East-West J. of Mathematics: Vol. 15, No 1 (2013) pp. 70-84

ON RIGID MODULES

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Abstract

A module is reduced (in the sense of Lee and Zhou) if and only if it is rigid and semicommutative. We record several conditions under which rigidity implies the reduced module property.

1 Introduction

Reduced rings, i.e., rings without nonzero nilpotent elements, have been studied by algebraists for over forty years (see [14]). The study of reduced modules was initiated by Lee and Zhou in [11] and was continued in [4], [5], [12] and [2]. In this paper we study rigid modules, a class of modules related to reduced modules.

All our rings are associative with identity, subrings and ring homomorphisms are unitary and - unless otherwise mentioned - modules are unitary left modules. Domains need not be commutative. R denotes a ring and M denotes an R-module. Module homomorphisms are written on the side opposite that of scalars. We may not mention which letters denote elements of rings and which of modules over them, when this is clear from the context. All our left-sided concepts and results have right-sided counterparts. For unexplained concepts and results we refer to [3] and [15].

Keywords: Reduced rings, reduced modules, rigid modules, semicommutative modules.
2000 AMS Classification: 16S36, 16U99.

A left R-module is reduced (respectively, rigid) if given $a \in R$ and $m \in M$ the condition $a^2m = 0$ implies $aRm = 0$ (respectively, $am = 0$). Reduced modules are certainly rigid, but the converse is not true as shown by Examples 2.20 and 2.21. Semiprime rigid modules are reduced (Theorem 2.43). All rigid modules are reduced over von Neumann regular rings (Proposition 2.37), and over semi-primary rings (Corollary 2.32).

The following terminology/known results will be used, often without explicit mention:

1. A ring is *left duo* (respectively, *left quo*) if every left ideal (respectively, maximal left ideal) is two-sided. Left duo rings are *semicommutative*, i.e.,whenever $ab = 0$ we have $acb = 0$ for each element c of the ring. The ring of 2×2 upper triangular matrices over a field is a well-known example of a left and right quo ring which is not left or right duo.

2. A ring R is *symmetric* if it satisfies the equivalent conditions (a)'for elements $a, b, c \in R$ given $abc = 0$ we have $bac = 0$ ', and (b)'given $abc = 0$ we have $acb = 0$'.

3. Reduced rings are symmetric; symmetric rings are *reversible*, i.e., for elements a, b of R whenever $ab = 0$ we have $ba = 0$; reversible rings are semicommutative, and semicommutative rings are *abelian*, namely, satisfy the 'idempotents are central' condition.

Throughout $Nil(R)$ denotes the set of nilpotent elements of R, $I(R)$ the set of its idempotents and $ann(M)$ the annihilator of the left R-module M.

2 Rigid Modules

We begin by recalling the definition of a reduced module [11].

Definition 2.1. A left R-module M is *reduced* if it satisfies the following equivalent conditions.

(1) If for $a \in R$ and $m \in M$, we have $a^2m = 0$, then $aRm = 0$.

(2) Whenever $am = 0$, then $aM \cap Rm = 0$.

Semicommutative modules were defined and studied in [7].

Definition 2.2. A left R-module M is *semicommutative* if the condition $am = 0$ implies $aRm = 0$.

Definition 2.3. A left R-module M is *rigid* if the condition $a^2m = 0$ (for elements $a \in R$ and $m \in M$) implies $am = 0$.

Remark 2.4. If the left R-module R is rigid, then it follows - by choosing $m = 1$ in Definition 2.3 - that the ring R is reduced. The converse being trivial, the term 'rigid ring' is clearly redundant.

Remark 2.5. A module M has any of the above three properties if and only if every cyclic submodule of M has the same property. This obvious fact will be used without mention.

We note the following characterization of rigid modules.

Proposition 2.6. *A left R-module M is rigid if and only if for each integer $k \geq 2$ the condition $a^k m = 0$ implies $am = 0$.*

Proof Assuming that $a^k m = 0$ holds we choose an integer n for which $2^n \geq k$. Clearly $a^{2^n} m = 0$ yields, on applying the rigidity hypothesis repeatedly, $am = 0$. □

For later use we record the following remark.

Remark 2.7. Let M be a rigid R-module and let $a \in Nil(R)$ so that for some integer $k \geq 1$ we have $a^k = 0$. By Proposition 2.6, we have $a \in ann(M)$, yielding $Nil(R) \subset ann(M)$.

