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CO-COHEN-MACAULAY MODULES IN DIMENSION $> s$

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Abstract

Let (R, m) be a Noetherian local ring and A an Artinian R-module. For an integer $s > -1$, we say that A is *co-Cohen-Macaulay in dimension* $> s$ if every system of parameters of A is an A-cosequence in dimension $> s$ introduced by Nhan-Hoang [NH]. In this paper, we give some characterizations for co-Cohen-Macaulay modules in dimension $> s$ in terms of the dimension of the local homology modules $H_i^m(A)$, the polynomial type $\mathrm{ld}(A)$ of A and the multiplicity $e(\underline{x}; A)$ of A with respect to a system of parameter $\underline{x}$.

1 Introduction

Throughout this paper, let (R, m) be a Noetherian local ring and A an Artinian R-module of Noetherian dimension d. Using the concept of an A-cosequence defined by Ooishi [O], Tang and Zakeri [TZ] introduced the class of modules satisfying the condition that every system of parameters (s.o.p. for short) of A is an A-cosequence called *co-Cohen-Macaulay module*. This class of modules plays an important role in the theory of Artinian modules and their structure are well-known in terms of multiplicity, local homology and Noetherian dimension (see [CNh1], [CN], [O]). There are some extensions of the concepts of A-cosequences and co-Cohen-Macaulay modules, among which are the notions of A-cosequences in dimension $> s$ introduced by Nhan-Hoang [NH] and co-filter modules defined by Dung [D1] which are in some senses dual

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to the notions of M-sequences in dimension $> s$ in sense of Brodmann-Nhan [BN] and f-modules defined by Cuong-Schenzel-Trung [CST].

Definition Let $s \geq -1$ be an integer. A sequence $(x_1, \ldots, x_k)$ of elements in m is called an *A-cosequence in dimension* $> s$ if $x_i \notin p$ for all attached primes $p \in \mathrm{Att}_R(0 :_A (x_1, \ldots, x_{i-1})R)$ satisfying $\dim(R/p) > s$ for all $i = 1, \ldots, k$.

Note that an A-cosequence in dimension $> -1, 0$ are exactly an A-cosequence in sense of A. Ooishi [O] and f-coregular sequence in sense of [D1], respectively.

The purpose of this paper is to introduce the class of co-Cohen-Macaulay modules in dimension $> s$ and give some their characterizations in terms of the dimension of the local homology modules $H_i^m(A)$ introduced by Cuong-Nam [CN], the polynomial type $\mathrm{ld}(A)$ of A given by Minh [MIN] and the multiplicity $e(\underline{x}; A)$ of A with respect to a s.o.p. $\underline{x}$ defined by [CNh1]. It is clear that if $s \geq d$ then A is always co-Cohen-Macaulay in dimension $> s$ and if $s > -1$ then a co-Cohen-Macaulay module in dimension > -1 is exactly a co-Cohen-Macaulay module. Therefore we only consider the case where $0 \leqslant s < d$.

The main result of this paper is the following theorem.

Main Theorem. *Suppose that* $0 \leqslant s < d$.
(i) The following statements are equivalent:
(a) $\dim_{\widehat{R}}(H_i^m(A)) \leqslant s$*, for all* $i < d$.
(b) $\mathrm{ld}(A) \leqslant s$.
(c) There exist a s.o.p. $\underline{x} = (x_1, \ldots, x_d)$ *of* A *and* $k_1, \ldots, k_s \in \{1, \ldots, d\}$ *such that*

$$I(y_1, \ldots, y_d; A) = I(x_1, \ldots, x_d; A),$$

where $y_j = x_j^2$ *if* $j \notin \{k_1, \ldots, k_s\}$ *and* $y_j = x_j$ *if* $j \in \{k_1, \ldots, k_s\}$.
(d) There exist a s.o.p. $\underline{x} = (x_1, \ldots, x_d)$ *of* A *and a constant* $C_{\underline{x}}$ *(not depending on* n*) such that for all integer* $n > 0$,

$$I(x_1^n, \ldots, x_d^n; A) \leqslant n^s C_{\underline{x}}.$$

(ii) If A is co-Cohen-Macaulay in dimension $> s$ then one of the conditions (a), (b), (c), (d) is satisfied.
(iii) If one of the conditions (a), (b), (c), (d) is satisfied then A is co-Cohen-Macaulay in dimension $> s$ as $\widehat{R}$-module.

