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OPEN CONDITION ON VARIETY OF COMPLEXES

Darmajid and Intan Muchtadi-Alamsyah

Algebra Research Division, Institut Teknologi Bandung
Jalan Ganesha no. 10 Bandung, Indonesia
Email: darmajid@students.itb.ac.id, ntan@math.itb.ac.id

Abstract

Let Λ be an algebra over an algebraically closed field. In this paper we study about the open condition on the collection of all Λ-homomorphisms from Λ-module M to Λ-module N and on the collection of variety of complexes. Moreover, we show that the function of rank and the function of dimension of homology are lower and upper semi continuous, respectively.

1 Introduction

Let $\mathbb{k}$ be an algebraically closed field and Λ be a finite dimensional $\mathbb{k}$-algebra. An affine variety is defined as the set of zeroes of collection of several polynomials over $\mathbb{k}$. It is a subset of the affine space $\mathbb{A}^n_{\mathbb{k}}$ that contain of n-tuples of elements in $\mathbb{k}$. The affine space $\mathbb{A}^n_{\mathbb{k}}$ has Zariski topology with affine variety as closed set [3]. Note that $Hom_\Lambda(M, N)$, the set of all Λ-homomorphisms from Λ-module M to Λ-module N, can be regard as affine spaces by parameterizing its elements into representative matrices for some $\mathbb{k}$-basis. Moreover, $comp^\Lambda_{\mathbf{d}}$, the collection of all chain complexes over fixed Λ-modules, is an affine variety [2]. It is an interesting task to study the open set in $Hom_\Lambda(M, N)$ and in $comp^\Lambda_{\mathbf{d}}$. But it is not easy to find open set of both collections in the topological context. In this paper we give the criteria of open conditions of both collection and study about the property of function of rank and function of dimension of homology in chain complexes of fixed Λ-modules.

Key words: open condition, variety of complexes, rank function, function of dimension of homology.
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2 Preliminaries

In this section we recall some facts on topology and affine varieties. For more details, we refer to [1],[3], and [4].

2.1 Topology

A *topological space* is a set X together with a collection of subsets of X, called *open sets*, such that $\emptyset$ and X are both open and such that arbitrary unions of finite intersections of open sets are open. The collection Ω of open sets of X is called *topology* on X. A subset C of a topological space X is said to be *closed* if the set $X - C$ is open.

Let X be a topological spaces with topology Ω. If Y is a subset of X the collection $\Omega_Y = \{Y \cap U | U \in \Omega\}$ is a topology on Y, called the *subspace topology*. The subset Z of X is called *locally closed* if Z is the intersection of open set and closed set of X.

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *continuous* if for each open set V of Y, the set $f^{-1}(V)$ is open in X. Note that the projections $\pi_1 : X \times Y \to X$, $\pi_1(x, y) = x$ and $\pi_2 : X \times Y \to Y$, $\pi_2(x, y) = y$ are continuous function. A function $f : X \to \mathbb{Z}$ on topological spaces X is said to be *upper* (resp. *lower*) *semicontinuous* if for each $n \in \mathbb{Z}$ the subspace $\{x \in X | f(x) \leq n\}$ is open (resp. closed) in X.

2.2 Affine Varieties

Let $\Bbbk$ be a fixed algebraically closed field. We define *n-dimensional affine space* over $\Bbbk$, denoted by $\mathbb{A}^n_{\Bbbk}$, to be the set of all n-tuples of elements of $\Bbbk$. The set $\mathcal{M}_{s\times t}(\Bbbk)$ of all $s \times t$-matrices over $\Bbbk$ can be regard as affine spaces $\mathbb{A}^{st}_{\Bbbk}$. Let $\Bbbk[X_1, \ldots, X_n]$ be ring of all polynomials over $\Bbbk$ with n variables. A subset V of $\mathbb{A}^n_{\Bbbk}$ is called an *affine variety* if there exists a subset $T \subseteq \Bbbk[X_1, \ldots, X_n]$ such that $V = \{P \in \mathbb{A}^n_{\Bbbk} | f(P) = 0, \forall f \in T\}$. We define the *Zariski topology* on $\mathbb{A}^n_{\Bbbk}$ by taking the open subsets to be the complements of the affine varieties. Zariski topology on $\mathbb{A}^n_{\Bbbk}$ also induces Zariski topology on any subset of $\mathbb{A}^n_{\Bbbk}$.

3 The Open Condition

In this section we give the criteria of open condition in $Hom_\Lambda(M, N)$ and in $comp^\Lambda_{\mathbf{d}}$ and the property of function of rank and function of dimension of homology. Before we study about it, we need some results about collection of matrices.