We have the following characterization of reduced modules, supplementing that given in Definition 2.1.

Proposition 2.8. *The following conditions are equivalent for a left R-module M.*

(1) M is reduced.
(2) If $aRam = 0$ holds, we have $aRm = 0$.
(3) M is rigid and semicommutative.
(4) If for some integer $k \geq 1$ we have $a(Ra)^k m = 0$, then $aRm = 0$.
(5) If for some integer $k \geq 2$ we have $a^k m = 0$, then $aRm = 0$.

Proof (1) $\Longrightarrow$ (2). The condition $aRam = 0$ yields $a^2 m = 0$, which implies, since M is reduced, $aRm = 0$.

(2) $\Longrightarrow$ (3). Since $am = 0$ yields $aRam = 0$, it follows that $aRm = 0$, and so M is semicommutative. Further $a^2 m = 0$ yields $aRam = 0$, by the semicommutativity of M, which yields $aRm = 0$.

Since the implication (3) $\Longrightarrow$ (1) is clear, and the equivalences (1) $\Longleftrightarrow$ (5) and (2) $\Longleftrightarrow$ (4) can be proved by the method of Proposition 2.6 the proof is complete. □

We shall use Propositions 2.9, 2.10 and 2.12 and also their analogues in the reduced and semicommutative cases (partially noted in [7] and [12]) without explicit mention.

Proposition 2.9. *Let $\theta : R \longrightarrow A$ be a ring homomorphism. Let M be a left A-module; then M is an R-module via $r.m = \theta(r)m$.*

(1) If ${}_A M$ is rigid, then so is ${}_R M$.

(2) If θ is onto and ${}_RM$ is rigid, then so is ${}_AM$.

(3) If θ is onto and ${}_RA$ is a rigid module, then A is a reduced ring and ${}_RA$ is a reduced module.

Proof (1). Let $a \in R$, $m \in M$ be such that $a^2m = 0$. Then $\theta(a)^2m = 0$. Since ${}_AM$ is rigid, we have $\theta(a)m = 0$ which implies $am = 0$. The proofs of (2) and (3) are similar. □

Proposition 2.10. *1. Let M be a left A-module, let R be a subring of A, and N an R-submodule of M. If M is rigid over A, then N is rigid over R.*

2. The class of rigid modules (over a given ring R) is closed under direct products, submodules and direct sums.

3. Let $\{R_i\}_{\{i\in I\}}$ be a family of rings and let $\{M_i\}_{\{i\in I\}}$ be a family of left R_i-modules. Write R_0 (respectively, M_0) for the direct product of the rings R_i (respectively, modules M_i). Then (with its natural structure) the R_0-module M_0 is rigid (respectively, reduced) if and only if each R_i-module M_i is rigid (respectively, reduced).

Example 2.11. Let p be a prime integer. The cyclic group of p elements is a rigid $\mathbb{Z}$-module. However, the cyclic group of p^2 elements is not rigid as a $\mathbb{Z}$-module. This shows that the class of rigid modules is not closed under module extensions.

An R-module M is *torsionless* if M is a submodule of a direct product of copies of R, equivalently, if given $m \in M$, $m \neq 0$, there exists $q \in M^* = Hom_R(M, R)$ such that $mq \neq 0$. If M is a faithful R-module, then R is a submodule of a direct product of copies of M. An application of Proposition 2.10(2) and Remark 2.4 yields the following proposition.

Proposition 2.12. *The following conditions are equivalent.*

1. *R is a reduced ring.*
2. *Every torsionless R-module is rigid.*
3. *Every submodule of a free R-module is rigid.*
4. *There exists a faithful, rigid R-module.*

Remark 2.13. For an R-module M, let $\overline{R}$ denote the ring $R/ann(M)$ and let $E(M)$ denote the ring of its R-endomorphisms. Consider the following conditions.

1. The left R-module M is rigid.

2. The left $\overline{R}$-module M is rigid.
3. $\overline{R}$ is a reduced ring.
4. The right $E(M)$-module M is rigid.
5. The ring $E(M)$ is reduced.

Then it can be checked that (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) and (4) $\Longrightarrow$ (5). However, there exist faithful, non-rigid modules over commutative reduced rings and so condition (3) does not imply (1); also, if M is a simple, non-rigid left R-module, then M is rigid as a right module over the division ring $E(M)$, and so (4) does not imply (1).

