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EXISTENCE RESULTS FOR NAVIER PROBLEMS WITH DEGENERATED (p, q)-LAPLACIAN AND (p, q)-BIHARMONIC OPERATORS IN WEIGHTED SOBOLEV SPACES

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Abstract

In this article, we prove the existence and uniqueness of solutions for the Navier problem

$$(P)\begin{cases}\Delta\big[\omega(x)\,|\Delta u|^{p-2}\Delta u+\nu(x)\,|\Delta u|^{q-2}\Delta u\big]-\operatorname{div}\big[\omega(x)\,|\nabla u|^{p-2}\nabla u+\nu(x)\,|\nabla u|^{q-2}\nabla u\big]\\ =f(x)-\operatorname{div}(G(x)),\quad\text{in}\;\;\Omega,\\ u(x)=\Delta u=0,\quad\text{in}\;\;\partial\Omega,\end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N\geq 2)$, $\dfrac{f}{\omega}\in L^{p'}(\Omega,\omega)$ and $\dfrac{G}{\nu}\in[L^{q'}(\Omega,\nu)]^N$.

1 Introduction

The main purpose of this paper (see Theorem 3.2) is to establish the existence and uniqueness of solutions for the Navier problem

$$(P)\begin{cases}Lu(x)=f(x)-\operatorname{div}(G(x)),\quad\text{in}\;\;\Omega,\\ u(x)=\Delta u(x)=0,\quad\text{in}\;\;\partial\Omega,\end{cases}$$

where

$Lu(x)=\Delta\big[\omega(x)\,|\Delta u|^{p-2}\Delta u+\nu(x)\,|\Delta u|^{q-2}\Delta u\big]-\operatorname{div}\big[\omega(x)|\nabla u|^{p-2}\nabla u+\nu(x)\,|\nabla u|^{q-2}\nabla u)\big]$,

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$\Omega\subset\mathbb{R}^N$ is a bounded open set, $\dfrac{f}{\omega}\in L^{p'}(\Omega,\omega)$, $\dfrac{G}{\nu}\in[L^{q'}(\Omega,\nu)]^N$, ω and ν are two weight functions (i.e., ω and ν are locally integrable functions on $\mathbb{R}^N$ such that $0<\omega(x)<\infty$ and $0<\nu(x)<\infty$ a.e. $x\in\mathbb{R}^N$), Δ is the Laplacian operator, $1<q<p<\infty$, $1/p+1/p'=1$ and $1/q+1/q'=1$.

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1], [4], [5], [7], [8] and [11]). The type of a weight depends on the equation type.

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B.Muckenhoupt in the early 1970's (see [8]). These classes have found many useful applications in harmonic analysis (see [9] and 10]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^N$ often belong to A_p (see [3] and [11]). There are, in fact, many interesting examples of weights (see [7] for p-admissible weights).

In the non-degenerate case (i.e. with $\omega(x)\equiv 1$), for all $f\in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u=f(x), \text{ in } \Omega\\ u(x)=0, \text{ in } \partial\Omega\end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ (see [6]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u=f(x), \text{ in } \Omega\\ u(x)=0, \text{ in } \partial\Omega\end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [2]), where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the degenerated p-Laplacian has been studied in [3].

The paper is organized as follow. In Section 2 we present the definitions and basic results. In Section 3 we prove our main result about existence and uniqueness of solutions for problem (P).

2 Definitions and basic results

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω positive and finite functions $\omega=\omega(x)$, $x\in\Omega$. Elements of $\mathcal{W}(\Omega)$ will be called weight functions. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will be denoted by μ_ω. Thus, $\mu_\omega(E)=\displaystyle\int_E\omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^N$.

Definition 2.1. Let $1 \le p < \infty$. A weight ω is said to be an A_p-weight, if there is a positive constant C such that, for every ball $B \subset \mathbb{R}^N$

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \le C, \quad \text{if} \quad p > 1,$$
$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{\omega(x)}\right) \le C, \quad \text{if} \quad p = 1,$$

where $|.|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$. The infimum over all such constants C is called the A_p - constant of ω and is dnotaded by $C_{p,\omega}$.

If $1 < q \le p$, then $A_q \subset A_p$ (see [5], [7] or [11] for more information about A_p-weights). As an example of an A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p-1)$ (see [11], Chapter IX, Corollary 4.4). If $\varphi \in BMO(\mathbb{R}^N)$, then $\omega(x) = \mathrm{e}^{\alpha\,\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [9]).

Remark 2.1. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \le C_{p,\omega}\,\frac{\mu_\omega(E)}{\mu_\omega(B)}$$

for all measurable subsets E of B (see *15.5 strong doubling property* in [7]). Therefore, $\mu_\omega(E) = 0$ if and only if $|E| = 0$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ a bounded open set, $\omega \in \mathcal{W}(\Omega)$ and $1 \le p < \infty$. We shall denote by $L^p(\Omega,\omega)$ the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\,dx\right)^{1/p} < \infty.$$

We denote $[L^p(\Omega,\omega)]^N = L^p(\Omega,\omega) \times \ldots \times L^p(\Omega,\omega)$.

Remark 2.2. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ (see [11], Remark 1.2.4). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, k be a non-negative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \le |\alpha| \le k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\,dx + \sum_{1\le|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^p \omega(x)\,dx\right)^{1/p}. \quad (2.1)$$

We also define the space $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1). We have that the spaces $W^{k,p}(\Omega,\omega)$ and $W_0^{k,p}(\Omega,\omega)$ are Banach spaces (see Proposition 2.1.2 in [11]).

