

## GENERALIZED IDEAL CO-TRANSFORMS

Tran Tuan Nam and My Vinh Quang

*Ho Chi Minh University of Pedagogy  
280 An Duong Vuong, District 5, Ho Chi Minh City, Vietnam  
Email: namtuantran@gmail.com*

### Abstract

We introduce the definition of generalized ideal co-transforms  $C_i^I(M, N)$  which is a generalization of the definition of ideal co-transforms  $C_i^I(M)$ . We also study some basic properties of ideal co-transforms of linearly compact modules.

## 1 Introduction

Throughout this paper,  $(R, m)$  is a local noetherian commutative ring and has a topological structure. Let  $I$  be an ideal of  $R$ , in [14] we defined the  $i$ -th ideal co-transform  $C_i^I(M)$  of  $M$  with respect to  $I$  (or  $i$ -th  $I$ -co-transform of  $M$ ) by

$$C_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(I^t; M).$$

Note that the definition of  $C_i^I(M)$  at degree zero ( $i=0$ ) is in some sense dual to Brodmann's definition of ideal transforms ([1]). Also in [14] we studied some basic properties of ideal co-transforms  $C_i^I(M)$  and found out relations to local homology modules  $H_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(R/I^t, M)$ .

In the paper we will define the  $i$ -th generalized ideal co-transform  $C_i^I(M, N)$  by

$$C_i^I(M, N) = \varprojlim_t \operatorname{Tor}_i^R(I^t M, N).$$

This definition is in fact a generalization of the definition of ideal co-transforms. The organization of the paper is as follows.

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**Key words:** Linearly compact module, (generalized) ideal co-transform, (generalized) local homology, (generalized) local cohomology.

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In section 2 we recall briefly some properties of linearly compact modules and generalized local homology modules that we shall use.

In section 3 we introduce the definition of generalized ideal co-transforms  $C_i^I(M, N)$ . If  $M$  is a finitely generated  $R$ -module and  $N$  is a linearly compact, then  $C_i^I(M, N)$  ( $i \geq 0$ ) is also linearly compact and the sequence of functors  $\{C_i^I(M, -)\}$  is a positive strongly connected sequence on the category of linearly compact modules and continuous homomorphisms. On the other hand, we have a long exact sequence of linearly compact  $R$ -modules

$$\cdots \rightarrow H_{i+1}^I(M, N) \rightarrow C_i^I(M, N) \rightarrow \text{Tor}_i^R(M, N) \rightarrow H_i^I(M, N) \rightarrow \cdots$$

$$\cdots \rightarrow H_1^I(M, N) \rightarrow C^I(M, N) \xrightarrow{\eta_{M,N}} M \otimes_R N \xrightarrow{\theta_{M,N}} \Lambda_I(M \otimes_R N) \rightarrow 0.$$

Moreover, if  $pd(M) = p < +\infty$ , then  $H_{p+1}^I(M, N) \cong C_p^I(M, N)$  (Theorem 3.4).

Section 4 is devoted to study the relations between generalized ideal co-transforms and co-localization of linearly compact modules. In Theorem 4.2 we show that  $C^I(M \otimes_R N) \cong C^I(M, N)$  and  $C^{aR}(M, N) \cong {}_a C^{aR}(M, N)$  for any  $a \in R$ . The section is close by Theorem 4.3 in which we prove the isomorphism  $C^{aR}(M, N) \cong {}_a(M \otimes_R N)$ .

## 2 PRELIMINARIES

Let us begin by recalling briefly the definition and basic properties of linearly compact modules by terminology of I. G. Macdonald ([10]) that we shall use.

Let  $M$  be a topological  $R$ -module.  $M$  is said to be *linearly topologized* if  $M$  has a base of neighborhoods of the zero element  $\mathcal{M}$  consisting of submodules.  $M$  is called *Hausdorff* if the intersection of all the neighborhoods of the zero element is 0. A Hausdorff linearly topologized  $R$ -module  $M$  is said to be *linearly compact* if  $\mathcal{F}$  is a family of closed cosets (i.e., cosets of closed submodules) in  $M$  which has the finite intersection property, then the cosets in  $\mathcal{F}$  have a non-empty intersection (see [10, 10]).

It is clear that Artinian  $R$ -modules are linearly compact and discrete. We have some following properties of linearly compact modules.