The proof of Main Theorem will be given in Section 3. In the next section, we recall some preliminaries which will be used later.

2 Preliminaries

We first recall the *Noetherian dimension* $\mathrm{N\text{-}dim}_R A$ of an Artinian R-module A defined by Kirby [K2] and Roberts [R]: if $A = 0$, we put $\mathrm{N\text{-}dim}\, A = -1$. For an integer $d \geq 0$, we put $\text{N-dim}_R A = d$ if $\text{N-dim}_R A < d$ is false, and for every ascending sequence $A_0 \subseteq A_1 \subseteq \ldots$ of submodules of A, there exists n_0 such that $\text{N-dim}_R(A_{n+1}/A_n) < d$ for all $n > n_0$.

Lemma 2.1. [CNh2] *(i) Let A be Artinian R-module. Then A has a natural structure $\widehat{R}$-module and*

$$\text{N-dim}_R A = \text{N-dim}_{\widehat{R}} A = \dim_{\widehat{R}}(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A) \leqslant \dim(R/\operatorname{Ann}_R A).$$

(ii) $\text{N-dim}\, A = 0$ *if and only if* $\dim_R A = 0$. *In this case, the length of A is finite and the ring $R/\operatorname{Ann}_R A$ is Artinian.*

(iii) Let I be an ideal of R and M a non zero f.g. R-module. Then $\text{N-dim}(H^i_m(M)) \leqslant i$ *and in particular,* $\text{N-dim}(H^d_m(M)) = d$.

The theory of secondary representation introduced by I. G. Macdonald [Mac] is in some sense dual to the more known theory of primary decomposition. It has shown in [Mac] that every Artinian R-module A has a secondary representation $A = A_1 + \ldots + A_n$ of p_i-secondary submodules A_i. The set $\{p_1, \ldots, p_n\}$ is independent of the minimal secondary representation of A and it is denoted by $\operatorname{Att}_R A$.

Lemma 2.2. *(i)* $A \neq 0$ *if and only if* $\operatorname{Att}_R A \neq \emptyset$. *In this case, the set of all minimal elements of* $\operatorname{Att}_R A$ *is exactly the set of all minimal prime ideals of* $Var(\operatorname{Ann}_R A)$.

(ii) $\text{N-dim}\, A \leqslant \dim\big(R/\operatorname{Ann}_R A\big) = \max\{\dim R/p : \ p \in \operatorname{Att}_R A\}$.

From the definition of A-cosequence in dimension $> s$, if denote by $\dim_R A$ the Krull dimension of the ring $R/\operatorname{Ann}_R A$ then we have the following result (see [ND]).

Lemma 2.3. *Let I be an ideal of R.*

(i) If $\dim_R(0 :_A I) \leqslant s$ *then there exists an A-cosequence in dimension $> s$ in I of length n for any integer $n > 0$.*

(ii) If $\dim_R(0 :_A I) > s$ *then each A-cosequence in dimension $> s$ in I can be extended to a maximal one and all maximal A-cosequences in dimension $> s$ in I have the same length, this common length is equal to the least integer i such that* $\dim_R(\operatorname{Tor}^R_i(R/I, A)) > s$.

The common length in Lemma 2.3 is called *the width in dimension* $> s$ in I with respect to A and denoted by $\operatorname{Width}_{>s}(I, A)$. In case $\dim_R(0 :_A I) \leqslant s$ we set $\operatorname{Width}_{>s}(I, A) = \infty$. Note that $\operatorname{Width}_{>-1}(I, A) = \operatorname{Width}(I, A)$, the width of A in I defined by A. Ooishi [O] (cf. [ND]).