The dual space of $W_0^{1,p}(\Omega,\omega)$ is the space $[W_0^{1,p}(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega)$,

$$W^{-1,p'}(\Omega,\omega) = \{T = f - \operatorname{div}(G) : G = (g_1,...,g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega,\omega)\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \leq \omega(x) \leq C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $\mathrm{W}^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{k,p}(\Omega)$). Consequently, we shall be interested in all above such weight functions ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basics results.

Theorem 2.3. *(The weighted Sobolev inequality) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p-weight, $1 < p < \infty$. Then there exists positive constants C_Ω and δ such that for all $u \in W_0^{1,p}(\Omega,\omega)$ and $1 \leq \eta \leq N/(N-1)+\delta$*

$$\|u\|_{L^{\eta p}(\Omega,\omega)} \leq C_\Omega \| |\nabla u| \|_{L^p(\Omega,\omega)}, \tag{2.2}$$

where C_Ω may be taken to depend only on N, the A_p- constant of ω, p and the diameter of Ω.

Proof. Its suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [4]). To extend the estimates (2.2) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^p(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.2). □

Lemma 2.4. *(a) Let $1 < p < \infty$, then exists a constant $C_p > 0$ such that for all $\xi, \eta \in \mathbb{R}^N$,*

$$\big| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \big| \leq C_p\, |\xi - \eta| (\, |\xi| + |\eta|)^{p-2}.$$

(b) Let $1 < p < \infty$. There exist two positive constants α_p and β_p such that for every $\xi, \eta \in \mathbb{R}^N$ $(N \geq 1)$

$$\alpha_p(\, |\xi| + |\eta|)^{p-2} |\xi - \eta|^2 \leq \langle\, |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\rangle \leq \beta_p(\, |\xi| + |\eta|)^{p-2} |\xi - \eta|,$$

where $\langle .,. \rangle$ denotes here the Euclidiean scalar product in $\mathbb{R}^N$.

Proof. See Proposition 17.2 and Proposition 17.3 in [2]. □

3 Weak Solutions

Let $\omega \in A_p$, $1 < p < \infty$. We denote by $X = W^{2,p}(\Omega,\omega) \cap W_0^{1,p}(\Omega,\omega)$ with the norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p \, \omega \, dx + \int_\Omega |\Delta u|^p \, \omega \, dx \right)^{1/p}.$$

In this section we prove the existence and uniqueness of weak solutions $u \in X$ to the Navier problem

$$(P) \begin{cases} Lu(x) = f(x) - \operatorname{div}(G(x)), & \text{in } \Omega, \\ u(x) = \Delta u = 0, & \text{in } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $\dfrac{f}{\omega} \in L^{p'}(\Omega,\omega)$ and $\dfrac{G}{\nu} \in [L^{q'}(\Omega,\nu)]^N$, $G = (g_1, ..., g_N)$.

Definition 3.1. We say that $u \in X$ is a weak solution for problem (P) if

$$\begin{aligned} &\int_\Omega |\Delta u|^{p-2} \Delta u \, \Delta\varphi \, \omega \, dx + \int_\Omega |\Delta u|^{q-2} \Delta u \, \Delta\varphi \, \nu \, dx \\ &+ \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla\varphi \rangle \, \omega \, dx + \int_\Omega |\nabla u|^{q-2} \langle \nabla u, \nabla\varphi \rangle \, \nu \, dx \\ &= \int_\Omega f \, \varphi \, dx + \int_\Omega \langle G, \nabla\varphi \rangle \, dx, \end{aligned} \tag{3.1}$$

for all $\varphi \in X$, with $f/\omega \in L^{p'}(\Omega,\omega)$ and $G/\nu \in [L^{q'}(\Omega,\nu)]^N$, where $\langle .,. \rangle$ denotes here the Euclidean scalar product in $\mathbb{R}^N$.

Remark 3.1. (i) Since $1 < q < p < \infty$ and if $\dfrac{\nu}{\omega} \in L^{p/(p-q)}(\Omega,\omega)$, there exists a constant $C_{p,q} > 0$ such that

$$\|u\|_{L^q(\Omega,\nu)} \leq C_{p,q} \|u\|_{L^p(\Omega,\omega)}, \tag{3.2}$$

where $C_{p,q} = \left[\displaystyle\int_\Omega \left(\frac{\nu}{\omega} \right)^{p/(p-q)} \omega \, dx \right]^{(p-q)/p\,q} = \|\nu/\omega\|_{L^{p/(p-q)}(\Omega,\omega)}^{1/q}$.

In fact, since $1 < q < p < \infty$, we have $r = p/q > 1$ and $r' = p/(p-q)$,

$$\begin{aligned} \|u\|_{L^q(\Omega,\nu)}^q &= \int_\Omega |u|^q \, \nu \, dx = \int_\Omega |u|^q \frac{\nu}{\omega} \, \omega \, dx \\ &\leq \left(\int_\Omega |u|^{q\,r} \, \omega \, dx \right)^{1/r} \left(\int_\Omega \left(\frac{\nu}{\omega} \right)^{r'} \omega \, dx \right)^{1/r'} \\ &= \left(\int_\Omega |u|^p \, \omega \, dx \right)^{q/p} \left(\int_\Omega \left(\frac{\nu}{\omega} \right)^{p/(p-q)} \omega \, dx \right)^{(p-q)/p}. \end{aligned}$$