Lemma 1. *The subset* $\mathcal{C}_{s\times t}^{\leq r}(\Bbbk) = \{M \in \mathcal{M}_{s\times t}(\Bbbk) \,|rank\,(M) \leq r\}$ *of* $\mathcal{M}_{s\times t}(\Bbbk)$ *is closed in* $\mathcal{M}_{s\times t}(\Bbbk)$.

Proof. We will construct a collection of polynomials over $\Bbbk$ with variables $X_{ij}, i \in \{1, \ldots, s\}, j \in \{1, \ldots, t\}$. Clearly, for $r < 0$, the set $\mathcal{C}_{s\times t}^{\leq r}(\Bbbk) = \emptyset$ is closed in $\mathcal{M}_{s\times t}(\Bbbk)$. For $r = 0$, we choose T_0 as collection of all polynomials in $\Bbbk[X_{ij}]$ which have zero constant. Hence, $\mathcal{C}_{s\times t}^{\leq 0}(\Bbbk) \subset \mathbb{A}_{\Bbbk}^{st}$ is the set of simultaneous zeroes of T_0. Consequently, $\mathcal{C}_{s\times t}^{\leq 0}(\Bbbk)$ closed in $\mathcal{M}_{s\times t}(\Bbbk)$. Now assume that $r > 0$. Let X be a $s \times t$-matrix with X_{ij} in the i^{th} row and j^{th} column and $\mathcal{X}$ be a set of all $(r+1) \times (r+1)$-sub matrices of X. We define

$$T_r = \{p(X_{ij}) \in \Bbbk[X_{ij}] \,|p(X_{ij}) = det(Y),\ Y \in \mathcal{X}\}.$$

Let $M \in \mathcal{C}_{s\times t}^{\leq r}(\Bbbk)$. Since $rank\,(M) \leq r$ then all of determinant of $(r+1) \times (r+1)$-submatrices of M are zero. Hence, $\mathcal{C}_{s\times t}^{\leq r}(\Bbbk)$ is the set of simultaneous zeroes of T_r. Consequently, $\mathcal{C}_{s\times t}^{\leq r}(\Bbbk)$ is closed in $\mathcal{M}_{s\times t}(\Bbbk)$ and this completes the proof. □

Corollary 2. *The subset* $\mathcal{C}_{s\times t}^{\geq r}(\Bbbk) = \{M \in \mathcal{M}_{s\times t}(\Bbbk) \,|rank\,(M) \geq r\}$ *of* $\mathcal{M}_{s\times t}(\Bbbk)$ *is open in* $\mathcal{M}_{s\times t}(\Bbbk)$.

Proof. For $r \leq 0$, we have $\mathcal{C}_{s\times t}^{\geq r}(\Bbbk) = \mathcal{M}_{s\times t}(\Bbbk)$ which is open in $\mathcal{M}_{s\times t}(\Bbbk)$. Now assume that $r > 0$. Note that $\mathcal{C}_{s\times t}^{\geq r}(\Bbbk) = \mathcal{M}_{s\times t}(\Bbbk) - \mathcal{C}_{s\times t}^{\leq r-1}(\Bbbk)$. By 1, $\mathcal{C}_{s\times t}^{\leq r-1}(\Bbbk)$ is closed in $\mathcal{M}_{s\times t}(\Bbbk)$. Hence, $\mathcal{C}_{s\times t}^{\geq r}(\Bbbk)$ is open in $\mathcal{M}_{s\times t}(\Bbbk)$. This completes the proof. □

3.1 Open Condition in $Hom_\Lambda(M, N)$

In this subsection we fix two Λ-modules M and N with $dim_{\Bbbk}(M) = t$ and $dim_{\Bbbk}(N) = s$. Let

$$Hom_\Lambda(M, N) = \{f : M \to N | f \text{ is } \Lambda\text{-modul homomorphism}\}.$$

Lemma 3. *The subset* $Hom_\Lambda^r(M, N) = \{f \in Hom_\Lambda(M, N) \,|rank\,(f) = r\}$ *is locally closed in* $Hom_\Lambda(M, N)$.

Proof. Note that $Hom_\Lambda^r(M, N)$ is intersection of two sets

$$Hom_\Lambda^{\leq r}(M, N) = \{f \in Hom_\Lambda(M, N) \,|rank\,(f) \leq r\} \qquad \text{and}$$

$$Hom_\Lambda^{\geq r}(M, N) = \{f \in Hom_\Lambda(M, N) \,|rank\,(f) \geq r\}.$$

By fixing $\Bbbk$-basis for M and N, we can parameterize $Hom_\Lambda(M, N)$ into $\mathcal{M}_{s\times t}(\Bbbk)$. By using Lemma 1 and Corollary 2, $Hom_\Lambda^{\leq r}(M, N)$ and $Hom_\Lambda^{\geq r}(M, N)$ are closed and open in $Hom_\Lambda(M, N)$, respectively, and the lemma follows. □

Lemma 4. *The function $\varphi : Hom_\Lambda(M,N) \to \mathbb{Z}$ given by sending f to $rank(f)$ is lower semicontinuous.*

Proof. By Lemma 1, for each $r \in \mathbb{Z}$, the set $\{f \in Hom_\Lambda(M,N) \,|rank(f) \le r\}$ closed in $Hom_\Lambda(M,N)$, and the lemma follows. □

Now, we will give some open conditions in $Hom_\Lambda(M,N)$.