The class of co-Cohen-Macaulay modules (co-CM for short) for Artinian modules is introduced by Tang and Zakeri [TZ] on the Noetherian local ring which is in some senses dual to the class of Cohen-Macaulay modules for Noetherian modules. Recall that an Arinian R-module A is called *co-Cohen-Macaulay* if $\mathrm{Width}(A) = \text{N-dim}\, A$. Now by the definition of A-coregular sequence in dimension $> s$, we introduce the new class of modules as follow.

Definition 2.4. An Artinian R-module A is called *co-Cohen-Macaulay in dimension $> s$* if every s.o.p. of A is an A-coregular sequence in dimension $> s$.

Note that co-CM modules in dimension $> -1, 0$ are exactly co-CM module introduced by Tang and Zakeri [TZ] and co-filter modules defined by [D1], respectively.

The multiplicity theory for Artinian modules is introduced by Cuong-Nhan [CNh1]. Let $\underline{x} = (x_1, \ldots, x_t) \subseteq m$ be a multiplicative system of A, i.e. it satisfies the condition $\ell(0 :_A \underline{x}R) < \infty$. A multiplicity system $\underline{x}$ is called a s.o.p. of A if $t = d = \text{N-dim}\, A$. Denote by $e(\underline{x}; A)$ the multiplicity of A w.r.p. to $\underline{x}$, it is proved that the number $e(\underline{x}; A)/d!$ is exactly the first cofficient of the Hilbert polynomial with respect to the s.o.p. $\underline{x}$ introduced by Kirby [K1]. The following result, see [CNh1], is used in the sequel.

Lemma 2.5. *Suppose that $\underline{x} = (x_1, \ldots, x_t)$ is a multiplicity system for A.*

(i) Let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be an exact sequence of Artinian R-modules. Then

$$e(\underline{x}; A) = e(\underline{x}; A') + e(\underline{x}; A'').$$

(ii) $0 \leqslant e(\underline{x}; A) \leqslant \ell(0 :_A \underline{x}R)$ and $e(\underline{x}; A) > 0$ if and only if $t = d = \text{N-dim}\, A$.

(iii) Let $n_1, \ldots, n_t$ be positive integers and put $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_t^{n_t})$. Then

$$e(\underline{x}(\underline{n}); A) = n_1 \ldots n_t e(\underline{x}; A).$$

(iv) Let $(x_1, \ldots, x_d)$ be a s.o.p. of A. For each $i = 1, \ldots, d$, we set $C_i = 0 :_A (x_1, \ldots, x_{i-1})R$. Then

$$\ell(0 :_A (x_1, \ldots, x_d)R) - e((x_1, \ldots, x_d); A) = \sum_{i=1}^{d} e(x_{i+1}, \ldots, x_d; C_i/x_iC_i).$$

The notion of local homology modules was defined by Cuong-Nam [CN]: Let I be an ideal of R and M an arbitrary $R-$module. The *i-th local homology module* $H_i^I(M)$ of M with respect to I is defined by $H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(R/I^t; M)$. It has been presented in [CN] many basic properties of local homology modules for Artinian modules, which show that this theory of local homology modules is in some sense dual to the well-known theory of local cohomology of A. Grothendieck for Noetherian modules.

Lemma 2.6. *(i) Let $f : R \longrightarrow R'$ be a homomorphism of Noetherian rings and I an ideal of R. Then there exists an isomorphism $H_i^I(A) \cong H_i^{IR'}(A)$ of $\Lambda_I(R)-$modules for all $i \geq 0$, where $\Lambda_I(-)$ is the $I-$adic completion functor.*

(ii) $H_i^I(A) = 0$, for all $i >$ N-dim A.

Recall that if $\ell_R(H_i^m(A)) < \infty$ for all $i < d$ then A is called *generalized co-Cohen-Macaulay* (g.CCM for short), (see [CDN]), where $H_i^m(A)$ are local homology modules. Now we recall some characterizations of g.CCM modules which are used in the sequel. From now on, for a s.o.p. $\underline{x} = (x_1, \ldots, x_d)$ of A, we set

$$I(\underline{x}; A) = \ell_R(0 :_A \underline{x}R) - e(\underline{x}; A).$$

Lemma 2.7. *The following statements are equivalent:*

(i) A is g. CCM.