Certainly, the rigidity of the left R-module M does not, in general, imply that the ring $E(M)$ is reduced. (Consider a vector space of dimension ≥ 2.) However, this does happen in the case of cyclic modules, as noted in Proposition 2.15 below.

Remark 2.14. Let B be a left ideal of a ring R. By the *idealizer* of B in R we mean the subring $L(B) := \{x \in R | Bx \leq B\}$ of R. $L(B)$ contains B as an ideal, and is the largest subring of R to do so. The endomorphism ring of R/B is isomorphic to $L(B)/B$. We use these ideas to prove the following proposition.

Proposition 2.15. *Suppose that the cyclic left R-module M is rigid. Then its endomorphism ring $E(M)$ is reduced.*

Proof Write $M \cong R/B$ where B is a left ideal of R. Consider the ring $L(B)/B$ in the notation of Remark 2.14. Since $L(B)/B$ is an $L(B)$-submodule of R/B, it is rigid as a left $L(B)$-module. By Proposition 2.9(3) $L(B)/B$ is a reduced ring. Hence, as noted in Remark 2.14, the ring $E(M)$ is reduced. □

In the next definition we introduce a class of modules which (over a given ring) contains all rigid modules. Recall that - extending the definition of a symmetric ring - an R-module M is called *symmetric* if given $x, y \in R$ and $m \in M$ the condition $xym = 0$ implies $yxm = 0$.

Definition 2.16. A left R-module M is *semisymmetric* if whenever $xy = 0$ for elements x, y of R, $yx \in ann(M)$.

Remarks 2.17. (a) Symmetric modules are clearly semisymmetric. Next, let M be a rigid R-module. Since, for elements $x, y \in R$, the condition $xy = 0$ implies $yx \in Nil(R)$, the module M is semisymmetric, by Remark 2.7 .

(b) Clearly, the ring R is reversible if and only if R is semisymmetric as a left (or, right) module over itself if and only if all left (or, right) R-modules are semisymmetric if and only if there exists a faithful semisymmetric R-module.

(c) By an argument similar to that given in the proof of Proposition 2.15 we can show that the endomorphism ring of a cyclic, semisymmetric module is reversible.

(d) For later use we record the result that if a left ideal J is semisymmetric (as a left R-module), then for $x, y \in R$ satisfying $xy = 0$ and $u, v \in J$ we have $yxu = 0$ yielding $xuyv = 0$.

(e) We note, in passing, that if a left ideal J is semicommutative (as a left R-module), then for $x, y \in R$ satisfying $xy = 0$, $u \in R$ and $v \in J$ we have $xyv = 0$ and so - using $yv \in J$ - we deduce $xuyv = 0$ - a property stronger than that noted in (d).

Following [1] an R-module M is *abelian* if given $a \in R, e \in I(R)$ and $m \in M$ we have $aem = eam$. (Classically, a module is called abelian if its endomorphism ring is abelian. A vector space is abelian in the sense of the 'classical' definition if and only if its dimension is 0 or 1, while all vector spaces are abelian in our sense.) It was noted in [1] that symmetric modules are abelian. Since reduced modules are symmetric, as noted in Proposition 2.2 of [12], it follows that if M is a reduced module, then M is abelian. In Corollary 2.19 we note that this last implication 'factors through' rigid modules.

Proposition 2.18. *Semisymmetric modules are abelian.*

Proof Let $a \in R, e \in I(R)$ and $m \in M$, a semisymmetric module. Since $e(a - ea) = 0$ we have $(a - ea)em = 0$, yielding $aem = eaem$. Similarly we get $eam = eaem$ yielding $aem = eam$. □

Corollary 2.19. *Rigid modules are abelian.*

Examples 2.20 and 2.21 show that rigid modules need not be reduced.

Example 2.20. Let $T = L < X, Y >$ be the ring of polynomials in non-commuting indeterminates X and Y over a field L and let M be the cyclic left T-module T/TX. We denote the residue class in M of a polynomial $w = w(X, Y) \in T$ by $\overline{w}$ and assume that for some nonzero element $f \in T$ we have $f^2\overline{w} = 0$. Write $f = a+gX+hY, w = b+uX+vY$, and $fw = c+kX+pY$ where $a, b, c \in L$ and $g, h, u, v, k, p \in T$. Then $f^2w = cf + fkX + fpY \in TX$ implies that the constant term of f^2w, namely ac, vanishes. So we have $0 = ac = a^2b$, yielding $c = ab = 0$. Hence $fpY \in TX$ which implies $fp = 0$, yielding (since $f \neq 0$) $p = 0$. It follows that $fw = kX \in TX$, implying $f\overline{w} = 0$. Finally, since $X\overline{1} = 0$, but $XY\overline{1} \neq 0$, the T-module M is not semicommutative, and hence is not reduced.