Lemma 5. *The set $Inj_\Lambda(M,N) = \{f \in Hom_\Lambda(M,N) \,|f \text{ is injective}\}$ is open in $Hom_\Lambda(M,N)$.*

Proof. By using the rank plus nullity Theorem, $f \in Inj_\Lambda(M,N)$ if and only if $rank(f) = dim_{\Bbbk}(M) = t$. This condition is satisfied if $min\{s,t\} = t \le s$. Hence,

$$\begin{aligned} Inj_\Lambda(M,N) &= \{f \in Hom_\Lambda(M,N) \,|rank(f) = t\} \\ &= \{f \in Hom_\Lambda(M,N) \,|rank(f) \ge t\} \end{aligned}$$

is open in $Hom_\Lambda(M,N)$ by Lemma 2. □

Lemma 6. *The set $Surj_\Lambda(M,N) = \{f \in Hom_\Lambda(M,N) \,|f \text{ is surjective}\}$ is open in $Hom_\Lambda(M,N)$.*

Proof. Note that, $f \in Surj_\Lambda(M,N)$ if and only if $rank(f) = dim_{\Bbbk}(N) = s$. This condition is satisfied if $min\{s,t\} = s \le t$. Hence,

$$\begin{aligned} Surj_\Lambda(M,N) &= \{f \in Hom_\Lambda(M,N) \,|rank(f) = s\} \\ &= \{f \in Hom_\Lambda(M,N) \,|rank(f) \ge s\} \end{aligned}$$

is open in $Hom_\Lambda(M,N)$ by Lemma 2. □

3.2 Open Condition on $comp_{\mathbf{d}}^{\Lambda}$

We will define an affine variety which parameterize chain of complexes of Λ-modules. Given a nonnegative integer d, let M_d be Λ-module with $dim_{\Bbbk}(M_d) = d$. For every sequence of non negative integers $\mathbf{d} : \mathbb{Z} \to \mathbb{N}_0$ for which $d_n = 0$ for all $n \gg 0$ and $n \ll 0$, we define $comp_{\mathbf{d}}^{\Lambda}$ to be the affine subvariety of $\prod\limits_{n\in\mathbb{Z}} Hom_\Lambda(M_n, M_{n-1})$ consisting of sequence of Λ-homomorphisms $(\partial_n : M_n \to M_{n-1})_{n\in\mathbb{Z}}$ such that $\partial_n \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. Clearly, $comp_{\mathbf{d}}^{\Lambda}$ parameterizes chain of complexes of Λ-modules with fixed dimension in each degree.

Let $exact_{\mathbf{d},l}^{\Lambda}$ be defined as the affine subvariety of $comp_{\mathbf{d}}^{\Lambda}$ such that $im(\partial_{l+1}) = ker(\partial_l)$, i.e. the sequence of Λ-modules

$$\cdots \xrightarrow{\partial_{l+2}} M_{l+1} \xrightarrow{\partial_{l+1}} M_l \xrightarrow{\partial_l} M_{l-1} \xrightarrow{\partial_{l-1}} \cdots$$

exact on M_l.

Theorem 7. *The set* $exact^{\Lambda}_{\mathbf{d},l}$ *is open in* $comp^{\Lambda}_{\mathbf{d}}$.

Proof. Let $(\partial_n)_{n\in\mathbb{Z}} \in comp^{\Lambda}_{\mathbf{d}}$. We define projection map

$$\pi_l : comp^{\Lambda}_{\mathbf{d}} \to Hom_{\Lambda}(M_{l+1}, ker(\partial_l)), \quad (\partial_n)_{n\in\mathbb{Z}} \mapsto \partial_{l+1},$$

where $Hom_{\Lambda}(M_{l+1}, ker(\partial_l)) \subseteq Hom_{\Lambda}(M_{l+1}, M_l)$. The condition $\partial_n\partial_{n+1} = 0$ ensures that $im(\partial_{l+1}) \subseteq ker(\partial_l)$. This means that $rank(\partial_{l+1}) \leq dim_{\Bbbk}(ker(\partial_l)) = z_l$. For $r \in \mathbb{Z}$, let $\mathcal{H}^{\geq r} = \{f \in Hom_{\Lambda}(M_{l+1}, ker(\partial_l)) \,|rank(f) \geq r\}$. By Lemma 2, $\mathcal{H}^{\geq r}$ is open in $Hom_{\Lambda}(M_{l+1}, ker(\partial_l))$. Hence, the set