**Lemma 2.1.** (see [10, §3]) (i) *If  $M$  is a Hausdorff linearly topologized  $R$ -module and  $N$  a closed submodule of  $M$ , then  $M$  is linearly compact if and only if  $N$  and  $M/N$  are linearly compact.*

(ii) *Let  $f : M \rightarrow N$  be a continuous homomorphism of Hausdorff linearly topologized  $R$ -modules. If  $M$  is linearly compact, then  $f(M)$  is linearly compact and  $f$  is a closed map.*

(iii) *The inverse limit of a system of linearly compact  $R$ -modules with continuous homomorphisms is linearly compact.*

**Lemma 2.2.** (see [8, 7.1]) *Let  $\{M_t\}$  be an inverse system of linearly compact modules with continuous homomorphisms. Then  $\varprojlim_t^i M_t = 0$  for all  $i > 0$ .*

Therefore, if

$$0 \longrightarrow \{M_t\} \longrightarrow \{N_t\} \longrightarrow \{P_t\} \longrightarrow 0$$

is a short exact sequence of inverse systems of  $R$ -modules, then the sequence of inverse limits

$$0 \longrightarrow \varprojlim_t M_t \longrightarrow \varprojlim_t N_t \longrightarrow \varprojlim_t P_t \longrightarrow 0$$

is exact.

**Lemma 2.3.** (see [3, §2]) *Let  $N$  be a finitely generated  $R$ -module and  $M$  a linearly compact  $R$ -module. Then for all  $i \geq 0$ ,  $\text{Tor}_i^R(N; M)$  is a linearly compact  $R$ -module. Moreover,*

(i) *If  $f : N \longrightarrow N'$  is a homomorphism of finitely generated  $R$ -modules, then the induced homomorphism  $f_{i,M} : \text{Tor}_i^R(N; M) \longrightarrow \text{Tor}_i^R(N'; M)$  is continuous.*

(ii) *If  $g : M \longrightarrow M'$  is a continuous homomorphism of linearly compact  $R$ -modules, then the induced homomorphism  $g_{N,i} : \text{Tor}_i^R(N; M) \longrightarrow \text{Tor}_i^R(N; M')$  is also continuous.*

Let  $I$  be an ideal of the ring  $R$  and  $M, N$   $R$ -modules. In [12], the  $i$ -th generalized local homology module  $H_i^I(M, N)$  of  $M, N$  with respect to  $I$  is defined by

$$H_i^I(M, N) = \varprojlim_t \text{Tor}_i^R(M/I^t M, N).$$

In the special case  $M = R$ ,  $H_i^I(R, N) = H_i^I(N)$  the  $i$ -th local homology module  $H_i^I(N)$  of  $N$  with respect to  $I$  ([2], [3]).

**Lemma 2.4.** ([13, 3.4]) *Let  $M$  be a finitely generated module and  $N$  a linearly compact  $R$ -module. If  $N$  is complete in  $I$ -adic topology (i.e.,  $\Lambda_I(N) \cong N$ ), then there is an isomorphism for all  $i \geq 0$ ,*

$$\text{Tor}_i^R(M, N) \cong H_i^I(M, N).$$

The  $i$ -th ideal co-transform  $C_i^I(M)$  of  $M$  with respect to  $I$  (or  $i$ -th  $I$ -co-transform of  $M$ ) is defined by

$$C_i^I(M) = \varprojlim_t \text{Tor}_i^R(I^t; M) \text{ ([14])}.$$

$C_0^I(M)$  is called the  $I$ -co-transform of  $M$  and denoted by  $C^I(M)$ .

**Lemma 2.5.** ([14, 3.5]) *Let  $M$  be a linearly compact  $R$ -module. There are two short exact sequences*

$$0 \longrightarrow H_1^I(M) \longrightarrow C^I(M) \longrightarrow \bigcap_{t>0} I^t M \longrightarrow 0,$$

$$0 \longrightarrow \bigcap_{t>0} I^t M \longrightarrow M \xrightarrow{\theta_M} \Lambda_I(M) \longrightarrow 0.$$

### 3 Generalized ideal Co-transforms

Let  $(R, m)$  be a local noetherian commutative ring. We suggest the following definition.