Hence, $\|u\|_{L^q(\Omega,\nu)} \leq C_{p,q}\,\|u\|_{L^p(\Omega,\omega)}$.
(ii) By (3.2), we have

$$\begin{aligned}
\left|\int_\Omega |\Delta u|^{q-2}\,\Delta u\,\Delta\varphi\,\nu\,dx\right| &\leq \int_\Omega |\Delta u|^{q-1}\,|\Delta\varphi|\,\nu\,dx \\
&\leq \left(\int_\Omega |\Delta u|^{(q-1)q'}\,\nu\,dx\right)^{1/q'}\left(\int_\Omega |\Delta\varphi|^q\,\nu\,dx\right)^{1/q} \\
&= \left(\int_\Omega |\Delta u|^q\,\nu\,dx\right)^{(q-1)/q}\left(\int_\Omega |\Delta\varphi|^q\,\nu\,dx\right)^{1/q} \\
&= \|\Delta u\|^{q-1}_{L^q(\Omega,\nu)}\,\|\Delta\varphi\|_{L^q(\Omega,\nu)} \\
&\leq C^{q-1}_{p,q}\,\|\Delta u\|^{q-1}_{L^p(\Omega,\omega)}\,C_{p,q}\,\|\Delta\varphi\|_{L^p(\Omega,\omega)} \\
&\leq C^q_{p,q}\,\|u\|^{q-1}_X\,\|\varphi\|_X,
\end{aligned}$$

and, analogously, we also have

$$\begin{aligned}
\left|\int_\Omega |\nabla u|^{q-2}\,\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx\right| &\leq \int_\Omega |\nabla u|^{q-1}\,|\nabla\varphi|\,\nu\,dx \\
&\leq C^q_{p,q}\|u\|^{q-1}_X\,\|\varphi\|_X.
\end{aligned}$$

Theorem 3.2. *(a) Let* $\omega\in A_p$, $\nu\in\mathcal{W}(\Omega)$, $1<q<p<\infty$ *and* $\dfrac{\nu}{\omega}\in L^{p/(p-q)}(\Omega,\omega)$*;*
(b) $f/\omega\in L^{p'}(\Omega,\omega)$ *and* $G/\nu\in[L^{q'}(\Omega,\nu)]^N$*.*
Then the problem (P) *has a unique solution* $u\in X$ *and*

$$\|u\|_X \leq \left[C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)}\right]^{1/(p-1)},$$

where C_Ω *is the constant in Theorem 2.3 and* $C_{p,q}$ *is the constant in Remark 3.1 (i).*

Proof. (I) *Existence.* By Theorem 2.3 (with $\eta=1$), we have that

$$\begin{aligned}
\left|\int_\Omega f\,\varphi\,dx\right| &\leq \left(\int_\Omega \left|\frac{f}{\omega}\right|^{p'}\omega\,dx\right)^{1/p'}\left(\int_\Omega |\,\varphi|^p\,\omega\,dx\right)^{1/p} \\
&\leq C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&\leq C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\|\varphi\|_X, \qquad (3.3)
\end{aligned}$$

and by Remark 3.1 (i)

$$
\begin{aligned}
\left|\int_\Omega \langle G, \nabla\varphi\rangle\, dx\right| dx &\leq \int_\Omega |\langle G, \nabla\varphi\rangle|\, dx \\
&\leq \int_\Omega |G||\nabla\varphi|\, dx \\
&= \int_\Omega \frac{|G|}{\nu}|\nabla\varphi|\,\nu\, dx \\
&\leq \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu)} \\
&\leq C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&\leq C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\varphi\|_X. \qquad (3.4)
\end{aligned}
$$

Define the functional $J : X \to \mathbb{R}$ by

$$
\begin{aligned}
J(\varphi) &= \frac{1}{p}\int_\Omega |\Delta\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\Delta\varphi|^q\,\nu\, dx \\
&+ \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\nu\, dx - \int_\Omega f\,\varphi\, dx - \int_\Omega \langle G, \nabla\varphi\rangle\, dx.
\end{aligned}
$$

Using (3.3), (3.4), Remark 3.1(i) and Young's inequality, we have that

$$
\begin{aligned}
J(\varphi) &\geq \frac{1}{p}\int_\Omega |\Delta\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\Delta\varphi|^q\,\nu\, dx \\
&+ \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\nu\, dx \\
&- \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_{L^p(\Omega,\omega)} - \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu)} \\
&\geq \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\nu\, dx \\
&- C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} - \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu)} \\
&\geq \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\, dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\nu\, dx \\
&- \frac{C_\Omega^{p'}}{p'}\left\|\frac{f}{\omega}\right\|^{p'}_{L^{p'}(\Omega,\omega)} - \frac{1}{p}\|\,|\nabla\varphi|\,\|^p_{L^p(\Omega,\omega)} - \frac{1}{q'}\left\|\frac{|G|}{\nu}\right\|^{q'}_{L^{q'}(\Omega,\nu)} - \frac{1}{q}\|\,|\nabla\varphi|\,\|^q_{L^q(\Omega,\nu)} \\
&\geq -\frac{C_\Omega^{p'}}{p'}\left\|\frac{f}{\omega}\right\|^{p'}_{L^{p'}(\Omega,\omega)} - \frac{1}{q'}\left\|\frac{|G|}{\nu}\right\|^{q'}_{L^{q'}(\Omega,\nu)}
\end{aligned}
$$

that is, J is bounded from below. Let $\{u_n\}$ be a minimizing sequence, that is, a sequence such that

$$J(u_n) \;\to\; \inf_{\varphi\,\in\, X} J(\varphi)\,.$$

Then for n large enough, we obtain

$$\begin{aligned}0 \geq J(u_n) \;\; &= \;\; \frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\Delta u_n|^q\,\nu\,dx \\ &+ \;\; \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla u_n|^q\,\nu\,dx \\ &- \;\; \int_\Omega f\,u_n\,dx - \int_\Omega \langle G, \nabla u_n\rangle\,dx,\end{aligned}$$

and we have

$$\begin{aligned}&\frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx \\ &\leq \frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\Delta u_n|^q\,\nu\,dx + \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla u_n|^q\,\nu\,dx \\ &\leq \int_\Omega f\,u_n\,dx + \int_\Omega \langle G, u_n\rangle\,dx. \qquad (3.5)\end{aligned}$$

