

# CO-COHEN-MACAULAY MODULES IN DIMENSION $> s$

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## Abstract

Let  $(R, m)$  be a Noetherian local ring and  $A$  an Artinian  $R$ -module. For an integer  $s > -1$ , we say that  $A$  is *co-Cohen-Macaulay in dimension  $> s$*  if every system of parameters of  $A$  is an  $A$ -cosequence in dimension  $> s$  introduced by Nhan-Hoang [NH]. In this paper, we give some characterizations for co-Cohen-Macaulay modules in dimension  $> s$  in terms of the dimension of the local homology modules  $H_i^m(A)$ , the polynomial type  $\text{ld}(A)$  of  $A$  and the multiplicity  $e(\underline{x}; A)$  of  $A$  with respect to a system of parameter  $\underline{x}$ .

## 1 Introduction

Throughout this paper, let  $(R, m)$  be a Noetherian local ring and  $A$  an Artinian  $R$ -module of Noetherian dimension  $d$ . Using the concept of an  $A$ -cosequence defined by Ooishi [O], Tang and Zakeri [TZ] introduced the class of modules satisfying the condition that every system of parameters (s.o.p. for short) of  $A$  is an  $A$ -cosequence called *co-Cohen-Macaulay module*. This class of modules plays an important role in the theory of Artinian modules and their structure are well-known in terms of multiplicity, local homology and Noetherian dimension (see [CNh1], [CN], [O]). There are some extensions of the concepts of  $A$ -cosequences and co-Cohen-Macaulay modules, among which are the notions of  $A$ -cosequences in dimension  $> s$  introduced by Nhan-Hoang [NH] and co-filter modules defined by Dung [D1] which are in some senses dual

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to the notions of  $M$ -sequences in dimension  $> s$  in sense of Brodmann-Nhan [BN] and  $f$ -modules defined by Cuong-Schenzel-Trung [CST].

**Definition** Let  $s \geq -1$  be an integer. A sequence  $(x_1, \dots, x_k)$  of elements in  $m$  is called an  $A$ -cosequence in dimension  $> s$  if  $x_i \notin p$  for all attached primes  $p \in \text{Att}_R(0 :_A (x_1, \dots, x_{i-1})R)$  satisfying  $\dim(R/p) > s$  for all  $i = 1, \dots, k$ .

Note that an  $A$ -cosequence in dimension  $> -1, 0$  are exactly an  $A$ -cosequence in sense of A. Ooishi [O] and  $f$ -coregular sequence in sense of [D1], respectively.

The purpose of this paper is to introduce the class of co-Cohen-Macaulay modules in dimension  $> s$  and give some their characterizations in terms of the dimension of the local homology modules  $H_i^m(A)$  introduced by Cuong-Nam [CN], the polynomial type  $\text{ld}(A)$  of  $A$  given by Minh [MIN] and the multiplicity  $e(\underline{x}; A)$  of  $A$  with respect to a s.o.p.  $\underline{x}$  defined by [CNh1]. It is clear that if  $s \geq d$  then  $A$  is always co-Cohen-Macaulay in dimension  $> s$  and if  $s > -1$  then a co-Cohen-Macaulay module in dimension  $> -1$  is exactly a co-Cohen-Macaulay module. Therefore we only consider the case where  $0 \leq s < d$ .

The main result of this paper is the following theorem.

**Main Theorem.** Suppose that  $0 \leq s < d$ .

(i) The following statements are equivalent:

(a)  $\dim_{\widehat{R}}(H_i^m(A)) \leq s$ , for all  $i < d$ .

(b)  $\text{ld}(A) \leq s$ .

(c) There exist a s.o.p.  $\underline{x} = (x_1, \dots, x_d)$  of  $A$  and  $k_1, \dots, k_s \in \{1, \dots, d\}$  such that

$$I(y_1, \dots, y_d; A) = I(x_1, \dots, x_d; A),$$

where  $y_j = x_j^2$  if  $j \notin \{k_1, \dots, k_s\}$  and  $y_j = x_j$  if  $j \in \{k_1, \dots, k_s\}$ .

(d) There exist a s.o.p.  $\underline{x} = (x_1, \dots, x_d)$  of  $A$  and a constant  $C_{\underline{x}}$  (not depending on  $n$ ) such that for all integer  $n > 0$ ,

$$I(x_1^n, \dots, x_d^n; A) \leq n^s C_{\underline{x}}.$$

(ii) If  $A$  is co-Cohen-Macaulay in dimension  $> s$  then one of the conditions (a), (b), (c), (d) is satisfied.