**Definition 3.1.** Let  $I$  be an ideal of  $R$  and  $M, N$   $R$ -modules. The  $i$ -th generalized ideal co-transform  $C_i^I(M, N)$  of  $M, N$  with respect to  $I$  is defined by

$$C_i^I(M, N) = \varprojlim_t \text{Tor}_i^R(I^t M, N).$$

Especially,  $C_0^I(M, N)$  is called the generalized  $I$ -co-transform of  $M, N$  and denoted by  $C^I(M, N)$  for simplicity. When  $M = R$ , we have  $C_i^I(R, N) = C_i^I(N)$  the  $i$ -th ideal co-transform  $C_i^I(M)$  of  $M$  with respect to  $I$  ([14]).

**Lemma 3.2.** *Let  $M$  be a finitely generated  $R$ -module. If  $N$  is a linearly compact  $R$ -module, then  $C_i^I(M, N)$  is also linearly compact for all  $i \geq 0$ .*

**Proof** It follows from 2.3 (i) that  $\{\text{Tor}_i^R(I^t M, N)\}$  ( $i \geq 0$ ) forms an inverse system of linearly compact modules which continuous homomorphisms. Therefore  $C_i^I(M, N)$  is also linearly compact by 2.1 (iii).  $\square$

Set  $D_i^I(M, N) = \varinjlim_t \text{Ext}_R^i(I^t M, N)$ . Note that

$$D_I^0(M, N) = \varinjlim_t \text{Hom}_R(I^t M, N) = D_I(M, N)$$

is the generalized  $I$ -transform of  $M, N$  ([15]).

Let  $D(M) = \text{Hom}_R(M, E(R/m))$  the Matlis dual of  $M$ .

**Proposition 3.3.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  an  $R$ -module. Then*

$$C_i^I(M, D(N)) \cong D(D_I^i(M, N))$$

for all  $i \geq 0$ .

**Proof** We first note that  $\mathrm{Tor}_i^R(I^t M, D(N)) \cong D(\mathrm{Ext}_R^i(I^t M, N))$  by [17, 3.4.14] and  $\varprojlim_t D(\mathrm{Ext}_R^i(I^t M, N)) \cong D(\varinjlim_t \mathrm{Ext}_R^i(I^t M, N))$  by [16, 2.27]. Therefore

$$\begin{aligned} C_i^I(M, D(N)) &\cong \varprojlim_t \mathrm{Tor}_i^R(I^t M, D(N)) \\ &\cong \varprojlim_t D(\mathrm{Ext}_R^i(I^t M, N)) \\ &\cong D(\varinjlim_t \mathrm{Ext}_R^i(I^t M, N)) = D(D_i^I(M, N)) \end{aligned}$$

as required.  $\square$

**Theorem 3.4.** *Let  $M$  be a finitely generated  $R$ -module.*

(i) *If  $N$  is a linearly compact  $R$ -module, then we have a long exact sequence of linearly compact  $R$ -modules*

$$\begin{aligned} \cdots \rightarrow H_{i+1}^I(M, N) \rightarrow C_i^I(M, N) \rightarrow \mathrm{Tor}_i^R(M, N) \rightarrow H_i^I(M, N) \rightarrow \cdots \\ \cdots \rightarrow H_1^I(M, N) \rightarrow C^I(M, N) \xrightarrow{\eta^{M, N}} M \otimes_R N \xrightarrow{\theta^{M, N}} \Lambda_I(M \otimes_R N) \rightarrow 0. \end{aligned}$$

Moreover, if  $\mathrm{pd}(M) = p < +\infty$ , then  $H_{p+1}^I(M, N) \cong C_p^I(M, N)$ ;

(ii) *If  $0 \rightarrow N'' \xrightarrow{f} N \xrightarrow{g} N' \rightarrow 0$  is a short exact sequence of linearly compact  $R$ -modules with continuous homomorphisms, then we have a long exact sequence of linearly compact  $R$ -modules*

$$\begin{aligned} \cdots \rightarrow C_i^I(M, N'') \xrightarrow{f_i} C_i^I(M, N) \xrightarrow{g_i} C_i^I(M, N') \rightarrow \\ \cdots \rightarrow C^I(M, N'') \xrightarrow{f_0} C^I(M, N) \xrightarrow{g_0} C^I(M, N') \rightarrow 0 \end{aligned}$$

in which the homomorphisms  $f_i, g_i$  are continuous for all  $i \geq 0$ .