Next we exhibit a left ideal of a ring, which is rigid but is not reduced as a left module over that ring. We construct it by applying the principle of idealization.

Example 2.21. Let T be a ring, M a left T-module and $E = E(M)$ the ring of T-endomorphisms of M. Recall that M has the canonical structure of a left T-, right E-bimodule. Consider the following ring.

$$U := \left\{ \begin{pmatrix} t & m \\ 0 & \alpha \end{pmatrix} | t \in T, m \in M, \alpha \in E \right\}$$

with usual formal matrix multiplication. Then

$$\tilde{M} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} | m \in M \right\}$$

is an ideal of U. Trivially, $\tilde{M}$ is rigid (respectively, reduced) as a left U-module if and only if M is rigid (respectively, reduced) as a left T-module. Hence choosing T and M as in Example 2.20 we get an example of a rigid left ideal which is not reduced.

Remark 2.22. In view of these examples it is of interest to consider conditions which are sufficient for rigid modules to be reduced, equivalently, as noted in Proposition 2.8, semicommutative. We shall say that a ring R has *property (P)* (on the left) if all rigid (equivalently, by Remark 2.5, cyclic rigid) left R-modules are reduced. Since, as noted in Proposition 2.11 of [7], a ring R is left duo if and only if all left R-modules are semicommutative, left duo rings certainly have property (P). By Example 2.20 domains need not have property (P).

We note a 'lifting condition' for property (P).

Proposition 2.23. *Suppose that the ring R has a nil ideal B such that $A := R/B$ has property (P).Then R has property (P).*

Proof Let M be a rigid R-module. By Remark 2.7 we have $B \subset Nil(R) \subset ann(M)$. Hence M can be regarded as an A-module, and - by change of rings results (Proposition 2.9) - it is a rigid A-module. Since A has property (P), M is a reduced module, over A, and by 'a change of rings', over R.

In Proposition 2.25 we prove that if a direct summand of the left R-module R is rigid, then it is reduced. □

Recall that the set of all idempotents of a ring R is denoted by $I(R)$. An element $e \in I(R)$ is *left semi-central* if $eRe = Re$, equivalently, if $ete = te$ for each $t \in R$.

Remarks 2.24. (a) If the left ideal J of a ring R is rigid as a left R-module, and if $k \in R, j \in J$ satisfy $k^2 = 0$ and $kj = k$, then $k^2 j = 0$ yields, by the rigidity of J, $k = kj = 0$.

(b) If for an element $e \in I(R)$ the R-module Re is rigid, then e is left semi-central. (This can be quickly seen as follows. Let $t \in R$. We have $(te - ete)^2 = 0$

and $(te - ete)e = te - ete$ yielding - by (a) - the condition $te - ete = 0$ and so e is left semi-central.)

Proposition 2.25. *If for some $e \in I(R)$ the left R-module Re is rigid, then it is reduced.*

Proof Let $a \in R$ and $te \in Re$ satisfy $ate = 0$. Let $s \in R$. Since Re is a semisymmetric left ideal, choosing $x = a, y = te, u = se$ and $v = e$ in Remark 2.17(d), we get $asetee = 0$. This implies, using the semi-centrality of e proved in Remark 2.24(b), that $aste = 0$, proving that Re is semicommutative as a left R-module. □

Various conditions related to the (von Neumann) regularity condition in rings have played an important role in the study of a large number of classes of rings. It is well-known that under the assumption of regularity, the conditions 'reduced, semicommutative, abelian, reversible, left or right duo, left or right quo' are equivalent in a ring and each of them is equivalent to the strong regularity condition. (A ring R is *strongly regular* if for each element a of R there exists an element b satisfying $a = ba^2$; it is well-known that this condition is left-right symmetric.) In view of this it is of interest to see what role regularity (and related) conditions in rings and modules play in the context of the rigidity assumption.

