cohn stability test. They also use matrices to design biorthogonal wavelet bases, analyse the differentiability of wavelets, and construct multidimensional nonseparable wavelet bases. These include two dimensional analogs of the one dimensional Haar wavelet bases where the wavelets are constant of self similar regions whose translates tile the plane and whose boundaries have fractal dimension. They use matrices to develop applications of wavelets to image compression, wavelet-Galerkin and wavelet-multigrid solution of elliptic boundary value problems, and modulation and channel coding.

$A(z)$ is a paraunitary m-channel filter bank if and only if $A(z) \in SU(m),\ z \in \mathbb{T}_c$, this means that $A(z)$ belongs to the loop group $C^\infty(\mathbb{T}_c, SU(m))$. Moreover, $A(z)$ is wavelet matrix that yields bases of sufficiently smooth wavelets if and only if the associated loop satisfies certain interpolatory conditions. Since $SU(m)$ is a semisimple Lie group, every loop $A(z)$ in $SU(m)$ can be uniformly approximated by a polynomial loop $A_0(z)$. A short proof of this amazing result is given by Pressley and Segal ([29], Proposition 3.5.3). In [19] we combined this result with methods from algebraic topology to show that the approximation of loops by polynomial loops can preserve the interpolatory conditions, thus providing a method to construct orthonormal wavelet bases of compactly supported wavelets of prescribed regularity by approximating orthonormal wavelet bases of wavelets with noncompact supports. The later include the Lemarie-Battle wavelet bases constructed from multiresolution analyses constructed from splines. They are described by Battle [3] , Meyer [26], and Daubechies [9]. Results using loop groups were obtained earlier (unpublished) by Hennings [14] and sharp approximation results extending classical Jackson-Bernstein theorems were derived later by Oswald and Shingel [27], [28].

5 Open Problems for Future Research

We discussed our current research in the areas of spectral factorization and almost periodic functions, and issues related to Lyapunov localization of vibrations in random media and their possible explanation by random products of matrices.

Noncompactly supported smooth wavelets on stratified nilpotent Lie groups, such as Heisenberg groups, we introduced in 1989 by Lemarié [24]. In 200 we introduced [18] compactly supported refinable distributions and scaling functions on these Lie groups. Little is understood about the properties, such as regularity and vanishing moments, of the compactly supported scaling functions and associated wavelets. Matrix methods will likely be useful to derive these properties.

In [20] and [21] we outlined approaches to solve an extension of the Ten Martini Problem for a class of chaotic quantum systems. The operators for these systems cannot be represented by tridiagonal matrices so the use of 2×2 transfer matrices is precluded. Loop group approximation methods provide approximate representations by banded matrices thus enabling the use of finite dimensional transfer matrices.

Progress in the above areas will likely involve relationships between these area and a deeper understanding of almost periodic functions, quasicrystals, chaos and perhaps even the distribution of roots of the Riemann's zeta function. We refer readers who are intrigued by provocative assertion to articles by Dyson [10] and Laaksonen [17].

References

[1] A. Avila and S. Jitomirskaya, *The ten martini problem*, Annals of Mathematics, (1) 170 (2009), 303–342. http://arxiv.org/abs/math/0503363

[2] N. I. Ahiezer, *On the theory of entire functions of finite degree*, Doklady Akad. Nauk, 63 (1948), 475–478, MR 10,289.

[3] G. Battle, *A block spin construction of ondelettes, Part I: Lemarié functions*, Communications on Mathematical Physics, 110 (1987), 601–615.

[4] Y. Chen, *On the Inverse Scattering Problem for the Helmholtz Equation in One Dimension*, PhD Dissertation, Yale University, 1992, and Research Report, YALEU/DCS/RR-913, June 22, 1992.

[5] A. Cohn, *Über die Anzahl der Wurzeln einer algebraische Gleichung in einem Kreise*, Math. Z., 14 (1922), 110–148.

[6] A.S. Besicovitch, *Almost Periodic Functions*, Cambridge University Press, 1932.

[7] R. P. Boas, *Entire Functions*, Academic Press, New York, 1954. MR 16, 914.

[8] H. Bohr, *Zur Theorie der fastperiodishen Funktionen*. Acta Mathematica, I. 45 (1924), II. 29–127, 46 (1925), III. 101–214, 47 (1926), 237–281.

[9] I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, 1992.

[10] F. Dyson, *Birds and frogs*, Notices of the Amer. Math. Soc., (2) 56 (2009), 212–223.

