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A NOTE ON JUMPS-FRACTIONAL PROCESSES

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Abstract

In this note we study some stochastic processes having jumps at some times $\tau_1, ..., \tau_n, ...$ and which, between two jumps, satify a stochastic differential equation driven by a fractional Brownian motion.

1. Introduction

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ on which we define a fractional Brownian motion $(W_t^H, t \geq 0)$, a Poisson process $(N_t, t \geq 0)$ with intensity λ, a sequence $(U_n, n \geq 1)$ of independent and identically distributed random variables taking value in $(0, +\infty)$ and a filtration $(\mathcal{F}_t, t \geq 0)$. Suppose that $(W_t^H), (N_t)$ and (U_n) are independent.

For convenience we take for $\mathcal{F}_t$ the σ-algebra generated by random variables W_s^H, N_s for $s \leq t$ and U_n for $1 \leq n \leq N_t$:

$$\mathcal{F}_t = \sigma(W_s^H, N_s, U_n, s \leq t, 1 \leq n \leq N_t).$$

Let $(X_t, t \geq 0)$ be a stochastic process adapted to filtration $(\mathcal{F}_t, t \geq 0)$ and having jumps at times $\tau_1, \tau_2, ..., \tau_n, ...$

Denote by ΔX_{τ_n} the jump size of X_t at time τ_n,

$$\Delta X_{\tau_n} = X_{\tau_n} - X_{\tau_n -} \tag{1.1}$$

where $X_{\tau_n -} = \lim_{t \nearrow \tau_n} X_t$.

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We take the relative jump size of X_t at τ_n as value of U_n,

$$U_n := \frac{X_{\tau_n} - X_{\tau_n -}}{X_{\tau_n -}},\ n = 1, 2, ...$$

thus

$$X_{\tau_n} = X_{\tau_n -}(1 + U_n). \tag{1.2}$$

Note that, since the random variables $(U_n, 1 \leq n \leq N_t)$ are $\mathcal{F}_t$-measurable, we see that at time t, the relative jump sizes taking place before t are known. Moreover, the times τ_n's are stopping times of $(\mathcal{F}_t)$ because $(\tau_n \leq t) = (N_t \geq j) \in \mathcal{F}_t$ (see [3]).

In this note, the considered processes are supposed, between jumps, to satisfy some fractional stochastic differential equations.

2. On the fractional stochastic calculus from L^2- semimatingale approximation approach

2.1 Fractional Brownian motion

A fractional Brownian motion $(W_t^H, t \geq 0)$ with Hurst index H, $(0 < H < 1)$ is a centered Gaussian process having covariance function $R(s,t) = E(W_s^H W_t^H)$ given by

$$R(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{2.1}$$

In the case $H = 1/2$, W_t^H becomes usual standard Brownian motion.
If $H \neq 1/2$, W_t^H is neither a Markov process nor a semimartingale. It is by this property one cannot apply the traditional Ito calculus to study stochastic dynamics driven by a fractional Brownian motion. Many approaches have been introduced to overcome this difficulty, such as these of T. E. Duncan et al., D. Nualart, B. Oksendal and many others. In this note, we will follow the approximate approach given by T. H. Thao (see [6-10]).

From this approach one considers the fractional Browinan motion of Liouville form B_t^H defined by

$$B_t^H = \int_0^t (t-s)^{H-1/2} dW_s, \tag{2.2}$$

where W_t is a standard Brownian motion. B_t^H is related to W_t^H by

$$W_t^H = \frac{1}{\Gamma(H+1/2)}(Z_t + B_t^H), \tag{2.3}$$

where Γ is the gamma function, Z_t is a process having absolutely continuous trajectories and therefore the long range dependence of the fractional Brownian motion W_t^H focusses at B_t^H.