An R-module M is *semiprime* [17] if given $m \in M$, $m \neq 0$, there exists $q \in M^* = Hom_R(M, R)$ such that $(mq)m \neq 0$. The ring R is semiprime (i.e., has no nonzero nilpotent ideals) if and only if the module ${}_RR$ is semiprime. A module M is Z-regular ('Zelmanowitz-regular')[16] if given $m \in M$, there exists $q \in M^*$ such that $(mq)m = m$. Semisimple, projective modules are Z-regular, Z-regular modules are semiprime, and semiprime modules are torsionless. (We use semisimple in the sense of Bourbaki [6].)

A module M is *E-regular* ('Elliger-regular' [9]), if every cyclic submodule of M is a direct summand of M. Semisimple modules as well as Z-regular modules are E-regular.

The *Jacobson radical* $Rad(M)$ of a module M is the intersection of all its maximal submodules. A module M is *semiprimitive* if $Rad(M) = 0$.

Let Q be a property of modules. We say a module M is *completely* Q (respectively, *cyclically* Q) if every factor module (respectively, cyclic submodule) of M has property Q. Let M denote the rationals (regarded as an additive $\mathbb{Z}$-module). Since $M^* = 0$, M is not semiprime; since $Rad(M) = M$ it is not semiprimitive. However, it is both cyclically semiprime and cyclically semiprimitive.

If a free left R-module M is rigid, then it certainly is reduced, since a nonzero free module has a copy of R as a direct summand. In what follows we prove analogues of this result for other classes of modules; namely, for semisimple modules, regular modules as defined by Zelmanowitz and Elliger, semiprime modules and some other related classes of modules.

We first consider semisimple modules. We begin with a lemma.

Lemma 2.26. *Suppose that for a left ideal $B \neq R$ of a ring R the left R-module R/B is rigid. If $x \in R$, then we have $B + Bx \neq R$.*

Proof Suppose that for some element $x \in R$ we have $B + Bx = R$, yielding the existence of elements $a, b \in B$ satisfying $1 = a + bx$. Then $b^2x = b - ba \in B$ yields, since R/B is rigid, $bx = 1 - a \in B$ implying $B = R$, a contradiction! □

Remark 2.27. Let R be a ring and let B be an ideal of R. By 'change of rings' results, the R-module R/B is certainly reduced whenever the ring R/B is reduced. This applies, trivially, when B is an intersection of some family of prime ideals of a commutative ring R. However, as the following example shows, even when B is a maximal left ideal of a left and right principal ideal domain R, the simple R-module R/B need not be rigid:

Example 2.28. Let H be the division ring of real (or rational) quaternions. Denote by R the ring $H[X]$, and let (with usual notation) $B = R(X + i)$, a maximal left ideal of R. Since $i(X + i) + k(X + i)j = -2$, a unit in R, we have $B + Bj = R$. Hence, by Lemma 2.26, the R-module R/B is not rigid.

We could not find a reference in the literature for the proof of the sufficiency of condition (2) in the next proposition for a maximal left ideal to be an ideal.

Proposition 2.29. *Let μ be a maximal left ideal of a ring R. Then the following conditions are equivalent:*

(1) The left R-module R/μ is rigid;
(2) Given $x, y \in R$ satisfying $x^2y \in \mu$, we have $xy \in \mu$;
(3) μ is an ideal;
(4) The left R-module R/μ is reduced.

Proof The equivalence of (1) and (2) is obvious. Next assume (1) and let $x \in R$. By Lemma 2.26 we have $\mu + \mu x \neq R$ yielding, since μ is maximal, $\mu x \leq \mu$, proving (3). Next, if (3) holds, the ring R/μ is a division ring and hence is a reduced ring. By 'change of rings' results R/μ must be a reduced R-module. □

Corollary 2.30. *If a simple (semisimple) module is rigid, then it must be reduced.*

Proof A simple, rigid R-module is reduced since it is of the form R/μ for some maximal left ideal μ of R. Every semisimple module is a direct sum of some family of its simple submodules. □

Corollary 2.31. *Semisimple rings have property (P).*

Proof Every module over a semisimple ring is, indeed, semisimple. □ A ring R is *semiprimary* if $Rad(R)$ is nilpotent and the factor ring $R/Rad(R)$ is semisimple (see [3], p.175). Left or right Artinian rings are semiprimary. An application of Proposition 2.23 and Corollary 2.31 yields the following result.

Corollary 2.32. *Semiprimary rings have property (P).*

Remark 2.33. A module-theoretic property which is preserved by direct products and submodules, and which holds for simple modules, also holds for semiprimitive modules, since such modules are submodules of direct products of simple modules.

