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GENERALIZED IDEAL CO-TRANSFORMS

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Abstract

We introduce the definition of generalized ideal co-transforms $C_i^I(M,N)$ which is a generalization of the definition of ideal co-transforms $C_i^I(M)$. We also study some basic properties of ideal co-transforms of linearly compact modules.

1 Introduction

Throughout this paper, (R,m) is a local noetherian commutative ring and has a topological structure. Let I be an ideal of R, in [14] we defined the *i-th ideal co-transform* $C_i^I(M)$ of M with respect to I (or $i-th$ $I-$*co-transform* of M) by

$$C_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(I^t; M).$$

Note that the definition of $C_i^I(M)$ at degree zero (i=0) is in some sense dual to Brodmann's definition of ideal transforms ([1]). Also in [14] we studied some basic properties of ideal co-transforms $C_i^I(M)$ and found out relations to local homology modules $H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(R/I^t, M)$.

In the paper we will define the *i-th generalized ideal co-transform* $C_i^I(M,N)$ by

$$C_i^I(M,N) = \varprojlim_t \mathrm{Tor}_i^R(I^t M, N).$$

This definition is in fact a generalization of the definition of ideal co-transforms. The organization of the paper is as follows.

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In section 2 we recall briefly some properties of linearly compact modules and generalized local homology modules that we shall use.

In section 3 we introduce the definition of generalized ideal co-transforms $C_i^I(M,N)$. If M is a finitely generated $R-$module and N is a linearly compact, then $C_i^I(M,N)$ $(i \geq 0)$ is also linearly compact and the sequence of functors $\{C_i^I(M,-)\}$ is a positive strongly connected sequence on the category of linearly compact modules and continuous homomorphisms. On the other hand, we have a long exact sequence of linearly compact $R-$modules

$$\cdots \to H_{i+1}^I(M,N) \to C_i^I(M,N) \to \mathrm{Tor}_i^R(M,N) \to H_i^I(M,N) \to \cdots$$

$$\cdots \to H_1^I(M,N) \to C^I(M,N) \xrightarrow{\eta_{M,N}} M\bigotimes\nolimits_R N \xrightarrow{\theta_{M,N}} \Lambda_I(M\bigotimes\nolimits_R N) \to 0.$$

Moreover, if $pd(M) = p < +\infty$, then $H_{p+1}^I(M,N) \cong C_p^I(M,N)$ (Theorem 3.4).

Section 4 is devoted to study the relations between generalized ideal co-transforms and co-localization of linearly compact modules. In Theorem 4.2 we show that $C^I(M\otimes_R N) \cong C^I(M,N)$ and $C^{aR}(M,N) \cong {}_aC^{aR}(M,N)$ for any $a \in R$. The section is close by Theorem 4.3 in which we prove the isomorphism $C^{aR}(M,N) \cong {}_a(M\bigotimes_R N)$.

2 PRELIMINARIES

Let us begin by recalling briefly the definition and basic properties of linearly compact modules by terminology of I. G. Macdonald ([10]) that we shall use.

Let M be a topological $R-$module. M is said to be *linearly topologized* if M has a base of neighborhoods of the zero element $\mathcal{M}$ consisting of submodules. M is called *Hausdorff* if the intersection of all the neighborhoods of the zero element is 0. A Hausdorff linearly topologized $R-$module M is said to be *linearly compact* if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection (see [10, 10]).

It is clear that Artinian $R-$modules are linearly compact and discrete. We have some following properties of linearly compact modules.

Lemma 2.1. (see [10, §3]) *(i) If M is a Hausdorff linearly topologized $R-$module and N a closed submodule of M, then M is linearly compact if and only if N and M/N are linearly compact.*
(ii) Let $f : M \longrightarrow N$ be a continuous homomorphism of Hausdorff linearly topologized $R-$modules. If M is linearly compact, then $f(M)$ is linearly compact and f is a closed map.
(iii) The inverse limit of a system of linearly compact $R-$modules with continuous homomorphisms is linearly compact.