(iii) If one of the conditions (a), (b), (c), (d) is satisfied then  $A$  is co-Cohen-Macaulay in dimension  $> s$  as  $\widehat{R}$ -module.

The proof of Main Theorem will be given in Section 3. In the next section, we recall some preliminaries which will be used later.

## 2 Preliminaries

We first recall the *Noetherian dimension*  $\text{N-dim}_R A$  of an Artinian  $R$ -module  $A$  defined by Kirby [K2] and Roberts [R]: if  $A = 0$ , we put  $\text{N-dim } A = -1$ . For

an integer  $d \geq 0$ , we put  $\text{N-dim}_R A = d$  if  $\text{N-dim}_R A < d$  is false, and for every ascending sequence  $A_0 \subseteq A_1 \subseteq \dots$  of submodules of  $A$ , there exists  $n_0$  such that  $\text{N-dim}_R(A_{n+1}/A_n) < d$  for all  $n > n_0$ .

**Lemma 2.1.** [CNh2] (i) *Let  $A$  be Artinian  $R$ -module. Then  $A$  has a natural structure  $\widehat{R}$ -module and*

$$\text{N-dim}_R A = \text{N-dim}_{\widehat{R}} A = \dim_{\widehat{R}}(\widehat{R}/\text{Ann}_{\widehat{R}} A) \leq \dim(R/\text{Ann}_R A).$$

(ii)  *$\text{N-dim} A = 0$  if and only if  $\dim_R A = 0$ . In this case, the length of  $A$  is finite and the ring  $R/\text{Ann}_R A$  is Artinian.*

(iii) *Let  $I$  be an ideal of  $R$  and  $M$  a non zero f.g.  $R$ -module. Then  $\text{N-dim}(H_m^i(M)) \leq i$  and in particular,  $\text{N-dim}(H_m^d(M)) = d$ .*

The theory of secondary representation introduced by I. G. Macdonald [Mac] is in some sense dual to the more known theory of primary decomposition. It has shown in [Mac] that every Artinian  $R$ -module  $A$  has a secondary representation  $A = A_1 + \dots + A_n$  of  $p_i$ -secondary submodules  $A_i$ . The set  $\{p_1, \dots, p_n\}$  is independent of the minimal secondary representation of  $A$  and it is denoted by  $\text{Att}_R A$ .

**Lemma 2.2.** (i)  *$A \neq 0$  if and only if  $\text{Att}_R A \neq \emptyset$ . In this case, the set of all minimal elements of  $\text{Att}_R A$  is exactly the set of all minimal prime ideals of  $\text{Var}(\text{Ann}_R A)$ .*

(ii)  $\text{N-dim} A \leq \dim(R/\text{Ann}_R A) = \max\{\dim R/p : p \in \text{Att}_R A\}$ .

From the definition of  $A$ -cosequence in dimension  $> s$ , if denote by  $\dim_R A$  the Krull dimension of the ring  $R/\text{Ann}_R A$  then we have the following result (see [ND]).

**Lemma 2.3.** *Let  $I$  be an ideal of  $R$ .*

(i) *If  $\dim_R(0 :_A I) \leq s$  then there exists an  $A$ -cosequence in dimension  $> s$  in  $I$  of length  $n$  for any integer  $n > 0$ .*

(ii) *If  $\dim_R(0 :_A I) > s$  then each  $A$ -cosequence in dimension  $> s$  in  $I$  can be extended to a maximal one and all maximal  $A$ -cosequences in dimension  $> s$  in  $I$  have the same length, this common length is equal to the least integer  $i$  such that  $\dim_R(\text{Tor}_i^R(R/I, A)) > s$ .*

The common length in Lemma 2.3 is called *the width in dimension  $> s$  in  $I$*  with respect to  $A$  and denoted by  $\text{Width}_{>s}(I, A)$ . In case  $\dim_R(0 :_A I) \leq s$  we set  $\text{Width}_{>s}(I, A) = \infty$ . Note that  $\text{Width}_{>-1}(I, A) = \text{Width}(I, A)$ , the width of  $A$  in  $I$  defined by A. Ooishi [O] (cf. [ND]).