2.2. L^2-semimartingale approximation.

Nualart D. [4] has introduced the semimartingale $B_t^{H,\epsilon}$ for every $\epsilon > 0$ given by

$$B_t^{H,\epsilon} = \int_0^t (t-s+\epsilon)^{H-\frac{1}{2}} dW_s \tag{2.4}$$

It can be shown that $B_t^{H,\epsilon}$ has the form

$$B_t^{H,\epsilon} = (H-1/2)\int_0^t \varphi_s^\epsilon ds + \epsilon^{H-1/2} W_t. \tag{2.5}$$

And T. H. Thao has proved that $B_t^{H,\epsilon}$ converges in $L^2(\Omega, \mathcal{F}, P)$ to B_t^H uniformly with respect to t in any finite interval $[0,T]$ as $\epsilon \to 0$. (Refer to [6], [7]. [10]).

$$B_t^H = L^2 - \lim_{\epsilon \to 0} B_t^{H,\epsilon}. \tag{2.6}$$

2.3. Fractional stochastic integration and differential equations

A simple definition of fractional stochastic integral was introduced in [10] , motivated by the fact that it was a result for the case of definition for integral of a process of bounded variation by a formula of integration - by - part.

Assume that $f(t,\omega)$ is an adapted process such that

$$E\int_0^t f^2(s,\omega) ds < \infty \tag{2.7}$$

then the fractional integral is defined as

$$\int_0^t f(s,\omega) dB_s^H = L^2 - \lim_{\epsilon \to 0} \int_0^t f(s,\omega) dB_s^{H,\epsilon}, \tag{2.8}$$

where the existence of the integral in the right hand side of (2.8) with respect to the semimartingale $B_t^{H,\epsilon}$ is obvious and the limit is in the sense of the convergence in $L^2(\Omega, \mathcal{F}, P)$.

A theorem of existence and uniqueness for solution of the equation of form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) dB_t^H$$

has been proved in [10] and some classes of fractional stochastic dynamics with explicit solutions have been introduced in[11].

3. Fractional geometric Brownian motion with jumps.

Consider the fractional stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t^H \tag{3.1}$$

where B_t^H is a fractional Brownian motion of Liouville form. From an approximate approach in L^2-space given by T.H.Thao [4] we know that the solution of (3.1) can be expressed by

$$X_t = X_0 \exp(\mu t + \sigma B_t^H). \tag{3.2}$$

Now let $(X_t)_t \geq 0$ be an adapted process such that
(a) It satisfies the equation (3.1) for each time interval $[\tau_n, \tau_{n+1})$,
(b) At times τ_n, the jump of X_t is given by

$$\Delta X_{\tau_n} = X_{\tau_n} - X_{\tau_n -} = X_{\tau_n -} U_n,$$

then $X_{\tau_n} = X_{\tau_n -}(1 + U_n)$ where U_n is the relative jump size at time τ_n.
We see that, for $t \in [0, \tau_1]$

$$X_t = X_0 \exp(\mu t + \sigma B_t^H),$$

and the left-hand limit at τ_1 is

$$X_{\tau_1 -} = X_0 \exp(\mu \tau_1 + \sigma B_{\tau_1}^H),$$

and

$$X_{\tau_1} = X_{\tau_1 -}(1 + U_1) = X_0(1 + U_1) \exp(\mu \tau_1 + \sigma B_{\tau_1}^H).$$

Then for $t \in [\tau_1, \tau_2]$,

$$\begin{aligned} X_t &= X_{\tau_1} \exp[\mu(t - \tau_1) + \sigma(B_t^H - B_{\tau_1}^H)] \\ &= X_{\tau_1 -}(1 + U_1) \exp[\mu(t - \tau_1) + \sigma(B_t^H - B_{\tau_1}^H)] \\ &= X_0(1 + U_1) \exp(\mu t + \sigma B_t^H). \end{aligned}$$

Notice that, since B_t^H is a process having stationary increments then in the calculation above we can replace $B_{t-\tau_1}^H$ by $B_t^H - B_{\tau_1}^H$.
Repeating this procedure we have

$$X_t = X_0 [\prod_{n=1}^{N_t} (1 + U_n)] \exp(\mu t + \sigma B_t^H). \tag{3.3}$$

Then process X_t given by (3.3) is called a fractional geometric Brownian motion with jumps.