**Proof** (i). For any positive integer  $t$  the short exact sequence

$$0 \rightarrow I^t M \rightarrow M \rightarrow M/I^t M \rightarrow 0$$

gives rise a long exact sequence of linearly compact  $R$ -modules

$$\begin{aligned} \cdots \rightarrow \mathrm{Tor}_i^R(I^t M, N) \rightarrow \mathrm{Tor}_i^R(M, N) \rightarrow \mathrm{Tor}_i^R(M/I^t M, N) \rightarrow \cdots \\ \cdots \rightarrow I^t M \otimes_R N \rightarrow M \otimes_R N \rightarrow (M/I^t M) \otimes_R N \rightarrow 0. \end{aligned}$$

Since  $M$  is the finitely generated  $R$ -module and  $N$  is the linearly compact  $R$ -module, the modules of the long exact sequence above are also linearly

compact. Thus, passing inverse limits, we have the long exact sequence of linearly compact  $R$ -modules as required. Note that  $\text{Tor}_i^R(M, N) = 0$  for all  $i > p$ , so  $H_{p+1}^I(M, N) \cong C_p^I(M, N)$ .

(ii). The short exact sequence of linearly compact modules  $0 \rightarrow N'' \xrightarrow{f} N \xrightarrow{g} N' \rightarrow 0$  gives rise to a long exact sequence of linearly compact modules for all  $t > 0$

$$\begin{aligned} \cdots \rightarrow \text{Tor}_i^R(I^t M; N'') \xrightarrow{f_{it}} \text{Tor}_i^R(I^t M; N) \xrightarrow{g_{it}} \text{Tor}_i^R(I^t M; N') \rightarrow \\ \cdots \rightarrow I^t M \otimes_R N'' \xrightarrow{f_{0t}} I^t M \otimes_R N \xrightarrow{g_{0t}} I^t M \otimes_R N' \rightarrow 0 \end{aligned}$$

in which homomorphisms  $f_{it}, g_{it}$  are continuous by 2.3, (ii). Then  $\text{Im } f_{it}, \text{Ker } f_{it}$  are linearly compact. Passing inverse limits, we get the long exact sequence as required. Since the homomorphisms  $f_{it}, g_{it}$  are continuous, the homomorphisms  $f_i, g_i$  induced on inverse limits are also continuous.  $\square$

**Corollary 3.5.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  a linearly compact  $R$ -module. Then the homomorphism*

$$\eta_i(M, N) : C_i^I(M, N) \rightarrow \text{Tor}_i^R(M, N)$$

*is an isomorphism if  $H_{i+1}^I(M, N) = H_i^I(M, N) = 0$ .*

**Proof** It is immediately induced from 3.4 (i).  $\square$

Let  $\Lambda_I(M) = \varprojlim_t M/I^t M$  the  $I$ -adic completion of  $M$ , we have the following lemma.

**Lemma 3.6.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  a linearly compact  $R$ -module. If  $N$  is complete in  $I$ -adic topology (i.e.,  $\Lambda_I(N) \cong N$ ), then  $C_i^I(M, N) = 0$  for all  $i \geq 0$ .*

**Proof** From 3.4 (i) we have a long exact sequence

$$\begin{aligned} \cdots \rightarrow H_{i+1}^I(M, N) \rightarrow C_i^I(M, N) \rightarrow \text{Tor}_i^R(M, N) \rightarrow H_i^I(M, N) \rightarrow \cdots \\ \cdots \rightarrow H_1^I(M, N) \rightarrow C^I(M, N) \rightarrow M \otimes_R N \rightarrow \Lambda_I(M \otimes_R N) \rightarrow 0. \end{aligned}$$

As  $N$  is complete in  $I$ -adic topology, we have  $H_i^I(M, N) \cong \text{Tor}_i^R(M, N)$  for all  $i \geq 0$  by 2.4. Thus  $C_i^I(M, N) = 0$  for all  $i \geq 0$ .  $\square$

Since  $H_i^I(N)$  is complete in  $I$ -adic topology for all  $i \geq 0$ , we have the immediate consequence.