Lemma 2.2. (see [8, 7.1]) *Let $\{M_t\}$ be an inverse system of linearly compact modules with continuous homomorphisms. Then $\varprojlim_t{}^i M_t = 0$ for all $i > 0$. Therefore, if*

$$0 \longrightarrow \{M_t\} \longrightarrow \{N_t\} \longrightarrow \{P_t\} \longrightarrow 0$$

is a short exact sequence of inverse systems of $R-$modules, then the sequence of inverse limits

$$0 \longrightarrow \varprojlim_t M_t \longrightarrow \varprojlim_t N_t \longrightarrow \varprojlim_t P_t \longrightarrow 0$$

is exact.

Lemma 2.3. (see [3, §2]) *Let N be a finitely generated $R-$module and M a linearly compact $R-$module. Then for all $i \geq 0$, $\mathrm{Tor}_i^R(N; M)$ is a linearly compact $R-$module. Moreover,*
(i) If $f : N \longrightarrow N'$ is a homomorphism of finitely generated $R-$modules, then the induced homomorphism $f_{i,M} : \mathrm{Tor}_i^R(N; M) \longrightarrow \mathrm{Tor}_i^R(N'; M)$ is continuous.
(ii) If $g : M \longrightarrow M'$ is a continuous homomorphism of linearly compact $R-$modules, then the induced homomorphism $g_{N,i} : \mathrm{Tor}_i^R(N; M) \longrightarrow \mathrm{Tor}_i^R(N; M')$ is also continuous.

Let I be an ideal of the ring R and M, N $R-$modules. In [12], the $i-$th generalized local homology module $H_i^I(M, N)$ of M, N with respect to I is defined by

$$H_i^I(M, N) = \varprojlim_t \mathrm{Tor}_i^R(M/I^t M, N).$$

In the special case $M = R$, $H_i^I(R, N) = H_i^I(N)$ the $i-$th local homology module $H_i^I(N)$ of N with respect to I ([2], [3]).

Lemma 2.4. ([13, 3.4]) *Let M be a finitely generated module and N a linearly compact $R-$module. If N is complete in $I-$adic topology (i.e., $\Lambda_I(N) \cong N$), then there is an isomorphism for all $i \geq 0$,*

$$\mathrm{Tor}_i^R(M, N) \cong H_i^I(M, N).$$

The *i-th ideal co-transform* $C_i^I(M)$ of M with respect to I (or *$i-$th $I-$co-transform* of M) is defined by

$$C_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(I^t; M) \ ([14]).$$

$C_0^I(M)$ is called the $I-$co-transform of M and denoted by $C^I(M)$.

Lemma 2.5. ([14, 3.5]) *Let M be a linearly compact $R-$module. There are two short exact sequences*

$$0 \longrightarrow H_1^I(M) \longrightarrow C^I(M) \longrightarrow \bigcap_{t>0} I^t M \longrightarrow 0,$$

$$0 \longrightarrow \bigcap_{t>0} I^t M \longrightarrow M \xrightarrow{\theta_M} \Lambda_I(M) \longrightarrow 0.$$

3 Generalized ideal Co-transforms

Let (R, m) be a local noetherian commutative ring. We suggest the following definition.

Definition 3.1. Let I be an ideal of R and M, N $R-$modules. The *i-th generalized ideal co-transform* $C_i^I(M,N)$ of M, N with respect to I is defined by

$$C_i^I(M,N) = \varprojlim_t \mathrm{Tor}_i^R(I^t M, N).$$

Especially, $C_0^I(M,N)$ is called the generalized $I-$co-transform of M, N and denoted by $C^I(M,N)$ for simplicity. When $M = R$, we have $C_i^I(R,N) = C_i^I(N)$ the i-th ideal co-transform $C_i^I(M)$ of M with respect to I ([14]).