(ii) There exists a constant $I(A)$ such that $I(\underline{x}; A) \leqslant I(A)$ for all s.o.p. $\underline{x}$ of A.

(iii) There exists s.o.p. $\underline{x}$ of A such that $I(x_1^2, \ldots, x_d^2; A) = I(\underline{x}; A)$.

(iv) There exists an integer $s > 0$ and a s.o.p $\underline{x}$ such that $I(x_1^n, \ldots, x_d^n; A) \leqslant s$ for all $n \geq 1$.

When A satisfies one of the above equivalent conditions, we have

$$I(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^m(A)).$$

A s.o.p. $\underline{x}$ satisfies Lemma 2.7, (iii) is called a *co-standard* s.o.p. of A. Note that if a s.o.p. $\underline{x}$ of A is co-standard then $I(x_1^{n_1}, \ldots, x_d^{n_d}; A) = I(\underline{x}; A)$ for all $n_1, \ldots, n_d \geqslant 1$ (see [CDN, Lemma 4.3]).

Let $\underline{n} = (n_1, \ldots, n_d)$ be d-tuple of d non negative integers and consider

$$I(\underline{x}(\underline{n}); A) := \ell_R(0 :_A (x_1^{n_1}, \ldots, x_d^{n_d})R) - n_1 \ldots n_d e(\underline{x}; A)$$

as a function on $n_1, \ldots, n_d$. It is shown in [MIN] that this function is not a polynomial on $n_1, \ldots, n_d$ (even when $n_1, \ldots, n_d$ large enough). However, it always takes non-negative values and bounded above by polynomials. The least degree of all polynomials in $n_1, \ldots, n_d$ bounding the above function $I(\underline{x}(\underline{n}); A)$ is independent of the choice of $\underline{x}$ and denoted by $\mathrm{ld}(A)$. If we stipulate that the degree of polynomial zero is $-\infty$ then A is co-Cohen-Macaulay if and only if $\mathrm{ld}(A) = -\infty$ (see [MIN, Theorem 4.11]).

Lemma 2.8. *(i) A is g. CCM if and only if* $\mathrm{ld}(A) \leqslant 0$.

(ii) Let $\mathrm{ld}(A) > 0$*. Then we have*

(a) $\mathrm{ld}(A) = \max\limits_{i<d} \dim_{\widehat{R}}(H_i^m(A))$.

(b) If $x \in m$ *such that* $x \notin p$ *for all* $p \in \bigcup\limits_{i=1}^{d} \mathrm{Ass}_{\widehat{R}}(H_i^m(A)) \setminus \{m\}$ *then*

$$\mathrm{ld}(0 :_A x) = \mathrm{ld}(A) - 1.$$

Proof. (i) By [CDN, Theorem 4.4] and [MIN, Corollary 4.9], we need to prove the sufficient condition. Since $\mathrm{ld}(A) \leqslant 0$, we have two cases. If $\mathrm{ld}(A) = -\infty$, then A is co-Cohen-Macaulay (see [MIN, Theorem 4.11]). Therefore we only consider to the case $\mathrm{ld}(A) = 0$. We prove by induction on $d = \text{N-dim}\, A$. Let $d = 1$. Then A is g.CCM by [CDN, Theorem 4.4] since the length of the local homology $\ell(H_0^m(A))$ is always finite. Now assume that $d > 1$ and the assertion is true for all Artinian R-module of Noetherian dimesion smaller than d. Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p. of A. The assumtion $\mathrm{ld}(A) = 0$ states that

$$\ell_R(0 :_A \underline{x}(\underline{n})R) - e(\underline{x}(\underline{n})R; A) \leqslant n_1 \ldots n_d(\ell_R(0 :_A \underline{x}R) - e(\underline{x}R; A)) = C < \infty,$$