Hence, by Theorem 2.3 (with $\eta = 1$), (3.5) and Remark 3.1(i), we obtain

$$\begin{aligned}&\|u_n\|_X^p = \int_\Omega |\Delta u_n|^p\,\omega\,dx + \int_\Omega |\nabla u_n|^p\,\omega\,dx \\ &\leq p\bigg(\int_\Omega f\,u_n\,dx + \int_\Omega \langle G, \nabla u_n\rangle\,dx\bigg) \\ &\leq p\left(\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|u_n\|_{L^p(\Omega,\omega)} + \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla u_n|\,\|_{L^q(\Omega,\nu)}\right) \\ &\leq p\left(C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\,|\nabla u_n|\,\|_{L^p(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla u_n|\,\|_{L^p(\Omega,\omega)}\right) \\ &\leq p\left(C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)}\right)\|u_n\|_X.\end{aligned}$$

Hence,

$$\|u_n\|_X \leq \left[p\left(C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)}\right)\right]^{1/(p-1)}.$$

Therefore $\{u_n\}$ is bounded in X. Since X is reflexive, there exists a subsequence, still denoted by $\{u_n\}$, and a function $u \in X$ such that $u_n \rightharpoonup u$ in X. Since,

$$X \ni \varphi \;\mapsto\; \int_\Omega f\,\varphi\,dx + \int_\Omega \langle G, \nabla\varphi\rangle\,dx,$$

and

$$X \ni \varphi \mapsto \|\Delta\varphi\|_{L^p(\Omega,\omega)} + \|\Delta\varphi\|_{L^q(\Omega,\nu)} + \| \, |\nabla\varphi| \, \|_{L^p(\Omega,\omega)} + \| \, |\nabla\varphi| \, \|_{L^q(\Omega,\nu)},$$

are continuous then J is continuous. Moreover since $1 < q < p < \infty$ we have that J is convex and thus lower semi-continuous for the weak convergence. It follows that

$$J(u) \leq \liminf_n J(u_n) \; = \; \inf_{\varphi \in X} J(\varphi),$$

and thus u is a minimizer of J on X (see Theorem 25.C and Corollary 25.15 in [12]). For any $\varphi \in X$ the function

$$\begin{aligned} \lambda \mapsto \quad & \frac{1}{p}\int_\Omega |\Delta(u+\lambda\varphi)|^p \, \omega \, dx + \frac{1}{q}\int_\Omega |\Delta(u+\lambda\varphi)|^q \, \nu \, dx \\ & + \frac{1}{p}\int_\Omega |\nabla(u+\lambda\,\varphi)|^p \, \omega \, dx + \frac{1}{q}\int_\Omega |\nabla(u+\lambda\varphi)|^q \, \nu \, dx \\ & - \int_\Omega (u+\lambda\,\varphi)\, f \, dx - \int_\Omega \langle G, \nabla(u+\lambda\,\varphi)\rangle \, dx \end{aligned}$$

has a minimum at $\lambda = 0$. Hence,

$$\frac{d}{d\lambda}\Big(J(u+\lambda\,\varphi) \Big)\Big|_{\lambda=0} = 0, \;\; \forall \, \varphi \in X.$$

We have

$$\frac{d}{d\,\lambda}\left(|\, \nabla(u+\lambda\,\varphi)|^p \, \omega \right) = p \, \{ |\nabla(u+\lambda\,\varphi)|^{p-2} (\langle \nabla u, \nabla\varphi\rangle + \lambda \, |\nabla\varphi|^2)\} \, \omega,$$

and

$$\frac{d}{d\,\lambda}\left(|\, \Delta(u+\lambda\,\varphi)|^p \, \omega \right) = p \, |\Delta u + \lambda\Delta\varphi|^{p-2} (\, \Delta u + \lambda\Delta\varphi) \, \Delta\varphi \, \omega,$$

and we obtain

$$\begin{aligned}
0 &= \frac{d}{d\lambda}\Big(J(u+\lambda\,\varphi)\Big)\Big|_{\lambda=0} \\
&= \Big[\frac{1}{p}\Big(p\int_\Omega |\nabla(u+\lambda\,\varphi)|^{p-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\,\omega\,dx \\
&+ p\int_\Omega |\Delta u+\lambda\Delta\varphi|^{p-2}(\,\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\omega\,dx\Big) \\
&+ \frac{1}{q}\Big(q\int_\Omega |\nabla(u+\lambda\,\varphi)|^{q-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\,\nu\,dx \\
&+ q\int_\Omega |\Delta u+\lambda\Delta\varphi|^{q-2}(\,\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\nu\,dx\Big) \\
&- \int_\Omega \varphi\,f\,dx-\int_\Omega\langle G,\nabla\varphi\rangle\,dx\Big]\Big|_{\lambda=0} \\
&= \int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx+\int_\Omega|\nabla u|^{p-2}\,\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx \\
&+ \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu\,dx+\int_\Omega|\nabla u|^{q-2}\,\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx \\
&- \int_\Omega f\,\varphi\,dx-\int_\Omega\langle G,\nabla\varphi\rangle\,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx+\int_\Omega|\nabla u|^{p-2}\langle\nabla u\,\nabla\varphi\rangle\,\omega\,dx \\
&+\int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu\,dx+\int_\Omega|\nabla u|^{q-2}\,\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx \\
&=\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,
\end{aligned}$$

for all $\varphi\in X$, that is, $u\in X$ is a solution of problem (P).