The class of co-Cohen-Macaulay modules (co-CM for short) for Artinian modules is introduced by Tang and Zakeri [TZ] on the Noetherian local ring

which is in some senses dual to the class of Cohen-Macaulay modules for Noetherian modules. Recall that an Artinian  $R$ -module  $A$  is called *co-Cohen-Macaulay* if  $\text{Width}(A) = \text{N-dim } A$ . Now by the definition of  $A$ -coregular sequence in dimension  $> s$ , we introduce the new class of modules as follow.

**Definition 2.4.** An Artinian  $R$ -module  $A$  is called *co-Cohen-Macaulay in dimension  $> s$*  if every s.o.p. of  $A$  is an  $A$ -coregular sequence in dimension  $> s$ .

Note that co-CM modules in dimension  $> -1, 0$  are exactly co-CM module introduced by Tang and Zakeri [TZ] and co-filter modules defined by [D1], respectively.

The multiplicity theory for Artinian modules is introduced by Cuong-Nhan [CNh1]. Let  $\underline{x} = (x_1, \dots, x_t) \subseteq m$  be a multiplicative system of  $A$ , i.e. it satisfies the condition  $\ell(0 :_A \underline{x}R) < \infty$ . A multiplicity system  $\underline{x}$  is called a s.o.p. of  $A$  if  $t = d = \text{N-dim } A$ . Denote by  $e(\underline{x}; A)$  the multiplicity of  $A$  w.r.p. to  $\underline{x}$ , it is proved that the number  $e(\underline{x}; A)/d!$  is exactly the first coefficient of the Hilbert polynomial with respect to the s.o.p.  $\underline{x}$  introduced by Kirby [K1]. The following result, see [CNh1], is used in the sequel.

**Lemma 2.5.** *Suppose that  $\underline{x} = (x_1, \dots, x_t)$  is a multiplicity system for  $A$ .*

(i) *Let  $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$  be an exact sequence of Artinian  $R$ -modules. Then*

$$e(\underline{x}; A) = e(\underline{x}; A') + e(\underline{x}; A'').$$

(ii)  *$0 \leq e(\underline{x}; A) \leq \ell(0 :_A \underline{x}R)$  and  $e(\underline{x}; A) > 0$  if and only if  $t = d = \text{N-dim } A$ .*

(iii) *Let  $n_1, \dots, n_t$  be positive integers and put  $\underline{x}(n) = (x_1^{n_1}, \dots, x_t^{n_t})$ . Then*

$$e(\underline{x}(n); A) = n_1 \dots n_t e(\underline{x}; A).$$

(iv) *Let  $(x_1, \dots, x_d)$  be a s.o.p. of  $A$ . For each  $i = 1, \dots, d$ , we set  $C_i = 0 :_A (x_1, \dots, x_{i-1})R$ . Then*

$$\ell(0 :_A (x_1, \dots, x_d)R) - e((x_1, \dots, x_d); A) = \sum_{i=1}^d e(x_{i+1}, \dots, x_d; C_i/x_i C_i).$$

The notion of local homology modules was defined by Cuong-Nam [CN]: Let  $I$  be an ideal of  $R$  and  $M$  an arbitrary  $R$ -module. The  $i$ -th local homology module  $H_i^I(M)$  of  $M$  with respect to  $I$  is defined by  $H_i^I(M) = \varprojlim_t \text{Tor}_i^R(R/I^t; M)$ .

It has been presented in [CN] many basic properties of local homology modules for Artinian modules, which show that this theory of local homology modules is in some sense dual to the well-known theory of local cohomology of  $A$ . Grothendieck for Noetherian modules.

**Lemma 2.6.** (i) Let  $f : R \longrightarrow R'$  be a homomorphism of Noetherian rings and  $I$  an ideal of  $R$ . Then there exists an isomorphism  $H_i^I(A) \cong H_i^{IR'}(A)$  of  $\Lambda_I(R)$ -modules for all  $i \geq 0$ , where  $\Lambda_I(-)$  is the  $I$ -adic completion functor.

(ii)  $H_i^I(A) = 0$ , for all  $i > \text{N-dim } A$ .