Lemma 3.2. *Let M be a finitely generated $R-$module. If N is a linearly compact $R-$module, then $C_i^I(M,N)$ is also linearly compact for all $i \geq 0$.*

Proof It follows from 2.3 (i)that $\{\mathrm{Tor}_i^R(I^t M, N)\}$ $(i \geq 0)$ forms an inverse system of linearly compact modules which continuous homomorphisms. Therefore $C_i^I(M,N)$ is also linearly compact by 2.1 (iii). □

Set $D_I^i(M,N) = \varinjlim_t \mathrm{Ext}_R^i(I^t M, N)$. Note that

$$D_I^0(M,N) = \varinjlim_t \mathrm{Hom}_R(I^t M, N) = D_I(M,N)$$

is the generalized $I-$transform of M, N ([15]).

Let $D(M) = \mathrm{Hom}_(M, E(R/m))$ the Matlis dual of M.

Proposition 3.3. *Let M be a finitely generated $R-$module and N an $R-$module. Then*

$$C_i^I(M, D(N)) \cong D(D_I^i(M,N))$$

for all $i \geq 0$.

Proof We first note that $\mathrm{Tor}_i^R(I^tM, D(N)) \cong D(\mathrm{Ext}_R^i(I^tM, N))$ by [17, 3.4.14] and $\varprojlim_t D(\mathrm{Ext}_R^i(I^tM, N)) \cong D(\varinjlim_t \mathrm{Ext}_R^i(I^tM, N))$ by [16, 2.27]. Therefore

$$\begin{aligned} C_i^I(M, D(N)) &\cong \varprojlim_t \mathrm{Tor}_i^R(I^tM, D(N)) \\ &\cong \varprojlim_t D(\mathrm{Ext}_R^i(I^tM, N)) \\ &\cong D(\varinjlim_t \mathrm{Ext}_R^i(I^tM, N)) = D(D_I^i(M, N)) \end{aligned}$$

as required. □

Theorem 3.4. *Let M be a finitely generated $R-$module.*

(i) If N is a linearly compact $R-$module, then we have a long exact sequence of linearly compact $R-$modules

$$\cdots \to H_{i+1}^I(M, N) \to C_i^I(M, N) \to \mathrm{Tor}_i^R(M, N) \to H_i^I(M, N) \to \cdots$$

$$\cdots \to H_1^I(M, N) \to C^I(M, N) \xrightarrow{\eta_{M,N}} M\bigotimes_R N \xrightarrow{\theta_{M,N}} \Lambda_I(M\bigotimes_R N) \to 0.$$

Moreover, if $pd(M) = p < +\infty$, then $H_{p+1}^I(M, N) \cong C_p^I(M, N)$;

(ii) If $0 \longrightarrow N" \xrightarrow{f} N \xrightarrow{g} N' \longrightarrow 0$ is a short exact sequence of linearly compact $R-$modules with continuous homomorphisms, then we have a long exact sequence of linearly compact $R-$modules

$$\cdots \longrightarrow C_i^I(M, N") \xrightarrow{f_i} C_i^I(M, N) \xrightarrow{g_i} C_i^I(M, N') \longrightarrow$$

$$\cdots \longrightarrow C^I(M, N") \xrightarrow{f_0} C^I(M, N) \xrightarrow{g_0} C^I(M, N') \longrightarrow 0$$

in which the homomorphisms f_i, g_i are continuous for all $i \geq 0$.

Proof (i). For any positive integer t the short exact sequence

$$0 \longrightarrow I^tM \longrightarrow M \longrightarrow M/I^tM \longrightarrow 0$$

gives rise a long exact sequence of linearly compact $R-$modules

$$\cdots \to \mathrm{Tor}_i^R(I^tM, N) \to \mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^R(M/I^tM, N) \to \cdots$$

$$\cdots \to I^tM\bigotimes_R N \longrightarrow M\bigotimes_R N \longrightarrow (M/I^tM)\bigotimes_R N \to 0.$$

Since M is the finitely generated $R-$module and N is the linearly compact $R-$module, the modules of the long exact sequence above are also linearly compact. Thus, passing inverse limits, we have the long exact sequence of linearly compact $R-$modules as required. Note that $\operatorname{Tor}_i^R(M,N)=0$ for all $i>p$, so $H^I_{p+1}(M,N)\cong C^I_p(M,N)$.