Proposition 2.34. *The following conditions are equivalent:*

(1) R is left quo.
(2) Every simple left R-module is reduced.
(3) Every simple left R-module is rigid.
(4) Every semiprimitive module is rigid.

Proof (1) $\Longleftrightarrow$ (2) is a part of Proposition 3.6 of [12].
(2) $\Longrightarrow$ (3) is trivial.
(3) $\Longrightarrow$ (1). This follows from Proposition 2.29.
(3) $\Longleftrightarrow$ (4) holds by the preceding remark. □

Remark 2.35. The structure of rigid (equivalently, by Corollary 2.31, reduced) modules over a semisimple ring can now be determined by using the Wedderburn structure theorem, which asserts that a semisimple ring R is a finite direct product of full matrix rings R_i over division rings D_i. When $R_i = M_{n_i}(D_i)$ with $n_i \geq 2$, the zero module is the only reduced R_i-module. On the other hand, when $R_i = D_i$, all left D_i-vector spaces are reduced modules. All reduced R-modules can now be described using Proposition 2.10(3).

In what follows we study the rigidity condition in the context of (von Neumann) regularity and related properties in modules and rings. We begin with an extension of a result from [12].

Proposition 2.36. *The following conditions are equivalent.*

(1) Every left R-module is reduced.
(2) Every left R-module is rigid.
(3) The ring R is strongly regular.

Proof The implication (1)$\Longrightarrow$(2) is trivial and (3)$\Longrightarrow$ (1) is proved in Theorem 2.16 of [12]. (2)$\Longrightarrow$(3). (For the sake of completeness we note the following analogue of an argument used in the proof of Theorem 2.16 of [12].) Let $a \in R$ and let $M = R/Ra^2$. Let $\overline{1}$ denote the residue class of the element 1 in M. Since $a^2\overline{1} = 0$ in M the rigidity assumption yields $a\overline{1} = 0$ in M. This yields $a = ba^2$ for some $b \in R$. □

Next we show that regular rings have property (P); this extends Corollary 2.31.

Proposition 2.37. *Over a regular ring all rigid modules are reduced.*

Proof By Remark 2.5 we may assume M to be cyclic. Write $\overline{R} = R/ann(M)$. Then the $\overline{R}$-module M is rigid and faithful. It follows, by Proposition 2.12, that the ring $\overline{R}$ is (regular and) reduced. Hence, by Proposition 2.36, M is reduced as an $\overline{R}$-module and also as an R-module. □ Z-regular modules have been defined in the remarks after Proposition 2.25.

Proposition 2.38. *If M is a Z-regular and rigid R-module, then M is reduced.*

Proof It is enough to prove Rm is reduced for each $m \in M$. Since M is Z-regular, there exists $q \in M^*$ satisfying $(mq)m = m$. This implies that $e := mq$ is an idempotent in R and the restriction of q to Rm is an R-module isomorphism from Rm to Re. Since M is rigid, Re is also rigid - and therefore, by Proposition 2.25, reduced - as an R-module. It follows that Rm is a reduced module. □

Remark 2.39. A module M is called *anti-regular* if for each nonzero element m of M, there is a nonzero element $q \in M^* = Hom_R(M, R)$ such that $q(mq) = q$. (The canonical right R-module structure on M^* is exploited here.) The motivation for this term comes from the theory of generalized inverses. Anti-regular modules were studied in a series of papers beginning with [8]. Z-regular modules are anti-regular; if M is an E-regular (respectively, an anti-regular) module, then every nonzero submodule of M contains a nonzero direct summand (respectively, a nonzero projective direct summand) of M. We shall call a module M satisfying the condition 'every nonzero submodule of M contains a nonzero direct summand of M' E-*antiregular*. While anti-regular modules can be seen to be semiprime, E-antiregular modules need not even be cyclically semiprime, as shown by the example of (for a prime integer p) the simple $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z}$.

Proposition 2.40. *(a) E-antiregular modules are semiprimitive.*

(b) If an E-antiregular module is rigid then it is reduced.

Proof (a) We record the proof for the sake of completeness. Let M be an E-antiregular module, and let m be a nonzero element of M. By hypothesis, the R-module Rm contains a nonzero direct summand W of M. Since W is a direct summand of Rm as well, it is cyclic and, therefore, contains a maximal submodule V. If W' is a complementary direct summand of W in M, it is easily seen that $V + W'$ is a maximal submodule of M satisfying $m \notin V + W'$ proving the semiprimitivity of M.