It is known from [6], [7] and [11] that $\exp(\mu t + \sigma B_t^H)$ is the $\mathbf{L^2}$- limit of $\exp(\mu t + \sigma B_t^{H,\epsilon})$, where

$$B_t^{H,\epsilon} = \alpha \int_0^t \varphi_s^\epsilon ds + \epsilon^\alpha W_t,$$

$\alpha = H - 1/2$, $\varphi_t^\epsilon = \int_0^t (t-s+\epsilon)^{\alpha-1} dW_s$ and W_t is a standard Brownian motion. Thus X_t in (3.3) is the L^2-limit of

$$X_t^\epsilon = X_0[\prod_{n=1}^{N_t}(1+U_n)] \exp\big(b(t,\epsilon) + \sigma\epsilon^\alpha W_t\big), \tag{3.4}$$

where $b(t,\epsilon) = \mu t + \alpha\sigma \int_0^t \varphi_s^\epsilon ds$.

According to [3], the process X_t^ϵ of the form (3.4) can be expressed as

$$X_t^\epsilon = X_0 + \int_0^t X_s^\epsilon (b(s,\epsilon) + \sigma\epsilon^\alpha dW_s) + \sum_{n=1}^{N_t} X_{\tau_n-}^\epsilon U_n. \tag{3.5}$$

Note that the continuous part of (3.5) is

$$\begin{aligned}\int_0^t X_s^\epsilon (b(s,\epsilon) + \sigma\epsilon^\alpha dW_s) &= \int_0^t X_s^\epsilon[\mu ds + \sigma(\alpha\varphi_s^\epsilon ds + \epsilon^\alpha dW_s)] \\ &= \int_0^t X_s^\epsilon[\mu ds + \sigma dB_s^{H,\epsilon}]. \end{aligned} \tag{3.6}$$

And

$$L^2 - \lim_{\epsilon\to 0} \int_0^t X_s^\epsilon(\mu ds + \sigma dB_s^{H,\epsilon}) = \int_0^t X_s(\mu ds + \sigma dB_s^H) \tag{3.7}$$

$$L^2 - \lim_{\epsilon\to 0} \sum_{n=1}^{N_t} X_{\tau_n-}^\epsilon U_n = \sum_{n=1}^{N_t} X_{\tau_n-} U_n. \tag{3.8}$$

Finally,

$$X_t = \text{Ł}^2 - \lim_{\epsilon\to 0} X_t^\epsilon = X_0 + \int_0^t X_s(\mu ds + \sigma dB_s^H) + \sum_{n=1}^{N_t} X_{\tau_n-} U_n. \tag{3.9}$$

Now we have the proposition to state.

Theorem 3.1 *Suppose $(X_t, t \geq 0)$ is an adapted process verifying two conditions (a) and (b) above, then it can be expressed as*

$$X_t = X_0[\prod_{n=1}^{N_t}(1+U_n)]e^{\mu t + \sigma B_t^H}. \tag{3.10}$$

Also, it satisfies the following equation

$$X_t = X_0 + \int_0^t X_s(\mu ds + \sigma dB_s^H) + \sum_{n=1}^{N_t} X_{\tau_n-} U_n. \tag{3.11}$$

4. Fractional Ornstein-Uhlenbeck process with jumps

Consider a fractional stochastic Langevin equation

$$dX_t = -bX_t dt + \sigma dB_t^H, \quad t >\geq 0 \tag{4.1}$$

where b and σ are positive constants.

This equation was studied in [8,11] by an approximate approach. Its solution is called a fractional Ornstein-Uhlenbeck process.

Theorem 4.1 *Suppose X_t is a process having jumps at times $\tau_1, ..., \tau_n$ such that:*

(a) It satisfies the equation (4.1) for each time interval $[\tau_n, \tau_{n+1})$.