(ii). The short exact sequence of linearly compact modules $0\longrightarrow N" \xrightarrow{f} N \xrightarrow{g} N' \longrightarrow 0$ gives rise to a long exact sequence of linearly compact modules for all $t>0$

$$\cdots\longrightarrow \operatorname{Tor}_i^R(I^tM;N") \xrightarrow{f_{it}} \operatorname{Tor}_i^R(I^tM;N) \xrightarrow{g_{it}} \operatorname{Tor}_i^R(I^tM;N') \longrightarrow$$

$$\cdots\longrightarrow I^tM\otimes_R N" \xrightarrow{f_{0t}} I^tM\otimes_R N \xrightarrow{g_{0t}} I^tM\otimes_R N' \longrightarrow 0$$

in which homomorphisms f_{it}, g_{it} are continuous by 2.3, (ii). Then $\operatorname{Im} f_{it}, \operatorname{Ker} f_{it}$ are linearly compact. Passing inverse limits, we get the long exact sequence as required. Since the homomorphisms f_{it}, g_{it} are continuous, the homomorphisms $f_i,\ g_i$ induced on inverse limits are also continuous. □

Corollary 3.5. *Let M be a finitely generated $R-$module and N a linearly compact $R-$module. Then the homomorphism*

$$\eta_i(M,N): C^I_i(M,N)\longrightarrow \operatorname{Tor}_i^R(M,N)$$

is an isomorphism if $H^I_{i+1}(M,N)=H^I_i(M,N)=0$.

Proof It is immediately induced from 3.4 (i). □

Let $\Lambda_I(M)=\varprojlim_t M/I^tM$ the $I-$adic completion of M, we have the following lemma.

Lemma 3.6. *Let M be a finitely generated $R-$module and N a linearly compact $R-$module. If N is complete in $I-$adic topology (i.e., $\Lambda_I(N)\cong N$), then $C^I_i(M,N)=0$ for all $i\geq 0$.*

Proof From 3.4 (i) we have a long exact sequence

$$\cdots\to H^I_{i+1}(M,N)\to C^I_i(M,N)\to \operatorname{Tor}_i^R(M,N)\to H^I_i(M,N)\to\cdots$$

$$\cdots\to H^I_1(M,N)\to C^I(M,N)\longrightarrow M\bigotimes_R N\longrightarrow \Lambda_I(M\bigotimes_R N)\to 0.$$

As N is complete in $I-$adic topology, we have $H^I_i(M,N)\cong \operatorname{Tor}_i^R(M,N)$ for all $i\geq 0$ by 2.4. Thus $C^I_i(M,N)=0$ for all $i\geq 0$. □

Since $H^I_i(N)$ is complete in $I-$adic topology for all $i\geq 0$, we have the immediate consequence.

Corollary 3.7. *Let M be a finitely generated $R-$module and N a linearly compact $R-$module. Then $C^I_i(M,H^I_j(N))=0$ for all $i,j\geq 0$.*

Note that $C_i^I(R,N) = C_i^I(N)$, so we also have the following immediate consequence.

