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DIRECT BLOCK METHOD FOR SOLVING SINGULAR PERTUBATION OF LINEAR BOUNDARY VALUE PROBLEMS

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Abstract

In this study, we present direct block method of Adams Moulton type for solving singular perturbation of linear two point boundary value problem directly. Most of the existence research involving second order singular linear boundary value problems will reduce the problem to a system of first order ordinary differential equations (ODEs). This approach will enlarge the system of first order ODEs and needs more computational work. The advantage of the direct block method in this research is its ability to obtain the solutions at two points simultaneously, and also the second order singular perturbation problems will be solved directly without reducing it to first order ODEs. Lagrange interpolation polynomial was applied in the derivation of the proposed method. The method will solve singular purturbation linear boundary value problems together with linear shooting technique using constant step size. The proposed method is examined by comparing the result with the existing method. Numerical result shows that the direct block method is more efficient and accurate compared to the existing method. The proposed direct method is suitable for solving singular perturbation linear boundary value problems.

Key words: boundary value problem, direct method, singular perturbation, linear bvp.
2010 AMS Classification: 65L06, 65L10

1 Introduction

Singular perturbation problems have arisen very frequently in many science and engineering applications and have been extensively studied by researcher in recent years. These problems are easily found in fluid motion including fluid dynamics, aero dynamics, plasma dynamics, rarefied gas dynamics and oceanography. Singular perturbation problems and boundary value problems have the same type of boundary condition, an additional small positive parameter ε in singular perturbation problems causing computation difficulties. We consider a class of linear singular perturbation problems of the form

$$\varepsilon u'' + P(x)u' + Q(x)u = R(x), x \in [a, b] \tag{1}$$

with boundary condition

$$\begin{aligned} u(a) &= \alpha \\ u(b) &= \beta \end{aligned} \tag{2}$$

where ε is a small parameter $0 < \varepsilon \leq 1$; α and β are constants; $P(x)$, $Q(x)$ and $R(x)$ are continuous on $[a, b]$.

Chakravarthy et al. [1] presented a seventh order numerical method to solve linear and non-linear singular perturbation problems with a boundary layer at one end point. This method requires transforming the second order singular perturbation problem into systems of first order ordinary differential equation. Kumar et al. [2] have discussed a new boundary value technique to solve singular perturbation problems. Similar to other methods, the problems will be partitioned into inner and outer solution of differential equation. The inner region problem is solved as a two-point boundary layer correction problem and the outer region problem of the differential equation is solved as initial-value problem with initial condition at end point. Singular perturbation problems can also be solved using uniform B-spline collocation method and the method is proposed by Kadalbajoo & Gupta [3].

The approach for solving higher order ordinary differential equation directly has been explored by many researcher such as Suleiman [4], Majid et al. [5] and Ismail et al. [6]. Recently, Majid et al. [7] has solved nonlinear two point boundary value problem directly using Adams Moulton method with Newton's shooting technique. Most of the existing methods require to reduce the order of differential equation or tranform the higher order differential equation into system of equation for solving singular perturbation problems. In this paper, we are extended the method in Majid et al. [7] together with linear shooting technique for solving singular perturbation problems directly using constant step size.

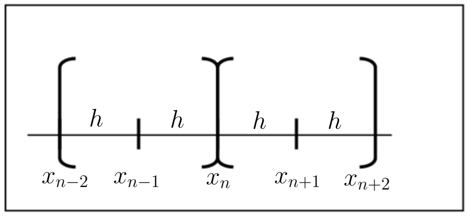

Figure 1: Two Step Direct Method

2 Formulation of the Method

The interval $[a, b]$ is divided into a series of blocks with each block containing two points as shown in Figure 1. Two approximate values are simultaneously found using the same back values i.e. y_{n+1} and y_{n+2}. The first point will be approximated by integrating Eq.(1) over the interval $[x_n, x_{n+1}]$ once and twice gives,

$$\int_{x_n}^{x_{n+1}} y''(x)dx = \int_{x_n}^{x_{n+1}} f(x, y, y')dx \tag{3}$$

and

$$\int_{x_n}^{x_{n+1}} \int_{x_n}^{x} y''(x)dxdx = \int_{x_n}^{x_{n+1}} \int_{x_n}^{x} f(x, y, y')dxdx. \tag{4}$$