(II) *Uniqueness.* If $u_1,u_2\in X$ are two weak solutions of problem (P), we have

$$\begin{aligned}
&\int_\Omega |\Delta u_1|^{p-2}\Delta u_1\,\Delta\varphi\,\omega\,dx+\int_\Omega|\Delta u_1|^{q-2}\Delta u_1\,\Delta\varphi\,\nu\,dx \\
&+\int_\Omega |\nabla u_1|^{p-2}\,\langle\nabla u_1,\nabla\varphi\rangle\,\omega\,dx+\int_\Omega|\nabla u_1|^{q-2}\,\langle\nabla u_1,\nabla\varphi\rangle\,\nu\,dx \\
&=\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,
\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega |\Delta u_2|^{p-2}\Delta u_2\,\Delta\varphi\,\omega\,dx+\int_\Omega |\Delta u_2|^{q-2}\Delta u_2\,\Delta\varphi\,\nu\,dx\\&+\int_\Omega |\nabla u_2|^{p-2}\,\langle\nabla u_2,\nabla\varphi\rangle\,\omega\,dx+\int_\Omega |\nabla u_2|^{q-2}\,\langle\nabla u_2,\nabla\varphi\rangle\,\nu\,dx\\&=\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,\end{aligned}$$

for all $\varphi\,X$. Hence

$$\begin{aligned}&\int_\Omega\Big(|\Delta u_1|^{p-2}\Delta u_1-|\Delta u_2|^{p-2}\Delta u_2\Big)\Delta\varphi\,\omega\,dx\\&+\int_\Omega\Big(|\Delta u_1|^{q-2}\Delta u_1-|\Delta u_2|^{q-2}\Delta u_2\Big)\Delta\varphi\,\nu\,dx\\&+\int_\Omega\Big(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla\varphi\rangle-|\nabla u_2|^{p-2}\langle\nabla u_2,\nabla\varphi\rangle\Big)\,\omega\,dx\\&+\int_\Omega\Big(|\nabla u_1|^{q-2}\langle\nabla u_1,\nabla\varphi\rangle-|\nabla u_2|^{q-2}\langle\nabla u_2,\nabla\varphi\rangle\Big)\,\nu\,dx=0.\end{aligned}$$

Taking $\varphi=u_1-u_2$, and using Lemma 2.4(b) there exist positive constants $\alpha_p,\tilde{\alpha}_p,\alpha_q,\tilde{\alpha}_q$ such that

$$\begin{aligned}0\ &=\ \int_\Omega\Big(|\Delta u_1|^{p-2}\Delta u_1-|\Delta u_2|^{p-2}\Delta u_2\Big)(\,\Delta u_1-\Delta u_2)\,\omega\,dx\\&+\ \int_\Omega\Big(|\Delta u_1|^{q-2}\Delta u_1-|\Delta u_2|^{q-2}\Delta u_2\Big)(\,\Delta u_1-\Delta u_2)\,\nu\,dx\\&+\ \int_\Omega\Big(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla u_1-\nabla u_2\rangle-|\nabla u_2|^{p-2}\langle\nabla u_2,\nabla u_1-\nabla u_2\rangle\Big)\,\omega\,dx\\&+\ \int_\Omega\Big(|\nabla u_1|^{q-2}\langle\nabla u_1,\nabla u_1-\nabla u_2\rangle-|\nabla u_2|^{q-2}\langle\nabla u_2,\nabla u_1-\nabla u_2\rangle\Big)\,\nu\,dx\\&=\ \int_\Omega\Big(|\Delta u_1|^{p-2}\Delta u_1-|\Delta u_2|^{p-2}\Delta u_2\Big)(\,\Delta u_1-\Delta u_2)\,\omega\,dx\end{aligned}$$

$$
\begin{aligned}
&+ \int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\Big)(\,\Delta u_1 - \Delta u_2)\,\nu\,dx \\
&+ \int_\Omega \langle\, |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle\,\omega\,dx \\
&+ \int_\Omega \langle\, |\nabla u_1|^{q-2}\nabla u_1 - |\nabla u_2|^{q-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle\,\nu\,dx \\
&\geq \alpha_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\,\omega\,dx \\
&+ \tilde{\alpha}_p \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{p-2} |\nabla u_1 - \nabla u_2|^2\,\omega\,dx \\
&+ \alpha_q \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{q-2} |\Delta u_1 - \Delta u_2|^2\,\nu\,dx \\
&+ \tilde{\alpha}_q \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{q-2} |\nabla u_1 - \nabla u_2|^2\,\nu\,dx \\
&\geq \alpha_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\,\omega\,dx \\
&+ \tilde{\alpha}_p \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{p-2} |\nabla u_1 - \nabla u_2|^2\,\omega\,dx
\end{aligned}
$$

Therefore $\Delta u_1 = \Delta u_2$ and $\nabla u_1 = \nabla u_2$ a.e. and since $u_1, u_2 \in X$, then $u_1 = u_2$ a.e. (by Remark 2.1).

(III) *Estimate for* $\|u\|_X$.