**Corollary 3.7.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  a linearly compact  $R$ -module. Then  $C_i^I(M, H_j^I(N)) = 0$  for all  $i, j \geq 0$ .*

Note that  $C_i^I(R, N) = C_i^I(N)$ , so we also have the following immediate consequence.

**Corollary 3.8.** *Let  $N$  be a linearly compact  $R$ -module. If  $N$  is complete in  $I$ -adic topology, then  $C_i^I(N) = 0$  for all  $i \geq 0$ .*

**Lemma 3.9.** *Let  $M, N$  be  $R$ -modules. Then*

$$\Lambda_I(C^I(M, N)) \cong C^I(M, \Lambda_I(N)).$$

**Proof** We have

$$\begin{aligned} \Lambda_I(C^I(M, N)) &\cong \varprojlim_t (R/I^t \otimes_R C^I(M, N)) \\ &= \varprojlim_t (R/I^t \otimes_R \varprojlim_s (I^s M \otimes_R N)) \\ &\cong \varprojlim_t \varprojlim_s (R/I^t \otimes_R (I^s M \otimes_R N)) \\ &\cong \varprojlim_s \varprojlim_t (I^s M \otimes_R (R/I^t \otimes_R N)) \\ &\cong \varprojlim_s (I^s M \otimes_R \varprojlim_t (R/I^t \otimes_R N)) \\ &\cong C^I(M, \Lambda_I(N)) \end{aligned}$$

as required.  $\square$

**Corollary 3.10.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  a linearly compact  $R$ -module. Then  $\Lambda_I(C^I(M, N)) = 0$ .*

**Proof** It follows from 3.7 and 3.9.  $\square$

**Lemma 3.11.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  a linearly compact  $R$ -module. Then  $C_i^I(M, C^I(N)) \cong C_i^I(M, N)$ .*

**Proof** From 2.5 we have two short exact sequences

$$0 \rightarrow H_1^I(N) \rightarrow C^I(N) \rightarrow \bigcap_{t>0} I^t N \rightarrow 0,$$

$$0 \rightarrow \bigcap_{t>0} I^t N \rightarrow N \rightarrow \Lambda_I(N) \rightarrow 0.$$

By 3.4 (ii) we get two exact sequences

$$\dots C_i^I(M, C^I(N)) \rightarrow C_i^I(M, \bigcap_{t>0} I^t N) \rightarrow C_{i-1}^I(M, H_1^I(N)) \dots,$$

$$\dots C_i^I(M, \bigcap_{t>0} I^t N) \rightarrow C_i^I(M, N) \rightarrow C_i^I(M, \Lambda_I(N)) \dots$$

Then the conclusion follows from 3.7.  $\square$

**Lemma 3.12.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  a linearly compact  $R$ -module. Then  $C^I(C^I(M, N)) \cong C^I(M, N)$ .*

**Proof** We have

$$\begin{aligned} C^I(C^I(M, N)) &\cong \varprojlim_t (I^t \otimes_R \varprojlim_s (I^s M \otimes_R N)) \\ &\cong \varprojlim_t \varprojlim_s (I^t \otimes_R (I^s M \otimes_R N)) \\ &\cong \varprojlim_s \varprojlim_t (I^s M \otimes_R (I^t \otimes_R N)) \\ &\cong \varprojlim_s (I^s M \otimes_R \varprojlim_t (I^t \otimes_R N)) \\ &\cong C^I(M, C^I(N)). \end{aligned}$$

Now the conclusion follows from 3.12.  $\square$

**Proposition 3.13.** *Let  $f : N' \rightarrow N$  be a homomorphism of linearly compact  $R$ -modules such that  $\text{Ker } f$  and  $\text{coker } f$  are complete in  $I$ -adic topology. Let  $\varphi : K \rightarrow N$  be a further homomorphism of linearly compact  $R$ -modules. Then*

(i) *The homomorphism  $C_i^I(M, f) : C_i^I(M, N') \rightarrow C_i^I(M, N)$  is an isomorphism for all  $i \geq 0$ .*

(ii) *There is a homomorphism  $\psi : C^I(M, K) \rightarrow M \otimes_R N'$  such that the diagram*

$$\begin{array}{ccccc} M \otimes_R N' & \xrightarrow{M \otimes_R f} & M \otimes_R N \\ \uparrow \psi & & \uparrow M \otimes_R \varphi \\ C^I(M, K) & \xrightarrow{\eta_{M, K}} & M \otimes_R K, \end{array}$$