(b). Let, if possible, M be an E-antiregular rigid R-module which is not reduced. Then there exist elements $a, t \in R$ and $m \in M$ which satisfy $am = 0$ and $atm \neq 0$. By hypothesis, there is a nonzero direct summand V of M which is contained in $Ratm$. Since V is a direct summand of Rm also, $Rm = V \oplus W$ for some submodule W of Rm. Write $m = v + w$ for elements $v \in V, w \in W$. Then we have $0 = am = av + aw$ yielding $av = 0$. Note that $v = ratm$ for some $r \in R$. Hence $0 = av = aratm = (ra)^2tm$ implying, since M is rigid, $v = ratm = 0$. So $m = w \in W$ implying that $V = 0$, a contradiction. □

Next we consider modules which are completely rigid, i.e., satisfy the condition 'all factor modules are rigid'.

Proposition 2.41. *If M is a cyclically semiprimitive, completely rigid module, then M is reduced.*

Proof It is enough to prove that for each nonzero element m of M, the R-module Rm is reduced. Suppose, if possible, for some elements $a, b, t \in R$ we have $abm = 0$ and $atbm \neq 0$. As Rm is semiprimitive, it has a maximal submodule W satisfying $atbm \notin W$. Now the simple R-module Rm/W is rigid, being a submodule of the rigid module M/W. Hence, by Corollary 2.30, the R-module Rm/W is reduced. However,with $\overline{m}$ denoting the residue class of m in M/W, we have $ab\overline{m} = 0$ and $atb\overline{m} \neq 0$. □

Recall that a left R-module M is a *co-semisimple* module (also called a *V-module*) if every simple R-module is M-injective in the sense of Azumaya.

Corollary 2.42. *A completely rigid, co-semisimple module is completely reduced.*

Proof Let M be a completely rigid, co-semisimple module. By Exercise 23 on p.216 of [3], a module is co-semisimple if and only if it is completely semiprimitive. Hence every factor module of M is a completely rigid, semiprimitive module and is, therefore, reduced, by Proposition 2.41. □

In our final result we add conditions (3) and (4) to those considered in Proposition 3.2 of [12].

Theorem 2.43. *Let M be a cyclically semiprime module. Then the following conditions are equivalent.*

(1)M is reduced.

(2)M is symmetric.

(3)M is rigid.

(4)M is semisymmetric.

(5)M is semicommutative.

Proof The equivalence of (1),(2) and (5) follows from Proposition 3.2 of [12]. Next (1) $\Longrightarrow$ (3) by the definitions and (3) $\Longrightarrow$ (4) has been noted in Remark 2.17(a). Finally, assume that (4) holds, and that for some $a, t \in R$ and $m \in M$ we have $am = 0$ and $atm \neq 0$. As M is cyclically semiprime, for some $q \in (Rm)^*$, we have $[(atm)q]atm \neq 0$. It follows that $(mq)aM \neq 0$. However, $am = 0$ yields $a(mq) = (am)q = 0$ which implies, since M is semisymmetric, $(mq)a \in ann(M)$, a contradiction. Hence (5) holds, and this concludes the proof of the theorem. □

3 Concluding Remarks

Remark 3.1. Some results in §2 follow from other results. For example, Corollary 2.31 is a special case of Proposition 2.37. Also Proposition 2.38 can be deduced from either Proposition 2.40(b) or Theorem 2.43 since Z-regular modules are both E-antiregular and semiprime. However, we have retained the proofs of Corollary 2.31 and Proposition 2.38 given here since these proofs as well as some intermediate results (like Propositions 2.25 and 2.29) seem to be of independent interest.

Remark 3.2. We record a few questions arising out of our study.

1. Consider the following conditions for a left R-module M.

(1) M is rigid.

(2) The left $R[x]$-module $M[x]$ is rigid.

(3) The left $R[[x]]$-module $M[[x]]$ is rigid.

The implications (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are consequences of Proposition 2.10(1). We do not know if (1)$\Longrightarrow$(3) holds; its analogue holds for reduced modules but is false for semicommutative modules.