Recall that if  $\ell_R(H_i^m(A)) < \infty$  for all  $i < d$  then  $A$  is called *generalized co-Cohen-Macaulay* (g.CCM for short), (see [CDN]), where  $H_i^m(A)$  are local homology modules. Now we recall some characterizations of g.CCM modules which are used in the sequel. From now on, for a s.o.p.  $\underline{x} = (x_1, \dots, x_d)$  of  $A$ , we set

$$I(\underline{x}; A) = \ell_R(0 :_A \underline{x}R) - e(\underline{x}; A).$$

**Lemma 2.7.** The following statements are equivalent:

(i)  $A$  is g. CCM.

(ii) There exists a constant  $I(A)$  such that  $I(\underline{x}; A) \leq I(A)$  for all s.o.p.  $\underline{x}$  of  $A$ .

(iii) There exists s.o.p.  $\underline{x}$  of  $A$  such that  $I(x_1^2, \dots, x_d^2; A) = I(\underline{x}; A)$ .

(iv) There exists an integer  $s > 0$  and a s.o.p.  $\underline{x}$  such that  $I(x_1^n, \dots, x_d^n; A) \leq s$  for all  $n \geq 1$ .

When  $A$  satisfies one of the above equivalent conditions, we have

$$I(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^m(A)).$$

A s.o.p.  $\underline{x}$  satisfies Lemma 2.7, (iii) is called a *co-standard* s.o.p. of  $A$ . Note that if a s.o.p.  $\underline{x}$  of  $A$  is co-standard then  $I(x_1^{n_1}, \dots, x_d^{n_d}; A) = I(\underline{x}; A)$  for all  $n_1, \dots, n_d \geq 1$  (see [CDN, Lemma 4.3]).

Let  $\underline{n} = (n_1, \dots, n_d)$  be  $d$ -tuple of  $d$  non negative integers and consider

$$I(\underline{x}(\underline{n}); A) := \ell_R(0 :_A (x_1^{n_1}, \dots, x_d^{n_d})R) - n_1 \dots n_d e(\underline{x}; A)$$

as a function on  $n_1, \dots, n_d$ . It is shown in [MIN] that this function is not a polynomial on  $n_1, \dots, n_d$  (even when  $n_1, \dots, n_d$  large enough). However, it always takes non-negative values and bounded above by polynomials. The least degree of all polynomials in  $n_1, \dots, n_d$  bounding the above function  $I(\underline{x}(\underline{n}); A)$  is independent of the choice of  $\underline{x}$  and denoted by  $\text{ld}(A)$ . If we stipulate that the degree of polynomial zero is  $-\infty$  then  $A$  is co-Cohen-Macaulay if and only if  $\text{ld}(A) = -\infty$  (see [MIN, Theorem 4.11]).

**Lemma 2.8.** (i)  $A$  is g. CCM if and only if  $\text{ld}(A) \leq 0$ .

(ii) Let  $\text{ld}(A) > 0$ . Then we have

$$(a) \text{ld}(A) = \max_{i < d} \dim_{\widehat{R}}(H_i^m(A)).$$

(b) If  $x \in m$  such that  $x \notin p$  for all  $p \in \bigcup_{i=1}^d \text{Ass}_{\widehat{R}}(H_i^m(A)) \setminus \{m\}$  then

$$\text{ld}(0 :_A x) = \text{ld}(A) - 1.$$

*Proof.* (i) By [CDN, Theorem 4.4] and [MIN, Corollary 4.9], we need to prove the sufficient condition. Since  $\text{ld}(A) \leq 0$ , we have two cases. If  $\text{ld}(A) = -\infty$ , then  $A$  is co-Cohen-Macaulay (see [MIN, Theorem 4.11]). Therefore we only consider to the case  $\text{ld}(A) = 0$ . We prove by induction on  $d = \text{N-dim } A$ . Let  $d = 1$ . Then  $A$  is g.CCM by [CDN, Theorem 4.4] since the length of the local homology  $\ell(H_0^m(A))$  is always finite. Now assume that  $d > 1$  and the assertion is true for all Artinian  $R$ -module of Noetherian dimension smaller than  $d$ . Let  $\underline{x} = (x_1, \dots, x_d)$  be a s.o.p. of  $A$ . The assumption  $\text{ld}(A) = 0$  states that

$$\ell_R(0 :_A \underline{x}(\underline{n})R) - e(\underline{x}(\underline{n})R; A) \leq n_1 \dots n_d (\ell_R(0 :_A \underline{x}R) - e(\underline{x}R; A)) = C < \infty,$$