where C is constant. So $\ell_R(0 :_A \underline{x}R) - e(\underline{x}R; A)) = C$ and hence $x_1 A \supseteq m^n A$ for some $n \in \mathbb{N}$, i.e. x_1 is a weak co-sequence by [CDN]. Therefore $\ell(A/x_1A) < \infty$ and hence $H_i^m(A/x_1A) = 0$, for all $i > 0$. Thus, form exact sequences

$$0 \longrightarrow x_1 A \longrightarrow A \longrightarrow A/x_1 A \longrightarrow 0;$$

$$0 \longrightarrow 0 :_A x_1 \longrightarrow A \xrightarrow{x_1} x_1 A \longrightarrow 0$$

we get the long exact sequences for $i = 1, \ldots, d-1$,

$$\ldots \longrightarrow H_i^m(0_A : x_1) \longrightarrow H_i^m(A) \xrightarrow{x_1} H_i^m(A) \longrightarrow H_{i-1}^m(0_A : x_1) \longrightarrow \ldots.$$

So, by using the induction hypothesis with respect to the s.o.p. $(x_2, \ldots, x_r)$ of $(0 :_A x_1)$, we have by [CDN, Theorem 4.4] that $\ell\big(H_i^m(0_A : x_1)\big) < \infty$ for all $i \leqslant d-2$. Therefore, we have $\ell\big(H_i^m(A)\big) < \infty$ for all $i \leqslant d-1$ and A is g.CCM by [CDN, Theorem 4.4].

(ii) Note that $\mathrm{ld}(A) = p(D(A))$, where $D(A)$ is a Noetherian $\widehat{R}$-module and $p(D(A))$ is a polynomial type of $D(A)$ defined by [C]. Hence from isomorphisms $H_m^i(D(A)) \cong D(H_i^m(A))$ and $0 :_A x \cong D(A)/xD(A)$ of $\widehat{R}$-modules, using Matlis duality, we get the result by [CMN, Lemma 3.1]. □

3 Proof of Main Theorem

(i). $(a) \Leftrightarrow (b)$ follows by Lemma 2.8,(ii).

$(a) \Rightarrow (c)$. Let $d = 1$. Then $s = 0$ and A is g.CCM. By Lemma 2.7(iii), there exists a standard s.o.p. x_1 of A, i.e. $I(x_1^2; A) = I(x_1; A)$. Therefore (c) is true.

Let $d > 1$. We prove the result by induction on s, where $0 \leqslant s < d$. Let $s = 0$. Then $\dim_{\widehat{R}} H_i^m(A) \leqslant 0$ for all $i < d$. By Lemma 2.1(ii), $\ell_{\widehat{R}}(H_i^m(A)) < \infty$ for all $i < d$, i.e. A is g.CCM by [CDN, Theorem 4.4]. Hence there exists by Lemma 2.7(iii) a s.o.p. $(x_1, \ldots, x_d)$ of A such that $I(x_1^2, \ldots, x_d^2; A) = I(x_1, \ldots, x_d; A)$. Therefore the condition (c) is true for $s = 0$. Let $1 \leqslant s < d$ and assume that the result is true for the case $s - 1$. If $\mathrm{ld}(A) \leqslant 0$ then A is g.CCM by Lemma 2.8,(i). Therefore there exists a standard s.o.p. $\underline{x} = (x_1, \ldots, x_d)$ of A. Thus by [CDN, Lemma 4.3] we have

$$I(\underline{x}; A) \leqslant I(y_1, \ldots, y_d; A) \leqslant I(x_1^2, \ldots, x_d^2; A) = I(\underline{x}; A),$$