Corollary 3.8. *Let N be a linearly compact $R-$module. If N is complete in $I-$adic topology, then $C_i^I(N) = 0$ for all $i \geq 0$.*

Lemma 3.9. *Let M, N be $R-$modules. Then*

$$\Lambda_I(C^I(M,N)) \cong C^I(M, \Lambda_I(N)).$$

Proof We have

$$\begin{aligned}
\Lambda_I(C^I(M,N)) &\cong \varprojlim_t (R/I^t \otimes_R C^I(M,N)) \\
&= \varprojlim_t (R/I^t \otimes_R \varprojlim_s (I^s M \otimes_R N)) \\
&\cong \varprojlim_t \varprojlim_s (R/I^t \otimes_R (I^s M \otimes_R N) \\
&\cong \varprojlim_s \varprojlim_t (I^s M \otimes_R (R/I^t \otimes_R N)) \\
&\cong \varprojlim_s (I^s M \otimes_R \varprojlim_t (R/I^t \otimes_R N)) \\
&\cong C^I(M, \Lambda_I(N))
\end{aligned}$$

as required. □

Corollary 3.10. *Let M be a finitely generated $R-$module and N a linearly compact $R-$module. Then $\Lambda_I(C^I(M,N)) = 0$.*

Proof It follows from 3.7 and 3.9. □

Lemma 3.11. *Let M be a finitely generated $R-$module and N a linearly compact $R-$module. Then $C_i^I(M, C^I(N)) \cong C_i^I(M,N)$.*

Proof From 2.5 we have two short exact sequences

$$0 \to H_1^I(N) \to C^I(N) \to \bigcap_{t>0} I^t N \to 0,$$

$$0 \to \bigcap_{t>0} I^t N \to N \to \Lambda_I(N) \to 0.$$

By 3.4 (ii) we get two exact sequences

$$\ldots C_i^I(M, C^I(N)) \to C_i^I(M, \bigcap_{t>0} I^t N) \to C_{i-1}^I(M, H_1^I(N)) \ldots,$$

$$\ldots C_i^I(M, \bigcap_{t>0} I^t N) \to C_i^I(M,N) \to C_i^I(M, \Lambda_I(N)) \ldots.$$

Then the conclusion follows from 3.7. $\square$

Lemma 3.12. *Let M be a finitely generated $R-$module and N a linearly compact $R-$module. Then $C^I(C^I(M,N)) \cong C^I(M,N)$.*

Proof We have

$$\begin{aligned} C^I(C^I(M,N)) &\cong \varprojlim_t (I^t \otimes_R \varprojlim_s (I^s M \otimes_R N)) \\ &\cong \varprojlim_t \varprojlim_s (I^t \otimes_R (I^s M \otimes_R N)) \\ &\cong \varprojlim_s \varprojlim_t (I^s M \otimes_R (I^t \otimes_R N)) \\ &\cong \varprojlim_s (I^s M \otimes_R \varprojlim_t (I^t \otimes_R N)) \\ &\cong C^I(M, C^I(N)). \end{aligned}$$

Now the conclusion follows from 3.12. $\square$

Proposition 3.13. *Let $f : N' \longrightarrow N$ be a homomorphism of linearly compact $R-$modules such that* $\operatorname{Ker} f$ *and* $\operatorname{coker} f$ *are complete in $I-$adic topology. Let $\varphi : K \longrightarrow N$ be a further homomorphism of linearly compact $R-$modules. Then*
(i) The homomorphism $C_i^I(M,f) : C_i^I(M,N') \longrightarrow C_i^I(M,N)$ is an isomorphism for all $i \geq 0$.
(ii) There is a homomorphism $\psi : C^I(M,K) \longrightarrow M \otimes_R N'$ such that the diagram

$$\begin{array}{ccc} M\otimes_R N' & \xrightarrow{M\otimes_R f} & M\otimes_R N \\ \uparrow \psi & & \uparrow M\otimes_R \varphi \\ C^I(M,K) & \xrightarrow{\eta_{M,K}} & M\otimes_R K, \end{array}$$

is commutative, i. e., $M\otimes_R f \circ \psi = M\otimes_R \varphi \circ \eta_{M,K}$.

Proof (i) We have short exact sequences of linearly compact modules

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow N' \xrightarrow{\alpha} \operatorname{Im} f \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} f \xrightarrow{\beta} N \longrightarrow \operatorname{coker} f \longrightarrow 0$$

in which $f = \beta\alpha$ and homomorphisms are continuous. It is therefore enough to show that $C_i^I(M,\alpha)$ and $C_i^I(M,\beta)$ are both isomorphisms.