where  $C$  is constant. So  $\ell_R(0 :_A \underline{x}R) - e(\underline{x}R; A) = C$  and hence  $x_1 A \supseteq m^n A$  for some  $n \in \mathbb{N}$ , i.e.  $x_1$  is a weak co-sequence by [CDN]. Therefore  $\ell(A/x_1 A) < \infty$  and hence  $H_i^m(A/x_1 A) = 0$ , for all  $i > 0$ . Thus, form exact sequences

$$0 \longrightarrow x_1 A \longrightarrow A \longrightarrow A/x_1 A \longrightarrow 0;$$

$$0 \longrightarrow 0 :_A x_1 \longrightarrow A \xrightarrow{x_1} x_1 A \longrightarrow 0$$

we get the long exact sequences for  $i = 1, \dots, d-1$ ,

$$\dots \longrightarrow H_i^m(0_A : x_1) \longrightarrow H_i^m(A) \xrightarrow{x_1} H_i^m(A) \longrightarrow H_{i-1}^m(0_A : x_1) \longrightarrow \dots$$

So, by using the induction hypothesis with respect to the s.o.p.  $(x_2, \dots, x_r)$  of  $(0 :_A x_1)$ , we have by [CDN, Theorem 4.4] that  $\ell(H_i^m(0_A : x_1)) < \infty$  for all  $i \leq d-2$ . Therefore, we have  $\ell(H_i^m(A)) < \infty$  for all  $i \leq d-1$  and  $A$  is g.CCM by [CDN, Theorem 4.4].

(ii) Note that  $\text{ld}(A) = p(D(A))$ , where  $D(A)$  is a Noetherian  $\widehat{R}$ -module and  $p(D(A))$  is a polynomial type of  $D(A)$  defined by [C]. Hence from isomorphisms  $H_m^i(D(A)) \cong D(H_i^m(A))$  and  $0 :_A x \cong D(A)/xD(A)$  of  $\widehat{R}$ -modules, using Matlis duality, we get the result by [CMN, Lemma 3.1].  $\square$

### 3 Proof of Main Theorem

(i). (a)  $\Leftrightarrow$  (b) follows by Lemma 2.8,(ii).

(a)  $\Rightarrow$  (c). Let  $d = 1$ . Then  $s = 0$  and  $A$  is g.CCM. By Lemma 2.7(iii), there exists a standard s.o.p.  $x_1$  of  $A$ , i.e.  $I(x_1^2; A) = I(x_1; A)$ . Therefore (c) is true.

Let  $d > 1$ . We prove the result by induction on  $s$ , where  $0 \leq s < d$ . Let  $s = 0$ . Then  $\dim_{\widehat{R}} H_i^m(A) \leq 0$  for all  $i < d$ . By Lemma 2.1(ii),  $\ell_{\widehat{R}}(H_i^m(A)) < \infty$  for all  $i < d$ , i.e.  $A$  is g.CCM by [CDN, Theorem 4.4]. Hence there exists by Lemma 2.7(iii) a s.o.p.  $(x_1, \dots, x_d)$  of  $A$  such that  $I(x_1^2, \dots, x_d^2; A) = I(x_1, \dots, x_d; A)$ . Therefore the condition (c) is true for  $s = 0$ . Let  $1 \leq s < d$  and assume that the result is true for the case  $s - 1$ . If  $\text{ld}(A) \leq 0$  then  $A$  is g.CCM by Lemma 2.8(i). Therefore there exists a standard s.o.p.  $\underline{x} = (x_1, \dots, x_d)$  of  $A$ . Thus by [CDN, Lemma 4.3] we have

$$I(\underline{x}; A) \leq I(y_1, \dots, y_d; A) \leq I(x_1^2, \dots, x_d^2; A) = I(\underline{x}; A),$$

where  $y_j = x_j^2$  if  $j \notin \{k_1, \dots, k_s\}$  and  $y_j = x_j$  if  $j \in \{k_1, \dots, k_s\}$ , for all  $j = 1, \dots, d$ . Hence  $I(\underline{x}; A) = I(y_1, \dots, y_d; A)$ , the result is true in this case.