$$\mathcal{H}^{z_l} = \{f \in Hom_{\Lambda}(M_{l+1}, ker(\partial_l)) \,|rank(f) = z_l\} = \mathcal{H}^{\geq z_l}$$

is open in $Hom_{\Lambda}(M_{l+1}, ker(\partial_l))$. Since π_l is continuous map the

$$exact^{\Lambda}_{\mathbf{d},l} = \pi_l^{-1}(\mathcal{H}^{z_l})$$

is open in $comp^{\Lambda}_{\mathbf{d}}$. This completes the proof. □

For a complex $M = (\partial_n)_{n\in\mathbb{Z}} \in comp^{\Lambda}_{\mathbf{d}}$, let $H_n(M) = \frac{ker(\partial_i)}{im(\partial_{l+1})}$ be n-th homology of M.

Theorem 8. *The function* $h_l : comp^{\Lambda}_{\mathbf{d}} \to \mathbb{Z}$ *defined by sending* $M = (\partial_n)_{n\in\mathbb{Z}}$ *to* $dim_{\Bbbk}(H_l(M))$ *is upper semi continuous.*

Proof. By Lemma 4, for each $l \in \mathbb{Z}$, the function

$$b_l : Hom_{\Lambda}(M_l, ker(\partial_{l-1})) \to \mathbb{Z},\ \partial_l \mapsto rank(\partial_l) = dim_{\Bbbk}(im(\partial_l))$$

is lower semi continuous. Using the continuity of projection

$$\pi_l : comp^{\Lambda}_{\mathbf{d}} \to Hom_{\Lambda}(M_l, ker(\partial_{l-1})), \quad (\partial_n)_{n\in\mathbb{Z}} \mapsto \partial_l,$$

we can lift b_l into

$$\overline{b_l} : comp^{\Lambda}_{\mathbf{d}} \to \mathbb{Z},\ (\partial_n)_{n\in\mathbb{Z}} \mapsto rank(\partial_l) = dim_{\Bbbk}(im(\partial_l)),$$

given by $\overline{b_l} = b_l \circ \pi_l$, such that $\overline{b_l}$ is also lower semi continuous. By the same way and since $dim_{\Bbbk}(ker(\partial_l)) = d_l - rank(\partial_l)$, the function

$$\overline{z_l} : comp^{\Lambda}_{\mathbf{d}} \to \mathbb{Z},\ (\partial_n)_{n\in\mathbb{Z}} \mapsto rank(\partial_l) = dim_{\Bbbk}(ker(\partial_l))$$

is upper semicontinuous for every $l \in \mathbb{Z}$. These mean that, $\forall a \in \mathbb{Z}$, both of the sets

$$\left\{(\partial_n)_{n\in\mathbb{Z}} \in comp^{\Lambda}_{\mathbf{d}} | dim_{\Bbbk}(ker(\partial_l)) \leq a\right\} \quad \text{and} \quad \left\{(\partial_n)_{n\in\mathbb{Z}} \in comp^{\Lambda}_{\mathbf{d}} | rank(\partial_{l+1}) \geq a\right\}$$

are open in $comp^{\Lambda}_{\mathbf{d}}$. Moreover, the set

$$\mathcal{D}^l_a = \left\{M = (\partial_n)_{n\in\mathbb{Z}} \in comp^{\Lambda}_{\mathbf{d}} | dim_{\Bbbk}(H_l(M)) \leq a\right\}$$

is the intersection of the open set $\left\{(\partial_n)_{n\in\mathbb{Z}} \in comp_{\mathbf{d}}^{\Lambda} | dim_{\mathbb{k}}\left(ker\left(\partial_l\right)\right) \leq b\right\}$ and the open set $\left\{(\partial_n)_{n\in\mathbb{Z}} \in comp_{\mathbf{d}}^{\Lambda} | rank\left(\partial_{l+1}\right) \geq c\right\}$ for some $b, c \in \mathbb{Z}$. Hence, $\mathcal{D}_a^l$ is open in $comp_{\mathbf{d}}^{\Lambda}$. So, h_l is upper semi continuous and the proof is complete. $\square$

4 Concluding Remarks

We have shown that the subset $Hom_{\Lambda}^{r}(M, N)$ is locally closed in $Hom_{\Lambda}(M, N)$, both of the set $Inj_{\Lambda}(M, N)$ and $Surj_{\Lambda}(M, N)$ are open in $Hom_{\Lambda}(M, N)$, and the set $exact_{\mathbf{d},l}^{\Lambda}$ is open in $comp_{\mathbf{d}}^{\Lambda}$. Also, we have proved that the function of rank Λ-homomorphism and the function of dimension of homology in chain complexes are lower and upper semi continuous, respectively.

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