The first short exact sequence above induces by 3.4 (ii) an exact sequence

$$\dots C_i^I(M, \operatorname{Ker} f) \to C_i^I(M, N') \xrightarrow{C_i^I(M,\alpha)} C_i^I(M, \operatorname{Im} f) \dots .$$

From 3.7 and the hypothesis $\operatorname{Ker} f \cong \Lambda_I(\operatorname{Ker} f)$ we have $C_i^I(M, \operatorname{Ker} f) = 0$. Hence $C_i^I(M, \alpha)$ is an isomorphism. Next, from the second short exact sequence we get an induced exact sequence

$$\dots C_i^I(M, \operatorname{Im} f) \xrightarrow{C_i^I(M,\beta)} C_i^I(M, N) \to C_i^I(M, \operatorname{coker} f) \dots .$$

We have $C_i^I(M, \operatorname{coker} f) \cong C_i^I(M, \Lambda_I(\operatorname{coker} f)) = 0$ for all $i \geq 0$ by 3.6. Therefore $C_i^I(M, \beta)$ is an isomorphism.
(ii) We have a commutative diagram

$$\begin{array}{ccccc}
M \otimes_R N' & \xrightarrow{M \otimes_R f} & M \otimes_R N & \xleftarrow{M \otimes_R \varphi} & M \otimes_R K \\
\uparrow \eta_{M,N'} & & \uparrow \eta_{M,N} & & \uparrow \eta_{M,K} \\
C^I(M, N') & \xrightarrow{C^I(M,f)} & C^I(M, N) & \xleftarrow{C^I(M,\varphi)} & C^I(M, K).
\end{array}$$

By (i), $C^I(M, f)$ is an isomorphism.
Set $\psi = \eta_{M,N'} \circ C^I(M, f)^{-1} \circ C^I(M, \varphi)$, we have

$$\begin{aligned}
M \otimes_R f \circ \psi &= M \otimes_R f \circ \eta_{M,N'} \circ C^I(M, f)^{-1} \circ C^I(M, \varphi) \\
&= \eta_{M,N} \circ C^I(M, \varphi) = M \otimes_R \varphi \circ \eta_{M,K}.
\end{aligned}$$

The proof is complete. □

4 CO-LOCALIZATION

Let S be multiplicative set of R. For an $R-$mdule M the module $\operatorname{Hom}_R(R_S; M)$ is called co-localization of M with respect to S (see [11]). We denote it briefly by ${}_SM$. If M is a linearly compact $R-$module, then ${}_SM$ is also a linearly compact $R-$module by [5, 2.4]. The following proposition says that the co-localization can "commute" to the generalized ideal co-transform of a linearly compact $R-$module.

Proposition 4.1. *Let M be a finitely generated $R-$module and N a linearly compact $R-$module. Then*

$${}_SC_i^I(M, N) \cong C_i^{IR_S}(M_S, {}_SN)$$

for all $i \geq 0$.

Proof We first note that

$${}_S(\mathrm{Tor}_i^R(I^tM,N)) \cong \mathrm{Tor}_i^{R_S}(I^tM_S;{}_SN)$$

by [4, 3.9]. Then

$$\begin{aligned}
{}_SC_i^I(M,N) &= \mathrm{Hom}_R(R_S,\varprojlim_t \mathrm{Tor}_i^R(I^tM,N)) \\
&\cong \varprojlim_t \mathrm{Hom}_R(R_S,\mathrm{Tor}_i^R(I^tM,N)) = \varprojlim_t {}_S(\mathrm{Tor}_i^R(I^tM,N)) \\
&\cong C_i^{IR_S}(M_S,{}_SN)
\end{aligned}$$

as required. □

Let a be an element in R, the notation ${}_aM$ means that the co-localization of M with respect to the multiplicative set $S=\{1,a,a^2,\ldots\}$.