In particular, for $\varphi = u \in X$ in Definition 3.1 we have

$$
\begin{aligned}
&\int_\Omega |\Delta u|^p\,\omega\,dx + \int_\Omega |\Delta u|^q\,\nu\,dx + \int_\Omega |\nabla u|^p\,\omega\,dx + \int_\Omega |\nabla u|^q\,\nu\,dx \\
&= \int_\Omega f\,u\,dx + \int_\Omega \langle G, \nabla u\rangle\,dx.
\end{aligned}
$$

Then, by Theorem 2.3 and Remark 3.1 (i), we obtain

$$
\begin{aligned}
\|u\|_X^p &= \int_\Omega |\Delta u|^p\,\omega\,dx + \int_\Omega |\nabla u|^p\,\omega\,dx \\
&\leq \int_\Omega |\Delta u|^p\,\omega\,dx + \int_\Omega |\Delta u|^q\,\nu\,dx + \int_\Omega |\nabla u|^p\,\omega\,dx + \int_\Omega |\nabla u|^q\,\nu\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega f\, u\, dx + \int_\Omega \langle G, \nabla u\rangle\, dx \\
&\le \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|u\|_{L^p(\Omega,\omega)} + \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla u|\,\|_{L^q(\Omega,\nu)} \\
&\le C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\,|\nabla u|\,\|_{L^p(\Omega,\omega)} + C_{p,q} \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla u|\,\|_{L^p(\Omega,\omega)} \\
&\le \left(C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \right) \|u\|_X.
\end{aligned}
$$

Therefore,

$$
\|u\|_X \le \left(C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \right)^{1/(p-1)}.
$$

□

Corollary 3.3. *Under the assumptions of Theorem 3.2 with $2 \le q < p < \infty$. If $u_1, u_2 \in X$ are solutions of*

$$
(P_1)\begin{cases} Lu_1(x) = f(x) - \operatorname{div}(G(x)), & \text{in } \Omega, \\ u_1(x) = \Delta u_1(x) = 0, & \text{in } \partial\Omega, \end{cases}
$$

and

$$
(P_2)\begin{cases} Lu_2(x) = \tilde f(x) - \operatorname{div}(\tilde G(x)), & \text{in } \Omega, \\ u_2(x) = \Delta u_2(x) = 0, & \text{in } \partial\Omega, \end{cases}
$$

then

$$
\|u_1 - u_2\|_X \le \frac{1}{\gamma^{1/(p-1)}} \left(C_\Omega \left\|\frac{f - \tilde f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\|\frac{|G - \tilde G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \right)^{1/(p-1)},
$$

where γ is a positive constant, C_Ω and $C_{p,q}$ are the same constants of Theorem 3.2.

Proof. If u_1 and u_2 are solutions of $(P1)$ and $(P2)$ then for all $\varphi \in X$ we have

$$
\begin{aligned}
& \int_\Omega |\Delta u_1|^{p-2} \Delta u_1\, \Delta\varphi\, \omega\, dx + \int_\Omega |\Delta u_1|^{q-2} \Delta u_1\, \Delta\varphi\, \nu\, dx \\
& + \int_\Omega |\nabla u_1|^{p-2} \langle \nabla u_1, \nabla\varphi\rangle\, \omega\, dx + \int_\Omega |\nabla u_1|^{q-2} \langle \nabla u_1, \nabla\varphi\rangle\, \nu\, dx \\
& - \Big(\int_\Omega |\Delta u_2|^{p-2} \Delta u_2\, \Delta\varphi\, \omega\, dx + \int_\Omega |\Delta u_2|^{q-2} \Delta u_2\, \Delta\varphi\, \nu\, dx \\
& + \int_\Omega |\nabla u_1|^{p-2} \langle \nabla u_2, \nabla\varphi\rangle\, \omega\, dx + \int_\Omega |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla\varphi\rangle\, \nu\, dx \Big) \\
= {} & \int_\Omega (f - \tilde f)\, \varphi\, dx + \int_\Omega \langle G - \tilde G, \nabla\varphi\rangle\, dx. \qquad (3.6)
\end{aligned}
$$

In particular, for $\varphi = u_1 - u_2$, we obtain
(i) Since $2 \le q < p < \infty$ and by Lemma 2.4 (b), there exist two positive constants α_p and α_q such that

$$\begin{aligned}
&\int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta_2|^{p-2}\Delta u_2 \Big)\, \Delta(u_1 - u_2)\, \omega\, dx \\
&\ge \alpha_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2| \Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\, \omega\, dx \\
&\ge \alpha_p \int_\Omega |\Delta u_1 - \Delta u_2|^{p-2} |\Delta u_1 - \Delta u_2|^2\, \omega\, dx = \alpha_p \int_\Omega |\Delta(u_1 - u_2)|^p\, \omega\, dx,
\end{aligned}$$

and analogously

$$\int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2 \Big)\, \Delta(u_1 - u_2)\, \nu\, dx \ge \alpha_q \int_\Omega |\Delta(u_1 - u_2)|^q\, \nu\, dx \ge 0.$$

(ii) Since $2 \le q < p < \infty$ and by Lemma 2.4 (b), there exit two positive constants $\tilde{\alpha}_p$ and $\tilde{\alpha}_q$ such that

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u_1|^{p-2} \langle \nabla u_1, \nabla(u_1 - u_2)\rangle - |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla(u_1 - u_2)\rangle \Big)\, \omega\, dx \\
&= \int_\Omega \langle |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla(u_1 - u_2)\rangle\, \omega\, dx \\
&\ge \tilde{\alpha}_p \int_\Omega (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2\, \omega\, dx \\
&\ge \tilde{\alpha}_p \int_\Omega |\nabla u_1 - \nabla u_2|^{p-2}\, |\nabla u_1 - \nabla u_2|^2\, \omega\, dx = \tilde{\alpha}_p \int_\Omega |\nabla(u_1 - u_2)|^p\, \omega\, dx,
\end{aligned}$$

and analogously,

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u_1|^{q-2} \langle \nabla u_1, \nabla(u_1 - u_2)\rangle - |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla(u_1 - u_2)\rangle \Big)\, \nu\, dx \\
&\ge \tilde{\alpha}_q \int_\Omega |\nabla(u_1 - u_2)|^q\, \nu\, dx \ge 0.
\end{aligned}$$