(b) At time τ_n, the jump of X_t is given by

$$\Delta X_{\tau_n} = X_{\tau_n} - X_{\tau_n -} = X_{\tau_n -}(1 + U_n),$$

where U_n is the relative jump size at time τ_n.

Then X_t can be expressed by:

(i) For $t \in [0, \tau_1)$

$$X_t = e^{-bt}(X_0 + \sigma \int_0^t e^{bs} dB_s^H). \tag{4.2}$$

(ii) For $t \in [\tau_n, \tau_{n+1})$,

$$X_t = e^{-b(t-\tau_n)}\big(X_{\tau_n} + \sigma \int_{\tau_n}^{t-\tau_n} e^{bs} dW_s^H\big), \tag{4.3}$$

where $X_{\tau_n} = X_{\tau_{n-1}}(1 + U_n)$.

(iii) For $t = \tau_{n+1}$,

$$X_{\tau_{n+1}} = e^{-b(\tau_{n+1}-\tau_n)}\big(X_{\tau_n} + \sigma \int_{\tau_n}^{\tau_{n+1}-\tau_n} e^{bs} dB_s^H\big)(1 + U_{n+1}). \tag{4.4}$$

Proof. According to results given in [7,11] , the solution of Equation (4.1) is given by

$$X_t = e^{-bt}(X_0 + \sigma \int_0^t e^{bs} dB_s^H) \tag{4.5}$$

then for the first interval $[0, \tau_1]$, X_t has the same form of (4.5).

The left-limit $X_{\tau_1 -} = \lim_{t \nearrow \tau_1} X_t$ is of the form

$$X_{\tau_1 -} = e^{-b\tau_1}(X_0 + \sigma \int_0^{\tau_1} e^{bs} dB_s^H). \tag{4.6}$$

And

$$X_{\tau_1} = X_{\tau_1 -}(1+U_1) = e^{-b\tau_1}(X_0 + \sigma \int_0^{\tau_1} e^{bs} dB_s^H)(1+U_1). \tag{4.7}$$

For $t \in [\tau_1, \tau_2)$ we have

$$\begin{aligned} X_t &= e^{-b(t-\tau_1)}(X_{\tau_1} + \sigma \int_{\tau_1}^{t-\tau_1} e^{bs} dB_s^H) \\ &= e^{-b(t-\tau_1)}\big[e^{-b\tau_1}(X_0 + \sigma \int_0^{\tau_1} e^{bs} dB_s^H)(1+U_1) + \sigma \int_{\tau_1}^{t-\tau_1} e^{bs} dB_s^H\big] \\ &= e^{-bt}\big[(X_0 + \sigma \int_0^{\tau_1} e^{bs} dB_s^H)(1+U_1) + \sigma e^{b\tau_1} \int_{\tau_1}^{t-\tau_1} e^{bs} dB_s^H\big] \end{aligned} \tag{4.8}$$

Repeating procedure yields for $t \in [\tau_n, \tau_{n+1})$,

$$X_t = e^{-b(t-\tau_n)}\big(X_{\tau_n} + \sigma \int_{\tau_n}^{t-\tau_n} e^{bs} dB_s^H\big) \tag{4.9}$$

where $X_{\tau_n} = X_{\tau_{n-1}}(1+U_n)$.

We have also for $t = \tau_{n+1}$

$$X_{\tau_{n+1}} = e^{-b(\tau_{n+1}-\tau_n)}\big(X_{\tau_n} + \sigma \int_{\tau_n}^{\tau_{n+1}-\tau_n} e^{bs} dB_s^H\big)(1+U_{n+1}) \tag{4.10}$$

Formulas (4.5), (4.7), (4.8), (4.9) and (4.10) give us recursive relations for determining the fractional Ornstein-Uhlenbeck process with jumps that complete the proof. $\square$