Let  $\text{ld}(A) > 0$ . Let  $x_1 \in m$  such that  $x_1 \notin p$  for all  $p \in \bigcup_{i=1}^d \text{Ass}_{\widehat{R}}(H_i^m(A)) \setminus \{m\}$ . Note that  $\text{ld}(A) \leq s$  by Lemma 2.8(ii). Therefore we get by Lemma 2.8(ii) that  $\text{ld}(0 :_A x_1) = \text{ld}(A) - 1 \leq s - 1$ . Hence  $\dim_{\widehat{R}} H_i^m(A) \leq s - 1$  for all  $i < d - 1$  by Lemma 2.8(ii). Applying the induction for  $(0 :_A x_1)$ , there exists a s.o.p.  $(x_2, \dots, x_d)$  of  $A$  and integers  $k_2, \dots, k_s \in \{2, \dots, d\}$  such that

$$I(y_2, \dots, y_d; 0 :_A x_1) = I(x_2, \dots, x_d; 0 :_A x_1),$$

where  $y_j = x_j^2$  if  $j \notin \{k_2, \dots, k_s\}$  and  $y_j = x_j$  if  $j \in \{k_2, \dots, k_s\}$ , for all  $j = 2, \dots, d$ . Without loss any generality we can assume that  $k_2 = 2, \dots, k_s = s$ , i.e.

$$I(x_2, \dots, x_s, x_{s+1}^2, \dots, x_d^2; 0 :_A x_1) = I(x_2, \dots, x_d; 0 :_A x_1). \quad (1)$$

By the choice of  $x_1$ , we have  $\text{N-dim}(A/x_1A) \leq 0$ . Since  $d > 1$ , we have

$$e(x_2, \dots, x_s, x_{s+1}^2, \dots, x_d^2; A/x_1A) = 0 = e(x_2, \dots, x_s, x_{s+1}, \dots, x_d; A/x_1A).$$

Therefore, we have

$$\begin{aligned} I(x_2, \dots, x_s, x_{s+1}^2, \dots, x_d^2; 0 :_A x_1) &= \ell_R(0 :_A (x_1, \dots, x_s, x_{s+1}^2, \dots, x_d^2)) \\ &\quad - e(x_1, \dots, x_s, x_{s+1}^2, \dots, x_d^2; A) + e(x_2, \dots, x_s, x_{s+1}^2, \dots, x_d^2; A/x_1A) \\ &= I(x_1, \dots, x_s, x_{s+1}^2, \dots, x_d^2; A), \end{aligned}$$

and

$$\begin{aligned} I(x_2, \dots, x_d; 0 :_A x_1) &= \ell_R(0 :_A (x_1, x_2, \dots, x_d)) - e(x_1, x_2, \dots, x_d; A) \\ &\quad + e(x_2, \dots, x_d; A/x_1A) = I(x_1, \dots, x_d; A). \end{aligned}$$

So, it follows by (1) that

$$I(x_1, \dots, x_s, x_{s+1}^2, \dots, x_d^2; A) = I(x_1, \dots, x_d; A),$$

and (c) is proved.

(c)  $\Rightarrow$  (d). Let  $d = 1$ . Then  $s = 0$  and  $A$  is g.CCM. So, there exists a standard s.o.p.  $x_1$  of  $A$  and we have  $I(x_1; A) = I(x_1^2; A) = I(x_1^n; A)$  for all  $n \in \mathbb{N}$  by [CDN]. Set  $C_{\underline{x}} = I(x_1; A)$ . Then  $I(x_1^n; A) = C_{\underline{x}} = n^0 C_{\underline{x}}$  for all  $n \geq 1$ . Hence (d) is true.

Let  $d > 1$ . We prove the result by induction on  $s$ , where  $0 \leq s < d$ . Let  $s = 0$ . From the hypothesis (c), there exists a s.o.p.  $\underline{x} = (x_1, \dots, x_d)$  of  $A$  such that

$$I(x_1^2, \dots, x_d^2; A) = I(x_1, \dots, x_d; A).$$

It implies  $A$  is g.CCM and  $\underline{x}$  is a standard s.o.p. of  $A$  by [CDN]. Set  $C_{\underline{x}} = I(x_1, \dots, x_d; A)$ . Then

$$I(x_1^n, \dots, x_d^n; A) = n^0 C_{\underline{x}}$$

for all  $n \geq 1$  and (d) is true for the case  $s = 0$ . Let  $s > 0$  and assume that the result is true for  $s - 1$ . Let  $\underline{x} = (x_1, \dots, x_d)$  be a s.o.p. of  $A$  satisfies (c). Without loss any generality we can assume that  $k_1 = d - s + 1, \dots, k_s = d$ , i.e.