where $y_j = x_j^2$ if $j \notin \{k_1, \ldots, k_s\}$ and $y_j = x_j$ if $j \in \{k_1, \ldots, k_s\}$, for all $j = 1, \ldots, d$. Hence $I(\underline{x}; A) = I(y_1, \ldots, y_d; A)$, the result is true in this case. Let $\mathrm{ld}(A) > 0$. Let $x_1 \in m$ such that $x_1 \notin p$ for all $p \in \bigcup_{i=1}^{d} \mathrm{Ass}_{\widehat{R}}(H_i^m(A)) \setminus \{m\}$. Note that $\mathrm{ld}(A) \leqslant s$ by Lemma 2.8(ii). Therefore we get by Lemma 2.8(ii) that $\mathrm{ld}(0 :_A x_1) = \mathrm{ld}(A) - 1 \leqslant s - 1$. Hence $\dim_{\widehat{R}} H_i^m(A) \leqslant s - 1$ for all $i < d - 1$ by Lemma 2.8(ii). Applying the induction for $(0 :_A x_1)$, there exists a s.o.p. $(x_2, \ldots, x_d)$ of A and integers $k_2, \ldots, k_s \in \{2, \ldots, d\}$ such that

$$I(y_2, \ldots, y_d; 0 :_A x_1) = I(x_2, \ldots, x_d; 0 :_A x_1),$$

where $y_j = x_j^2$ if $j \notin \{k_2, \ldots, k_s\}$ and $y_j = x_j$ if $j \in \{k_2, \ldots, k_s\}$, for all $j = 2, \ldots, d$. Without loss any generality we can assume that $k_2 = 2, \ldots, k_s = s$, i.e.

$$I(x_2, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; 0 :_A x_1) = I(x_2, \ldots, x_d; 0 :_A x_1). \tag{1}$$

By the choice of x_1, we have $\mathrm{N\text{-}dim}(A/x_1A) \leqslant 0$. Since $d > 1$, we have

$$e(x_2, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; A/x_1A) = 0 = e(x_2, \ldots, x_s, x_{s+1}, \ldots, x_d; A/x_1A).$$

Therefore, we have

$$\begin{aligned} I(x_2, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; 0 :_A x_1) &= \ell_R(0 :_A (x_1, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2)) \\ &\quad - e(x_1, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; A) + e(x_2, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; A/x_1A) \\ &= I(x_1, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; A), \end{aligned}$$

and

$$\begin{aligned} I(x_2, \ldots, x_d; 0 :_A x_1) &= \ell_R(0 :_A (x_1, x_2, \ldots, x_d)) - e(x_1, x_2, \ldots, x_d; A) \\ &\quad + e(x_2, \ldots, x_d; A/x_1A) = I(x_1, \ldots, x_d; A). \end{aligned}$$

So, it follows by (1) that

$$I(x_1, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; A) = I(x_1, \ldots, x_d; A),$$

and (c) is proved.

$(c) \Rightarrow (d)$. Let $d = 1$. Then $s = 0$ and A is g.CCM. So, there exists a standard s.o.p. x_1 of A and we have $I(x_1; A) = I(x_1^2; A) = I(x_1^n; A)$ for all $n \in \mathbb{N}$ by [CDN]. Set $C_{\underline{x}} = I(x_1; A)$. Then $I(x_1^n; A) = C_{\underline{x}} = n^0 C_{\underline{x}}$ for all $n \geqslant 1$. Hence (d) is true.

Let $d > 1$. We prove the result by induction on s, where $0 \leqslant s < d$. Let $s = 0$. From the hypothesis (c), there exists a s.o.p. $\underline{x} = (x_1, \ldots, x_d)$ of A such that

$$I(x_1^2, \ldots, x_d^2; A) = I(x_1, \ldots, x_d; A).$$

It implies A is g.CCM and $\underline{x}$ is a standard s.o.p. of A by [CDN]. Set $C_{\underline{x}} = I(x_1, \ldots, x_d; A)$. Then

$$I(x_1^n, \ldots, x_d^n; A) = n^0 C_{\underline{x}}$$

for all $n \geqslant 1$ and (d) is true for the case $s = 0$. Let $s > 0$ and assume that the result is true for $s - 1$. Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p. of A satisfies (c). Without loss any generality we can assume that $k_1 = d - s + 1, \ldots, k_s = d$, i.e.