(iii) By Remark 3.1 (i) we have

$$\begin{aligned}
&\left| \int_\Omega (f - \tilde{f})\, (u_1 - u_2)\, dx + \int_\Omega \langle G - \tilde{G}, \nabla(u_1 - u_2)\rangle\, dx \right| \\
&\le \left(C_\Omega \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right) \|u_1 - u_2\|_X.
\end{aligned}$$

Hence, with $\gamma = \min\{\alpha_p, \tilde{\alpha}_p\}$, we obtain in (3.6)

$$\begin{aligned}\gamma \|u_1 - u_2\|_X^p &\leq \alpha_p \int_\Omega |\Delta(u_1 - u_2)|^p \,\omega\, dx + \tilde{\alpha}_p \int_\Omega |\nabla(u_1 - u_2)|^p \,\omega\, dx \\ &\leq \left(C_\Omega \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right) \|u_1 - u_2\|_X.\end{aligned}$$

Therefore,

$$\|u_1 - u_2\|_X \leq \frac{1}{\gamma^{1/(p-1)}} \left(C_\Omega \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right)^{1/(p-1)}.$$

□

Corollary 3.4. *Assume $2 \leq q < p < \infty$. Let the assumptions of Theorem 3.2 be fulfilled, and let $\{f_m\}$ and $\{G_m\}$ be sequences of functions satisfying $\dfrac{f_m}{\omega} \to \dfrac{f}{\omega}$ in $L^{p'}(\Omega,\omega)$ and $\left\| \dfrac{|G_m - G|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \to 0$ as $m \to \infty$. If $u_m \in X$ is a solution of the problem*

$$(P_m) \begin{cases} Lu_m(x) = f_m(x) - \operatorname{div}(G_m(x)), & \text{in } \Omega, \\ u_m(x) = \Delta u_m(x) = 0, & \text{in } \partial\Omega, \end{cases}$$

then $u_m \to u$ in X and u is a solution of problem (P).

Proof. By Corollary 3.3 we have

$$\|u_m - u_r\|_X \leq \frac{1}{\gamma^{1/(p-1)}} \left(C_\Omega \left\| \frac{f_m - f_r}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G_m - G_r|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right)^{1/(p-1)}.$$

Therefore $\{u_m\}$ is a Cauchy sequence in X. Hence, there is $u \in X$ such that $u_m \to u$ in X. We have that u is a solution of problem (P). In fact, since u_m

is a solution of (P_m), for all $\varphi \in X$ we have

$$\begin{aligned}
&\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu\,dx \\
&+ \int_\Omega |\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx + \int_\Omega |\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx \\
&= \int_\Omega \Big(|\Delta u|^{p-2}\Delta u - |\Delta_m|^{p-2}\Delta u_m\Big)\,\Delta\varphi\,\omega\,dx \\
&+ \int_\Omega \Big(|\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,\nu\,dx \\
&+ \int_\Omega \Big(|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\omega\,dx \\
&+ \int_\Omega \Big(|\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{q-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\nu\,dx \\
&+ \int_\Omega |\Delta u_m|^{p-2}\Delta u_m\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u_m|^{q-2}\Delta u_m\,\Delta\varphi\,\nu\,dx \\
&+ \int_\Omega |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\,\omega\,dx + \int_\Omega |\nabla u_m|^{q-2}\langle\nabla u_m,\nabla\varphi\rangle\,\nu\,dx \\
&= I_1 + I_2 + I_3 + I_4 + \int_\Omega f_m\varphi\,dx + \int_\Omega \langle G_m,\nabla\varphi\rangle\,dx \\
&= I_1 + I_2 + I_3 + I_4 + \int_\Omega f\,\varphi\,dx + \int_\Omega \langle G,\nabla\varphi\rangle\,dx \\
&+ \int_\Omega (f_m - f)\,\varphi\,dx + \int_\Omega \langle G_m - G,\nabla\varphi\rangle\,dx, \qquad (3.7)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_\Omega \Big(|\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\Big)\,\Delta\varphi\,\omega\,dx, \\
I_2 &= \int_\Omega \Big(|\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,\nu\,dx, \\
I_3 &= \int_\Omega \Big(|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\omega\,dx, \\
I_4 &= \int_\Omega \Big(|\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{q-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\nu\,dx.
\end{aligned}$$

We have that:

(1) By Lemma 2.4 (a) there exists $C_p > 0$ such that

$$\begin{aligned} |I_1| &\leq \int_\Omega \big||\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\big| \, |\Delta\varphi| \, \omega \, dx \\ &\leq C_p \int_\Omega |\Delta u - \Delta u_m| \, (|\Delta u| + |\Delta u_m|)^{p-2} \, |\Delta\varphi| \, \omega \, dx. \end{aligned}$$

Let $r = p/(p-2)$. Since $\dfrac{1}{p} + \dfrac{1}{p} + \dfrac{1}{r} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned} &|I_1| \\ &\leq C_p \left(\int_\Omega |\Delta u - \Delta u_m|^p \, \omega \, dx \right)^{1/p} \left(\int_\Omega |\Delta\varphi|^p \, \omega \, dx \right)^{1/p} \left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(p-2)r} \, \omega \, dx \right)^{1/r} \\ &\leq C_p \|u - u_m\|_X \, \|\varphi\|_X \||\Delta u| + |\Delta u_m|\|_{L^p(\Omega,\omega)}^{(p-2)}. \end{aligned}$$