$$I(x_1^2, \dots, x_{d-s}^2, x_{d-s+1}, \dots, x_d; A) = I(x_1, \dots, x_d; A). \quad (2)$$

We have by the property of multiplicity that

$$\begin{aligned} I(x_1^2, \dots, x_{d-s}^2, x_{d-s+1}, \dots, x_d; A) &= I(x_1^2, \dots, x_{d-s}^2, x_{d-s+1}, \dots, x_{d-1}; 0 :_A x_d) \\ &\quad + 2^{d-s} e(x_1, \dots, x_{d-1}; A/x_1 A). \end{aligned}$$

and

$$I(x_1, \dots, x_d; A) = I(x_1, \dots, x_{d-1}; 0 :_A x_d) + e(x_1, \dots, x_{d-1}; A/x_1 A).$$

Note that  $I(x_1^2, \dots, x_{d-s}^2, x_{d-s+1}, \dots, x_{d-1}; 0 :_A x_d) \geq I(x_1, \dots, x_{d-1}; 0 :_A x_d)$  by [CDN, Lemma 4.3]. Since  $s < d$ , we have

$$2^{d-s} e(x_1, \dots, x_{d-1}; A/x_d A) \geq e(x_1, \dots, x_{d-1}; A/x_d A).$$

Therefore it follows by (2) that  $e(x_1, \dots, x_{d-1}; A/x_1 A) = 0$  and

$$I(x_1, \dots, x_{d-s}, x_{d-s+1}, \dots, x_{d-1}; 0 :_A x_d) = I(x_1^2, \dots, x_{d-s}^2, x_{d-s+1}, \dots, x_{d-1}; 0 :_A x_d).$$

Thus,  $\text{N-dim}(A/x_d A) \leq d - 2$  and hence  $e(x_1^n, \dots, x_{d-1}^n; A/x_d A) = 0$  for all  $n > 0$ . Therefore, by applying the induction assumption for  $(0 :_A x_d)$ , there exists a constant  $C_{\underline{x}}$  such that

$$\begin{aligned} I(x_1^n, \dots, x_d^n; A) &\leq n I(x_1^n, \dots, x_{d-1}^n, x_d; A) \\ &= n (I(x_1^n, \dots, x_{d-1}^n; 0 :_A x_d) + e(x_1^n, \dots, x_{d-1}^n; A/x_d A)) \\ &= n (I(x_1^n, \dots, x_{d-1}^n; 0 :_A x_d)) \leq n n^{s-1} C_{\underline{x}} = n^s C_{\underline{x}} \end{aligned}$$



for all integer  $n > 0$ . Thus (d) is proved.

(d)  $\Rightarrow$  (b). Since  $I(x_1^n, \dots, x_d^n; A) \leq n^s I(\underline{x}; A)$  for all integers  $n$ , from the definition of the polynomial type  $\text{ld}(A)$  we have  $\text{ld}(A) \leq s$ .

(ii). Suppose that  $A$  is a co-CM  $R$ -module in dimension  $> s$ . Since each  $A$ -cosequence in dimension  $> s$  in  $I$  is always an  $A$ -cosequence in dimension  $> s$  in  $I\widehat{R}$  by Nhan-Dung [ND], we have that  $A$  is also a co-CM  $\widehat{R}$ -module in dimension  $> s$ . Therefore the Matlis dual  $D(A)$  of  $A$  is a CM  $\widehat{R}$ -module in dimension  $> s$ . It follows that  $D(A)$  is a CM  $R$ -module in dimension  $> s$  by [Z, Proposition 2.6] and hence  $\text{N-dim}_R(H_m^i(D(A))) \leq s$ , for all  $i < d$  by the Main Theorem, (iii) in [D2]. Since there is an isomorphism  $H_m^i(D(A)) \cong D(H_i^m(A))$  of  $\widehat{R}$ -modules, we have by Lemma 2.1 that

$$\dim_{\widehat{R}}(H_i^m(A)) = \text{N-dim}_{\widehat{R}}(H_m^i(D(A))) = \text{N-dim}_R(H_m^i(D(A))) \leq s$$

and (a) is satisfied.

(iii) Suppose that (a) is true, i.e.  $\dim_{\widehat{R}}(H_i^m(A)) \leq s$ . By using the Matlis dual and with similar arguments in (ii) we have  $D(A)$  is a CM  $\widehat{R}$ -module in dimension  $> s$  and hence  $A$  is a co-CM  $\widehat{R}$ -module in dimension  $> s$ .