2. The study of reduced modules in [11] was partly motivated by the relationship of reduced rings with Armendariz rings, a notion introduced in [13]. A ring R is *Armendariz* if given polynomials $f(x) = \Sigma a_i x^i$ and $g(x) = \Sigma b_j x^j$ with coefficients in R, the condition $f(x)g(x) = 0$ implies $a_i b_j = 0$ for every i and j. It was pointed out in 4.7 of [13] that this concept can be extended to modules and to the power series situation.

In [10] Kim, Lee and Lee studied the power series analogue of the 'Armendariz ring' concept; they called such rings *power-serieswise Armendariz rings.* Extending this concept to modules an R-module M is *ps-Armendariz* if whenever $f(x) = \sum a_i x^i \in R[[x]]$, $g(x) = \sum m_j x^j \in M[[x]]$ (the power series module) satisfy $f(x)g(x) = 0$, we have $a_i m_j = 0$, $\forall i$ and $\forall j$.

We have the following implications in the module case: rigid + semicommutative $\Longrightarrow$ ps-Armendariz $\Longrightarrow$ Armendariz + semicommutative. Against this background we can ask whether semicommutativity can be 'dropped' from the first and third conditions noted here, i.e., we can ask:

2.1. Are all rigid modules Armendariz?

The following question also arises in a natural manner.

2.2. Are all rigid, Armendariz modules reduced?

Since rigid modules need not be reduced, at least one of these two questions has a negative answer.

3. A number of questions involving property (P) arise naturally from the above study. We record a few:

3.1. Is the condition 'R has property (P)' left-right symmetric?

3.2. Do left Ore domains (or at least left principal ideal domains)/left quo rings/quasi-simple rings have property (P)?

4. In Propositions 2.40 and 2.41 we have given some evidence to suggest that the following question may have an affirmative answer: Are all semiprimitive, rigid modules reduced?

Acknowledgment.

We thank the referee for helpful suggestions, which resulted in an improvement in presentation.

References

[1] N. Agayev, G. Güngöroğlu, A. Harmanci and S.Halicioğlu, *Abelian modules*, Acta Math. Univ. Comenianae **78(2)**(2009), 235-244.

[2] N.Agayev, S.Halicioğlu and A.Harmanci, *On Rickart modules*, Accepted for publication in Bulletin of the Iranian Mathematical Society.

[3] F.W.Anderson and K.R.Fuller, *Rings and Categories of Modules*, Springer-Verlag, 1974.

[4] M.Baser and N.Agayev, *On reduced and semi-commutative modules*, Turk J.Math. **30**(2006), 285-291.

[5] M.Baser and A.Harmanci, *Reduced and p.q.-Baer modules*, Taiwanese J.Math. **11(1)**(2007), 267-275.

[6] N.Bourbaki, *Modules et anneaux semi-simples*, Hermann et Cie, Paris, 1958.

[7] A.M.Buhphang and M.B.Rege, *Semi-commutative modules and Armendariz modules*, Arab J.Math.Sc. **8**(2002), 53-65.

[8] I.Choudhuri, M.I.Devi and M.B.Rege, *Anti-regularity in modules*, Nat. Acad. Sci. Letters (Allahabad) **13(6)**(1990), 253-255.

[9] S.Elliger, *Interdirekte Summen von Moduln*, J.Algebra **18**(1971), 271-303.

[10] N.K.Kim, K.H.Lee and Y.Lee, *Power series rings satisfying a zero divisor property*, Comm. Algebra **34**(2006),2205-2218; doi:10.1080/00927870600549782.

[11] T.K.Lee, Y.Zhou, *Reduced modules*, in: Rings, modules, algebras and abelian groups, A.Facchini et al (Eds.), Lecture Notes in Pure Appl. Math. **236**, Dekker, New York, 2004, 365-377.

[12] M.B.Rege and A.M.Buhphang, *On reduced modules and rings*, International Electronic Journal of Algebra **3**(2008), 58-74.

[13] M.B.Rege and S.C.Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), 14 - 17.

[14] G.Renault, *Anneaux reduits noncommutatifs*, J.Math. pures et appl. **46**(1967) , 203 - 214.

[15] B.Stenström, *Rings of Quotients: An Introduction to the Methods of Ring Theory*, Springer-Verlag, New York, 1975.

[16] J.Zelmanowitz, *Regular modules*, Trans. Amer. Math. **163**(1972), 341-355.

[17] J.Zelmanowitz, *Semiprime modules with maximum conditions*, J. Algebra **25**(1973), 554-574.