5. Jumps fractional stochastic differential equation

Consider the equation

$$dX_t + g(t, X_t)dt + \gamma(t) X_t dB_t^H + X_{t-} U_{N_t} dN_t, \tag{5.1}$$

where (B_t^H) is a fractional Brownian of Liouville form, (N_t) is a standard Poisson process and $(U_k, k \geq 1)$ is an i.i.d. sequence of random variables, $g(t,x)$ and $\gamma(t)$ are some regular real functions such that there exists a unique solution for the equation

$$dX_t = g(t, X_t)dt + \gamma(t) X_t dB_t^H. \tag{5.2}$$

Suppose that (B_t^H), (N_t) and (U_k) are independent.

In fact, the process (X_t) satisfying (5.1) is a process having jumps at some times $\tau_1, \tau_2, ...$ such that the number of jumps in time interval $[0, t]$ is N_t and between jumps, it satisfies (5.2). Also, U_k is the relative jump size of (X_t)at time τ_n.

An application of Thao's approximation approach leads us to the approximate equation corresponding to (5.2)

$$dX_t^\epsilon = g(t, X_t^\epsilon)dt + \gamma(t)X_t^\epsilon dB_t^{H,\epsilon}, \tag{5.3}$$

where $B_t^{H,\epsilon}$ is a semimartingale

$$dB_t^{H,\epsilon} = \alpha\varphi_t^\epsilon dt + \epsilon^\alpha dW_t, \ \alpha = H - 1/2.$$

Thus

$$dX_t^\epsilon = f(t, X_t^\epsilon)dt + c(t)X_t^\epsilon dW_t, \tag{5.4}$$

where

$$f(t, X_t^\epsilon) = g(t, X_t^\epsilon) + \alpha c(t)\varphi_t^\epsilon X_t^\epsilon,$$
$$c(t) = \epsilon^\alpha \gamma(t).$$

For the convenience we put $X_0^\epsilon = X_0$.

Now, we denote

$$G_t = \exp(-\int_0^t c(s)dW_s + \frac{1}{2}\int_0^t c^2(s)ds), \tag{5.5}$$

and

$$Y_t^\epsilon = G_t X_t^\epsilon \tag{5.6}$$

We have

$$\begin{aligned} d(G_t X_t^\epsilon) &= X_t^\epsilon dG_t + G_t dX_t^\epsilon + dG_t dX_t^\epsilon \\ &= X_t^\epsilon(c^2(t)G_t - cG_t dW_t) + G_t(f(t, X_t^\epsilon) + c(t)X_t^\epsilon dW_t) - c^2(t)X_t^\epsilon G_t dt \\ &= G_t f(t, X_t^\epsilon)dt. \end{aligned}$$

Then $dY_t^\epsilon = G_t f(t, X_t^\epsilon)dt$ or Y_t^ϵ is the solution of an ordinary differential equation

$$\frac{dY_t^\epsilon}{dt} = G_t f(t, X_t^\epsilon). \tag{5.7}$$

Since $X_t^\epsilon = G_t^{-1}Y_t^\epsilon$ we see that the solution of the approximate equation is defined by

$$X_t^\epsilon = Y_t^\epsilon \exp(\int_0^t c(s)dW_s - \frac{1}{2}\int_0^t c^2(s)ds), \tag{5.8}$$

where Y_t^ϵ is the solution of (5.7) and $c(t) = \epsilon^\alpha\gamma(t)$.

By a similar way as in Sections 3 and 4 we can get at last

Theorem 5.1 *The solution of (5.1) can be given by*

$$X_t = L^2 - \lim_{\epsilon \to 0} X_t^\epsilon, \tag{5.9}$$

where

$$X_t^\epsilon = X_0 Y_t^\epsilon \prod_{k=1}^{N_t}(1+U_k)\exp(\int_0^t c(s)dW_s - \frac{1}{2}\int_0^t c^2(s)ds), \tag{5.10}$$

$c(t) = \epsilon^{H-1/2}\gamma(t)$ *and* Y_t^ϵ *is the solution of (5.7).*

6. Jumps fractional stochastic differential equation

In this section we discuss the general form

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t^H + c(t, X_{t-})dN_t \tag{6.1}$$

where $a(t,x)$, $b(t,x)$, $c(t,x)$ are coefficients which satisfy some conditions that assure the existence and uniqueness for solution of (6.1), $c(t,x)$ is the jump size coefficient.