Now, since $u_m \to u$ in X, then exists a constant $M > 0$ such that $\|u_m\|_X \leq M$. Hence,

$$\||\Delta u| + |\Delta u_m|\|_{L^p(\Omega,\omega)} \leq \|u\|_X + \|u_m\|_X \leq 2M. \qquad (3.8)$$

Therefore,

$$\begin{aligned} |I_1| &\leq C_p \, (2M)^{p-2} \, \|u - u_m\|_X \, \|\varphi\|_X \\ &= C_1 \, \|u - u_m\|_X \, \|\varphi\|_X. \end{aligned}$$

Analogously, there exists a constant C_3 such that

$$|I_3| \leq C_3 \|u - u_m\|_X \, \|\varphi\|_X.$$

(2) By Lemma 2.4 (a) there exists a positive constant C_q such that

$$\begin{aligned} |I_2| &\leq \int_\Omega \big||\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\big| \, |\Delta\varphi| \, \nu \, dx \\ &\leq C_q \int_\Omega |\Delta u - \Delta u_m| \, (|\Delta u| + |\Delta u_m|)^{q-2} \, |\Delta\varphi| \, \nu \, dx. \end{aligned}$$

Let $s = q/(q-2)$ (if $2 < q < p < \infty$). Since $\dfrac{1}{q} + \dfrac{1}{q} + \dfrac{1}{s} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned} &|I_2| \\ &\leq C_q \left(\int_\Omega |\Delta u - \Delta u_m|^q \, \nu \, dx \right)^{1/q} \left(\int_\Omega |\Delta\varphi|^q \, \nu \, dx \right)^{1/q} \left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(q-2)s} \, \nu \, dx \right)^{1/s} \\ &= C_q \, \|\Delta u - \Delta u_m\|_{L^q(\Omega,\nu)} \, \|\Delta\varphi\|_{L^q(\Omega,\nu)} \||\Delta u| + |\Delta u_m|\|_{L^q(\Omega,\nu)}^{q-2}. \end{aligned}$$

Now, by Remark 3.1 (i) and (3.8) we have

$$\begin{aligned}|I_2| &\le C_q\ C_{p,q}\|\Delta u-\Delta u_m\|_{L^p(\Omega,\omega)}\,C_{p,q}\|\Delta\varphi\|_{L^p(\Omega,\omega)}C_{p,q}^{q-2}\||\Delta u|+|\Delta u_m|\|_{L^p(\Omega,\omega)}^{q-2}\\ &\le C_q\ C_{p,q}^q\|u-u_m\|_X\|\varphi\|_X\ (2M)^{q-2}\\ &= C_2\,\|u-u_m\|_X\ \|\varphi\|_X.\end{aligned}$$

Analogously, there exists a positive constant C_4 such that

$$|I_4|\le C_4\,\|u-u_m\|_X\ \|\varphi\|_X.$$

In case $q=2$, we have $|I_2|,|I_4|\le C_{p,2}^2\,\|u-u_m\|_X\,\|\varphi\|_X$.
Therefore, we have $I_1,I_2,I_3,I_4\to 0$ when $m\to\infty$.

(3) We also have

$$\begin{aligned}&\left|\int_\Omega (f_m-f)\,\varphi\,dx+\int_\Omega\langle G_m-G,\nabla\varphi\rangle\,dx\right|\\ &\left(C_\Omega\left\|\frac{f_m-f}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+C_{p,q}\left\|\frac{|G_m-G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)}\right)\|\varphi\|_X\\ &\to 0,\end{aligned}$$

when $m\to\infty$.

Therefore, in (3.7), we obtain when $m\to\infty$ that

$$\begin{aligned}&\int_\Omega|\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx+\int_\Omega|\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu\,dx\\ &+\int_\Omega|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx+\int_\Omega|\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx\\ &=\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,\end{aligned}$$

i.e., u is a solution of problem (P). □

Example Let $\Omega=\{(x,y)\in\mathbb{R}^2\,:\,x^2+y^2<1\}$, $w(x,y)=(x^2+y^2)^{-1/2}$ ($\omega\in A_4$, $p=4$ and $q=3$), $\nu(x,y)=(x^2+y^2)^{-1/3}$, $f(x,y)=\dfrac{\cos(xy)}{(x^2+y^2)^{1/6}}$ and
$G(x,y)=\left(\dfrac{\sin(x+y)}{(x^2+y^2)^{1/6}},\dfrac{\sin(xy)}{(x^2+y^2)^{1/6}}\right)$. By Theorem 3.2 , the problem

$$\begin{cases}\Delta\Big[(x^2+y^2)^{-1/2}\,|\Delta u|^2\Delta u+(x^2+y^2)^{-1/3}|\Delta u|\Delta u\Big]\\ -\operatorname{div}\Big[(x^2+y^2)^{-1/2}|\nabla u|^2\nabla u+(x^2+y^2)^{-1/3}|\nabla u|\nabla u\Big]\\ =f(x)-\operatorname{div}(G(x)),\quad\text{in}\ \ \Omega\\ u(x)=\Delta u=0,\quad\text{in}\ \ \partial\Omega\end{cases}$$

has a unique solution $u\in W^{2,4}(\Omega,\omega)\cap W_0^{1,4}(\Omega,\omega)$.

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