$$I(x_1^2, \ldots, \ldots, x_{d-s}^2, x_{d-s+1}, \ldots, x_d; A) = I(x_1, \ldots, x_d; A). \tag{2}$$

We have by the property of multiplicity that

$$\begin{aligned} I(x_1^2, \ldots, x_{d-s}^2, x_{d-s+1}, \ldots, x_d; A) = I(x_1^2, \ldots, x_{d-s}^2, x_{d-s+1}, \ldots, x_{d-1}; 0 :_A x_d) \\ + 2^{d-s} e(x_1, \ldots, x_{d-1}; A/x_1A). \end{aligned}$$

and

$$I(x_1, \ldots, x_d; A) = I(x_1, \ldots, x_{d-1}; 0 :_A x_d) + e(x_1, \ldots, x_{d-1}; A/x_1A).$$

Note that $I(x_1^2, \ldots, x_{d-s}^2, x_{d-s+1}, \ldots, x_{d-1}; 0 :_A x_d) \geqslant I(x_1, \ldots, x_{d-1}; 0 :_A x_d)$ by [CDN, Lemma 4.3]. Since $s < d$, we have

$$2^{d-s} e(x_1, \ldots, x_{d-1}; A/x_dA) \geqslant e(x_1, \ldots, x_{d-1}; A/x_dA).$$

Therefore it follows by (2) that $e(x_1, \ldots, x_{d-1}; A/x_1A) = 0$ and

$$I(x_1, \ldots, x_{d-s}, x_{d-s+1}, \ldots, x_{d-1}; 0 :_A x_d) = I(x_1^2, \ldots, x_{d-s}^2, x_{d-s+1}, \ldots, x_{d-1}; 0 :_A x_d).$$

Thus, N-dim$(A/x_dA) \leqslant d - 2$ and hence $e(x_1^n, \ldots, x_{d-1}^n; A/x_dA) = 0$ for all $n > 0$. Therefore, by applying the induction assumption for $(0 :_A x_d)$, there exists a constant $C_{\underline{x}}$ such that

$$\begin{aligned} I(x_1^n, \ldots, x_d^n; A) &\leqslant nI(x_1^n, \ldots, x_{d-1}^n, x_d; A) \\ &= n\big(I(x_1^n, \ldots, x_{d-1}^n; 0 :_A x_d) + e(x_1^n, \ldots, x_{d-1}^n; A/x_dA)\big) \\ &= n\big(I(x_1^n, \ldots, x_{d-1}^n; 0 :_A x_d)\big) \leqslant nn^{s-1} C_{\underline{x}} = n^s C_{\underline{x}} \end{aligned}$$

for all integer $n > 0$. Thus (d) is proved.

$(d) \Rightarrow (b)$. Since $I(x_1^n, \ldots, x_d^n; A) \leqslant n^s I(\underline{x}; A)$ for all integers n, from the definition of the polynomial type $\mathrm{ld}(A)$ we have $\mathrm{ld}(A) \leqslant s$.

(ii). Suppose that A is a co-CM R-module in dimension $> s$. Since each A-cosequence in dimension $> s$ in I is always an A-cosequence in dimension $> s$ in $I\widehat{R}$ by Nhan-Dung [ND], we have that A is also a co-CM $\widehat{R}$-module in dimension $> s$. Therefore the Matlis dual $D(A)$ of A is a CM $\widehat{R}$-module in dimension $> s$. It follows that $D(A)$ is a CM R-module in dimension $> s$ by [Z, Proposition 2.6] and hence $\text{N-dim}_R(H_m^i(D(A))) \leqslant s$, for all $i < d$ by the Main Theorem, (iii) in [D2]. Since there is an isomorphism $H_m^i(D(A)) \cong D(H_i^m(A))$ of $\widehat{R}$-modules, we have by Lemma 2.1 that

$$\dim_{\widehat{R}}(H_i^m(A)) = \text{N-dim}_{\widehat{R}}(H_m^i(D(A))) = \text{N-dim}_R(H_m^i(D(A))) \leqslant s$$

and (a) is satisfied.