The approximation of (6.1) is

$$dX_t^\epsilon = a(t, X_t^\epsilon)dt + b(t, X_t^\epsilon)dB_t^{H,\epsilon} + c(t, X_{t-}^\epsilon)dN_t \tag{6.2}$$

where

$$B_t^{H,\epsilon} = \int_0^t (t-s+\epsilon)^{H-1/2}dW_t.$$

Suppose that under some conditions imposed on $a(t,x)$, $b(t,x)$ and $c(t,x)$ as in [11], the solution of (6.1) can be considered as L^2-limit of the solution of (6.2) when $\epsilon \to 0$.

There is no explicit solution of (6.2) in general, so a numerical method is needed. We follow the method of [1] to present a numerical scheme for calculating the solution of (6.2). We have already,

$$dB_t^{H,\epsilon} = \alpha\varphi_t^\epsilon dt + \epsilon^\alpha dW_t \text{ with } \varphi_t^\epsilon = \int_0^t (t-s+\epsilon)^{\alpha-1}dW_s.$$

Denote

$$\overline{a}(t, X_t^\epsilon) = a(t, X_t^\epsilon) + \alpha\varphi_t^\epsilon b(t, X_t^\epsilon)$$
$$\overline{b}(t, X_t^\epsilon) = \epsilon^\alpha b(t, X_t^\epsilon).$$

We have

$$dX_t^\epsilon = \overline{a}(t, X_t^\epsilon)dt + \overline{b}(t, X_t^\epsilon)dW_t + c(t, X_{t-}^\epsilon)dN_t \tag{6.3}$$

Suppose jumps appear at times $\tau_1, \tau_2, ..., \tau_n, ...$

On the intervals $[\tau_{n-1}, \tau_n)$, X_t is continuous and satisfies the equation

$$dX_t^\epsilon = \overline{a}(t, X_t^\epsilon)dt + \overline{b}(t, X_t^\epsilon)dW_t.$$

At $t = \tau_n$,

$$\Delta X_t^\epsilon = c(t, X_{t-}^\epsilon)\Delta N_t \text{ or } X_t^\epsilon = X_{t-}^\epsilon + c(t, X_{t-}^\epsilon),$$

where ΔN_t is the integer jump in N_t at time t, and $c(t, X^{\epsilon}_{t-})$ is the size of the jump.

Now we try to use MAPLE procedures to find the numerical solution for (6.3).

The scheme is considered with $t \in [0, T]$ and a partition $0 < t_1 < t_2 < ... < t_{N_T} = T$.

Denote

$$\Delta_n = t_{n+1} - t_n$$
$$\Delta W_n = W_{n+1} - W_n$$
$$\Delta N_n = N_{n+1} - N_n$$

and Y_n is the approximation to the solution of X^{ϵ}_t.

We use the Euler scheme to scalar jump-diffusion of (6.2)

$$Y_{n+1} = Y_n + \overline{a}(t_n, Y_n)\Delta_n + \overline{b}(t_n, Y_n)\Delta W_n + c(t, Y_n)\Delta N_n,$$

for $n = 0, 1, 2, ...N_T - 1$.

$$\begin{aligned}
&Euler - jump := proc(\overline{a} : algebraic, \overline{b} : algebraic, c : algebraic)\\
&\qquad\qquad localtemp, h;\\
&\qquad\qquad temp := Y[n+1] = Y[n] + \overline{a} * dt + \overline{b} * dW[n] + c * dN[n];\\
&\qquad\qquad temp := subs(x = Y[n], temp)\\
&\qquad end :
\end{aligned}$$

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