[11] H. Furstenberg, *Noncommuting random products*, Tran. Amer. Math. Soc. 108 (1963), 377–429.

[12] T. L. Heath, *The Thirteen Books of Euclid's Elements*, Dover, 1956.

[13] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math., 99 (1) (1958), 165–202. II. 106 (3,4) (1961), 175–213.

[14] A. Hennings, *Polynomiale Untergruppen der unitären Scheifengruppe*, unpublished manusript received from Universität Siegen, February 2004.

[15] K. Ishii, *Localization of eigenstates and transport phenomena in the one-dimensional disordered system*, Supplement of the Progress of Theoretical Physics, Number 53, (1973).

[16] A. Ya. Khinchin, *Continued Fractions*, University of Chicago Press, 1964.

[17] N. Laaksonen, *Quantum chaos and the Riemann hypothesis*, submitted for publication, 18 January 2011. http://uclmaths.org/images/a/a8/Laaksonen-qchaos-18Jan.pdf

[18] W. Lawton, *Infinite convolution products and refinable distributions on Lie groups*, Transactions of the American Mathematical Society, (6) 352 (2000), 2913–2936.

[19] W. Lawton, *Hermite interpolation in loop groups and conjugate quadrature filter approximation*, Acta Applicandae Mathematicae, 84 (2004), 315–349.

[20] W. Lawton, A. Mouritzen, J. Wang, J. Gong, *Spectral relationships between kicked Harper and on-resonance double kicked rotor operators*, Journal of Mathematical Physics, (3)50 (2009), Article 032103. http://arxiv.org/find/all/1/all:+AND+Lawton+Wayne/0/1/0/all/0/1

[21] W. Lawton, *Bose and Einstein meet Newton*, Contributions in Mathematics and Applications IV, East-West J. of Mathematics, a special volume, (2012), 1–14. http://arxiv.org/find/all/1/all:+AND+Lawton+Wayne/0/1/0/all/0/1

[22] W. Lawton, *Spectral factorization of trigonometric polynomials and lattice geometry*, Acta Arithmetica (3) 155 (2012), 339–347. http://arxiv.org/find/all/1/all:+AND+Lawton+Wayne/0/1/0/all/0/1

[23] W. Lawton, *Spectral factorization and entire functions*, submitted January 2013. http://arxiv.org/find/all/1/all:+AND+Lawton+Wayne/0/1/0/all/0/1

[24] P. G. Lemarié, *Bases d'ondelettes sur les groups de Lie stratifiés*, Bull. Soc. Math. France, 117 (1989), 211–232.

[25] S. Lin, *Wave Propagation and Lyapunov Localization in Random Media*, Honors Thesis, Department of Mathematics, National University of Singapore, 2007.

[26] Y. Meyer, *Wavelets and Operators*, Cambridge University Press, 1992.

[27] P. Oswald and T. Shingel, *Splitting methods for SU(N) loop approximation*, Journal of Approximation Theory, (1) 161 (2009), 174–186.

[28] P. Oswald and T. Shingel, *Close to optimal bounds for SU(N) loop apprxoximation*, Journal of Approximation Theory, (9) 162 (2010), 1511–1517.

[29] A. Pressley and G. Segal, *Loop Groups*, Oxford University Press, 1986.

[30] J. G. Proakis and D. G. Manolakis, *Digital Signal Processing*, Macmillan, 1992.

[31] H. L. Resnikoff and R. O. Wells, Jr., *Wavelet Analysis*, Springer, 1998.

[32] J. F. Riccati *Animadversiones in aequationes differentiales secundi gradus*, Acta Eruditorum Lipsiae, (1724). Translated and annotated by Ian Bruce, *Observations regarding differential equations of the second order*, (2007). http://www.17centurymaths.com/contents/euler/rictr.pdf

[33] E. A. Robinson, *Predictive decomposition of time series with applications to seismic exploration*, PhD Thesis, Department of Geology and Geophysics, Massachusetts Institute of Technology, Cambridge, Mass., 1954, also Geophysics, 32 (1967), 418–484.

[34] E. A. Robinson and S. Treitel, *Digital Signal Processing in Geophysics*, p. 439–491 in Applications of Digital Signal Processing, ed. A. V. Oppenheim, Prentice-Hall, 1978.

[35] I. Schur, *Über Polynome, die nur in Innern des Einheitkreis verschwinden*, 148 (1918), 122–145.