It should be mentioned that  $\text{Width}_{>s}(I, A) \leq \text{Width}_{>s}(I\widehat{R}, A)$  since each  $A$ -cosequence in dimension  $> s$  in  $I$  is an  $A$ -cosequence in dimension  $> s$  in  $I\widehat{R}$ . In case  $s \leq 0$ , the above inequality becomes equality. However, this is not the case when  $s > 0$ . A counter example given in [ND] shows that there exists a Noetherian local ring  $(S, n)$ , an ideal  $I$  of  $S$  and an Artinian  $S$ -module  $A$  such that  $\text{Width}_{>1}(I, A) < \text{Width}_{>1}(I\widehat{S}, A)$ , where  $\widehat{S}$  is the  $n$ -adic completion of  $S$  (cf. Corollary 3.3 and Example 3.4, [ND]). Therefore, in general, an Artinian  $\widehat{R}$ -module in dimension  $> s$  is not an Artinian  $R$ -module in dimension  $> s$ .

Below, by constructing similarly to the Example 3.4 in Nhan-Dung [ND], we can give a counter example for this comment.

**Example 3.1.** *There exists an Artinian module  $A$  over local ring  $(S, n)$  such that  $A$  is a co-CM  $\widehat{S}$ -module in dimension  $> 1$ , but  $A$  is not a co-CM  $S$ -module in dimension  $> 1$ , where  $\widehat{S}$  is the  $n$ -adic completion of  $S$ .*

*Proof.* Let  $(R, m)$  be the Noetherian local domain of dimension 2 constructed by D. Ferrand and M. Raynaud [FR] such that there exists an associated prime  $\widehat{p} \in \text{Ass } \widehat{R}$  satisfying  $\dim(\widehat{R}/\widehat{p}) = 1$ . Let  $S = R[[x]]$  be the ring of all formal power series in one variable  $x$  with coefficients in  $R$ . Then  $S$  is a Noetherian local domain of dimension 3,  $\text{depth } S = 2$ , the unique maximal ideal of  $S$  is  $n = (m, x)R[[x]]$  and  $\widehat{S}$  is the  $n$ -adic completion of  $S$ . Now, choose  $I = xS$  and  $A = H_n^2(S)$ . Then  $A$  is an Artinian  $S$ -module,  $\dim_S A = 3$ ,  $\dim_{\widehat{S}} A = 2 = \text{N-dim } A$ ,  $\dim_S(0 :_A I) = 2$ ,  $\dim_{\widehat{S}}(0 :_A I) = 1$  (see [ND, Example 3.4]).

Let  $(a, b)$  be a s.o.p of  $A$  in  $I\widehat{S}$ . Then  $a \notin \widehat{p}$ , for all  $\widehat{p} \in \text{Att}_{\widehat{S}}(A)$  such that  $\dim \widehat{S}/\widehat{p} = 2 > 1$ . Hence  $x$  is an  $A$ -cosequence in dimension  $> 1$  in  $I\widehat{S}$ .

Since  $b$  is a s.o.p of  $0 :_A a$ , we have  $b \notin \widehat{p}$ , for all  $\widehat{p} \in \text{Att}_{\widehat{S}}(0 :_A a)$  such that  $\dim \widehat{S}/\widehat{p} = 1$ , i.e.  $b \notin \widehat{p}$ , for all  $\widehat{p} \in \text{Att}_{\widehat{S}}(0 :_A a)$  such that  $\dim \widehat{S}/\widehat{p} > 1$ . Therefore  $b$  is also an  $0 :_A a$ -cosequence in dimension  $> 1$  in  $I\widehat{S}$  and hence  $(a, b)$  is an  $A$ -cosequence in dimension  $> 1$  in  $I\widehat{S}$ . Thus by the definition,  $A$  is a co-CM  $\widehat{S}$ -module in dimension  $> 1$ .

However,  $A$  is not a co-CM  $S$ -module in dimension  $> 1$ . In fact, let  $(a, b)$  be a s.o.p of  $A$  in  $IS$ . Since  $\text{Width}_{>1}(IS, A) = 1$  by [ND, Example 3.4],  $(a, b)$  can not be an  $A$ -cosequence in dimension  $> 1$  in  $IS$ .  $\square$

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