*is commutative, i. e.,  $M \otimes_R f \circ \psi = M \otimes_R \varphi \circ \eta_{M, K}$ .*

**Proof** (i) We have short exact sequences of linearly compact modules

$$0 \rightarrow \text{Ker } f \rightarrow N' \xrightarrow{\alpha} \text{Im } f \rightarrow 0$$

and

$$0 \rightarrow \text{Im } f \xrightarrow{\beta} N \rightarrow \text{coker } f \rightarrow 0$$

in which  $f = \beta\alpha$  and homomorphisms are continuous. It is therefore enough to show that  $C_i^I(M, \alpha)$  and  $C_i^I(M, \beta)$  are both isomorphisms.



The first short exact sequence above induces by 3.4 (ii) an exact sequence

$$\dots C_i^I(M, \text{Ker } f) \rightarrow C_i^I(M, N') \xrightarrow{C_i^I(M, \alpha)} C_i^I(M, \text{Im } f) \dots$$

From 3.7 and the hypothesis  $\text{Ker } f \cong \Lambda_I(\text{Ker } f)$  we have  $C_i^I(M, \text{Ker } f) = 0$ . Hence  $C_i^I(M, \alpha)$  is an isomorphism. Next, from the second short exact sequence we get an induced exact sequence

$$\dots C_i^I(M, \text{Im } f) \xrightarrow{C_i^I(M, \beta)} C_i^I(M, N) \rightarrow C_i^I(M, \text{coker } f) \dots$$

We have  $C_i^I(M, \text{coker } f) \cong C_i^I(M, \Lambda_I(\text{coker } f)) = 0$  for all  $i \geq 0$  by 3.6. Therefore  $C_i^I(M, \beta)$  is an isomorphism.

(ii) We have a commutative diagram

$$\begin{array}{ccccccc} M \otimes_R N' & \xrightarrow{M \otimes_R f} & M \otimes_R N & \xleftarrow{M \otimes_R \varphi} & M \otimes_R K \\ \uparrow \eta_{M, N'} & & \uparrow \eta_{M, N} & & \uparrow \eta_{M, K} \\ C^I(M, N') & \xrightarrow{C^I(M, f)} & C^I(M, N) & \xleftarrow{C^I(M, \varphi)} & C^I(M, K). \end{array}$$

By (i),  $C^I(M, f)$  is an isomorphism.

Set  $\psi = \eta_{M, N'} \circ C^I(M, f)^{-1} \circ C^I(M, \varphi)$ , we have

$$\begin{aligned} M \otimes_R f \circ \psi &= M \otimes_R f \circ \eta_{M, N'} \circ C^I(M, f)^{-1} \circ C^I(M, \varphi) \\ &= \eta_{M, N} \circ C^I(M, \varphi) = M \otimes_R \varphi \circ \eta_{M, K}. \end{aligned}$$

The proof is complete.  $\square$

## 4 CO-LOCALIZATION

Let  $S$  be multiplicative set of  $R$ . For an  $R$ -module  $M$  the module  $\text{Hom}_R(R_S; M)$  is called co-localization of  $M$  with respect to  $S$  (see [11]). We denote it briefly by  ${}_S M$ . If  $M$  is a linearly compact  $R$ -module, then  ${}_S M$  is also a linearly compact  $R$ -module by [5, 2.4]. The following proposition says that the co-localization can "commute" to the generalized ideal co-transform of a linearly compact  $R$ -module.

**Proposition 4.1.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  a linearly compact  $R$ -module. Then*

$${}_S C_i^I(M, N) \cong C_i^{I R_S}(M_S, {}_S N)$$

for all  $i \geq 0$ .

**Proof** We first note that

$${}_S(\mathrm{Tor}_i^R(I^t M, N)) \cong \mathrm{Tor}_i^{RS}(I^t M_S; {}_S N)$$

by [4, 3.9]. Then

$$\begin{aligned} {}_S C_i^I(M, N) &= \mathrm{Hom}_R(R_S, \varprojlim_t \mathrm{Tor}_i^R(I^t M, N)) \\ &\cong \varprojlim_t \mathrm{Hom}_R(R_S, \mathrm{Tor}_i^R(I^t M, N)) = \varprojlim_t {}_S(\mathrm{Tor}_i^R(I^t M, N)) \\ &\cong C_i^{IRS}(M_S, {}_S N) \end{aligned}$$

as required.  $\square$

Let  $a$  be an element in  $R$ , the notation  ${}_a M$  means that the co-localization of  $M$  with respect to the multiplicative set  $S = \{1, a, a^2, \dots\}$ .