(iii) Suppose that (a) is true, i.e. $\dim_{\widehat{R}}(H_i^m(A)) \leqslant s$. By using the Matlis dual and with similar aguments in (ii) we have $D(A)$ is a CM $\widehat{R}$-module in dimension $> s$ and hence A is a co-CM $\widehat{R}$-module in dimension $> s$.

It should be mentioned that $\mathrm{Width}_{>s}(I, A) \leqslant \mathrm{Width}_{>s}(I\widehat{R}, A)$ since each A-cosequence in dimension $> s$ in I is an A-cosequence in dimension $> s$ in $I\widehat{R}$. In case $s \leqslant 0$, the above inequality becomes equality. However, this is not the case when $s > 0$. A counter example given in [ND] shows that there exists a Noetherian local ring (S, n), an ideal I of S and an Artinian S-module A such that $\mathrm{Width}_{>1}(I, A) < \mathrm{Width}_{>1}(I\widehat{S}, A)$, where $\widehat{S}$ is the n-adic completion of S (cf. Corollary 3.3 and Example 3.4, [ND]). Therefore, in general, an Artinian $\widehat{R}$-module in dimension $> s$ is not an Artinian R-module in dimension $> s$.

Below, by constructing similarly to the Example 3.4 in Nhan-Dung [ND], we can give a counter example for this comment.

Example 3.1. *There exists an Artinian module A over local ring (S, n) such that A is a co-CM $\widehat{S}$-module in dimension > 1, but A is not a co-CM S-module in dimension > 1, where $\widehat{S}$ is the n-adic completion of S.*

Proof. Let (R, m) be the Noetherian local domain of dimension 2 constructed by D. Ferrand and M. Raynaud [FR] such that there exists an associated prime $\widehat{p} \in \mathrm{Ass}\, \widehat{R}$ satisfying $\dim(\widehat{R}/\widehat{p}) = 1$. Let $S = R[[x]]$ be the ring of all formal power series in one variable x with coefficients in R. Then S is a Noetherian local domain of dimension 3, $\operatorname{depth} S = 2$, the unique maximal ideal of S is $n = (m, x)R[[x]]$ and $\widehat{S}$ is the n-adic completion of S. Now, choose $I = xS$ and $A = H_n^2(S)$. Then A is an Artinian S-module, $\dim_S A = 3$, $\dim_{\widehat{S}} A = 2 = \text{N-dim}\, A$, $\dim_S(0 :_A I) = 2$, $\dim_{\widehat{S}}(0 :_A I) = 1$ (see [ND, Example 3.4]).

Let (a, b) be a s.o.p of A in $I\widehat{S}$. Then $a \notin \widehat{p}$, for all $\widehat{p} \in \mathrm{Att}_{\widehat{S}}(A)$ such that $\dim \widehat{S}/\widehat{p} = 2 > 1$. Hence x is an A-cosequence in dimension > 1 in $I\widehat{S}$.

Since b is a s.o.p of $0 :_A a$, we have $b \notin \widehat{p}$, for all $\widehat{p} \in \mathrm{Att}_{\widehat{S}}(0 :_A a)$ such that $\dim \widehat{S}/\widehat{p} = 1$, i.e. $b \notin \widehat{p}$, for all $\widehat{p} \in \mathrm{Att}_{\widehat{S}}(0 :_A a)$ such that $\dim \widehat{S}/\widehat{p} > 1$. Therefore b is also an $0 :_A a$-cosequence in dimension > 1 in $I\widehat{S}$ and hence (a, b) is an A-cosequence in dimension > 1 in $I\widehat{S}$. Thus by the definition, A is a co-CM $\widehat{S}$-module in dimension > 1.

However, A is not a co-CM S-module in dimension > 1. In fact, let (a, b) be a s.o.p of A in IS. Since $\mathrm{Width}_{>1}(IS, A) = 1$ by [ND, Example 3.4], (a, b) can not be an A-cosequence in dimension > 1 in IS. □

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