Theorem 4.2. *Let M be a finitely generated R-module and N a linearly compact R-module. Then*

(i) $C^I(M\otimes_R N)\cong C^I(M,N)$;

(ii) $C^{aR}(M,N)\cong {}_aC^{aR}(M,N)$ *for any element $a\in R$.*

Proof (i). We have an exact sequence by 3.4 (i)

$$H_1^I(M,N)\xrightarrow{\varphi} C^I(M,N)\xrightarrow{\eta_{M,N}} M\bigotimes_R N\xrightarrow{\theta_{M,N}}\Lambda_I(M\bigotimes_R N)\to 0.$$

It induces two short exact sequences

$$0\to \mathrm{Im}\,\varphi\to C^I(M,N)\longrightarrow \mathrm{Im}\,\eta_{M,N}\to 0,$$

$$0\to \mathrm{Im}\,\eta_{M,N}\to M\bigotimes_R N\xrightarrow{\theta_{M,N}}\Lambda_I(M\bigotimes_R N)\to 0.$$

Combining 3.4 (ii) with 3.8 yields induced exact sequences

$$0\to C^I(\mathrm{Im}\,\varphi)\to C^I(C^I(M,N))\longrightarrow C^I(\mathrm{Im}\,\eta_{M,N})\to 0,$$

$$0\to C^I(\mathrm{Im}\,\eta_{M,N})\to C^I(M\bigotimes_R N)\xrightarrow{\theta_{M,N}} C^I(\Lambda_I(M\bigotimes_R N))\to 0.$$

Note that $\mathrm{Im}\,\varphi$ and $\Lambda_I(M\bigotimes_R N)$ are both complete in $I-$adic topology, so

$$C^I(M\otimes_R N)\cong C^I(C^I(M,N))\cong C^I(M,N)$$

by 3.8 and 3.12.

(ii). It follows from [18, 10.8.3] that $a^tM\otimes_R\mathrm{Hom}(S^{-1}R,N)\cong\mathrm{Hom}_R(S^{-1}R,a^tM\otimes_R N)$. On the other hand, $C^{aR}(M,N)\cong C^{aR}(M,C^{aR}(N))$ by 3.12 and $C^{aR}(N)\cong {}_aN$ by [14, 4.4]. Therefore

$$\begin{aligned}
C^{aR}(M,N) &\cong C^{aR}(M,C^{aR}(N))\\
&\cong C^{aR}(M,{}_aN)\\
&\cong \varprojlim_t a^tM\otimes_R \mathrm{Hom}(S^{-1}R,N)\\
&\cong \varprojlim_t \mathop{\mathrm{Hom}}_R(S^{-1}R,a^tM\otimes_R N)\\
&\cong \mathop{\mathrm{Hom}}_R(S^{-1}R,\varprojlim_t a^tM\otimes_R N)\\
&= {}_aC^{aR}(M,N)
\end{aligned}$$

as required. □

Theorem 4.3. *Let M be a finitely generated $R-$module and N an artinian. There is an isomorphism*

$$C^{aR}(M,N)\cong {}_a(M\bigotimes_R N)$$

for any element $a\in R$.

Proof It follows from [6, 2.7] that $D_{aR}(M,N)\cong\mathrm{Hom}_R(M,N)_a$. Moreover, $C^{aR}(M,D(N))\cong D(D_{aR}(M,N))$ by 3.3. Hence

$$\begin{aligned}
C^{aR}(M,N) &\cong C^{aR}(M,DD(N))\\
&\cong D(D_{aR}(M,D(N))\\
&\cong D(\mathrm{Hom}_R(M,D(N))_a)\\
&\cong {}_aD(\mathrm{Hom}_R(M,D(N))\\
&\cong {}_a(M\otimes_R DD(N))\\
&\cong {}_a(M\otimes_R N))
\end{aligned}$$

as required. □

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