**Theorem 4.2.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  a linearly compact  $R$ -module. Then*

- (i)  $C^I(M \otimes_R N) \cong C^I(M, N)$ ;
- (ii)  $C^{aR}(M, N) \cong {}_a C^{aR}(M, N)$  for any element  $a \in R$ .

**Proof** (i). We have an exact sequence by 3.4 (i)

$$H_1^I(M, N) \xrightarrow{\varphi} C^I(M, N) \xrightarrow{\eta_{M,N}} M \otimes_R N \xrightarrow{\theta_{M,N}} \Lambda_I(M \otimes_R N) \rightarrow 0.$$

It induces two short exact sequences

$$0 \rightarrow \mathrm{Im} \varphi \rightarrow C^I(M, N) \rightarrow \mathrm{Im} \eta_{M,N} \rightarrow 0,$$

$$0 \rightarrow \mathrm{Im} \eta_{M,N} \rightarrow M \otimes_R N \xrightarrow{\theta_{M,N}} \Lambda_I(M \otimes_R N) \rightarrow 0.$$

Combining 3.4 (ii) with 3.8 yields induced exact sequences

$$0 \rightarrow C^I(\mathrm{Im} \varphi) \rightarrow C^I(C^I(M, N)) \rightarrow C^I(\mathrm{Im} \eta_{M,N}) \rightarrow 0,$$

$$0 \rightarrow C^I(\mathrm{Im} \eta_{M,N}) \rightarrow C^I(M \otimes_R N) \xrightarrow{\theta_{M,N}} C^I(\Lambda_I(M \otimes_R N)) \rightarrow 0.$$

Note that  $\mathrm{Im} \varphi$  and  $\Lambda_I(M \otimes_R N)$  are both complete in  $I$ -adic topology, so

$$C^I(M \otimes_R N) \cong C^I(C^I(M, N)) \cong C^I(M, N)$$

by 3.8 and 3.12.

(ii). It follows from [18, 10.8.3] that  $a^t M \otimes_R \text{Hom}(S^{-1}R, N) \cong \text{Hom}_R(S^{-1}R, a^t M \otimes_R N)$ . On the other hand,  $C^{aR}(M, N) \cong C^{aR}(M, C^{aR}(N))$  by 3.12 and  $C^{aR}(N) \cong {}_a N$  by [14, 4.4]. Therefore

$$\begin{aligned}
C^{aR}(M, N) &\cong C^{aR}(M, C^{aR}(N)) \\
&\cong C^{aR}(M, {}_a N) \\
&\cong \varprojlim_t a^t M \otimes_R \text{Hom}(S^{-1}R, N) \\
&\cong \varprojlim_t \text{Hom}_R(S^{-1}R, a^t M \otimes_R N) \\
&\cong \text{Hom}_R(S^{-1}R, \varprojlim_t a^t M \otimes_R N) \\
&= {}_a C^{aR}(M, N)
\end{aligned}$$

as required.  $\square$

**Theorem 4.3.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  an artinian. There is an isomorphism*

$$C^{aR}(M, N) \cong {}_a(M \otimes_R N)$$

for any element  $a \in R$ .

**Proof** It follows from [6, 2.7] that  $D_{aR}(M, N) \cong \text{Hom}_R(M, N)_a$ . Moreover,  $C^{aR}(M, D(N)) \cong D(D_{aR}(M, N))$  by 3.3. Hence

$$\begin{aligned}
C^{aR}(M, N) &\cong C^{aR}(M, DD(N)) \\
&\cong D(D_{aR}(M, D(N))) \\
&\cong D(\text{Hom}_R(M, D(N))_a) \\
&\cong {}_a D(\text{Hom}_R(M, D(N))) \\
&\cong {}_a(M \otimes_R DD(N)) \\
&\cong {}_a(M \otimes_R N)
\end{aligned}$$

as required.  $\square$

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