The function $f(x, y, y')$ in Eq.(3) and Eq.(4) will be replaced by Lagrange interpolating polynomial, P. The interpolating points involved in two step direct method of order four (2PD4) are $(x_{n-1}, f_{n-1}), (x_n, f_n), (x_{n+1}, f_{n+1})$ and (x_{n+2}, f_{n+2}), we will obtain the Lagrange interpolating polynomial:

$$\begin{aligned} P_3 \quad = \quad & \frac{(x - x_{n-1})(x - x_n)(x - x_{n+1})}{(x_{n+2} - x_{n-1})(x_{n+2} - x_n)(x_{n+2} - x_{n+1})} f_{n+2} \\ & \frac{(x - x_{n-1})(x - x_n)(x - x_{n+2})}{(x_{n+1} - x_{n-1})(x_{n+1} - x_n)(x_{n+1} - x_{n+2})} f_{n+1} \\ & \frac{(x - x_{n-1})(x - x_{n+1})(x - x_{n+2})}{(x_n - x_{n-1})(x_n - x_{n+1})(x_n - x_{n+2})} f_n \\ & \frac{(x - x_n)(x - x_{n+1})(x - x_{n+2})}{(x_{n-1} - x_n)(x_{n-1} - x_{n+1})(x_{n-1} - x_{n+2})} f_{n-1}. \end{aligned} \tag{5}$$

Let $s = \dfrac{x - x_{n+2}}{h}$ and by replacing $dx = hds$ and changing the limit of integration from -2 to -1. Eq.(8) and Eq.(9) can be written as:

$$y'_{n+1} - y'_n = \int_{-2}^{-1} P_3 hds \tag{6}$$

$$y_{n+1} - y_n - 2hy'_n = \int_{-2}^{-1} -(s+1)P_3 h^2 ds. \tag{7}$$

Apply the same process to find the second point, y_{n+2} of the two point direct method. Eq.(1) will be integrate over the interval $[x_n, x_{n+2}]$ once and twice gives,

$$\int_{x_n}^{x_{n+2}} y''(x)dx = \int_{x_n}^{x_{n+2}} f(x, y, y')dx \tag{8}$$

and

$$\int_{x_n}^{x_{n+2}} \int_{x_n}^{x} y''(x)dxdx = \int_{x_n}^{x_{n+2}} \int_{x_n}^{x} f(x, y, y')dxdx. \tag{9}$$

Let $s = \dfrac{x - x_{n+2}}{h}$ and by replacing $dx = hds$ and changing the limit of integration from -2 to 0. Eq.(8) and Eq.(9) can be written as:

$$y'_{n+2} - y'_n = \int_{-2}^{0} P_3 hds \tag{10}$$

$$y_{n+2} - y_n - hy'_n = \int_{-2}^{0} -(s)P_3 h^2 ds. \tag{11}$$

Evaluate these integrals using MAPLE, we obtain the corrector formulae for two step direct method of Adams Moulton type (2PD4):

$$\begin{aligned}
y'_{n+1} &= y'_n + \frac{h}{24}(-f_{n-1} + 13f_n + 13f_{n+1} - f_{n+2}) && (12)\\
y_{n+1} &= y_n + hy'_n + \frac{h^2}{360}(-8f_{n-1} + 129f_n + 66f_{n+1} - 7f_{n+2}) && (13)\\
y'_{n+2} &= y'_n + \frac{h}{3}(f_n + 4f_{n+1} + f_{n+2}) && (14)\\
y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{45}(-2f_{n-1} + 36f_n + 54f_{n+1} + 2f_{n+2}). && (15)
\end{aligned}$$

This method is the combination of predictor of one order less than the corrector. The same process is apply to find the predictor formulae of the two step direct method. For calculation of the initial points, two methods are involved which is Euler and Modified Euler method. These methods will be used at the beginning of the code to find the starting initial values. Both methods will solve the problems directly. The predictor and corrector direct method then can be implemented until the end of boundary interval by 3 iterations, i.e. P(EC)^3, where P is predictor, E is Evaluate and C is corrector. In order to get better approximation at the initial steps, the step size h will be reduce to $h/4$. While predictor and corrector direct method will remain using the chosen step size h.

3 Implementation of the Method

Consider a singular perturbation problem,

$$\varepsilon u'' - u' = 1$$

subject to $u(0) = 0, u(1) = -1$
the exact solution

$$u = -1 + \frac{1}{1 - e^{2/\sqrt{\varepsilon}}}(e^{2/\sqrt{\varepsilon} - x/\sqrt{\varepsilon}} - e^{x/\sqrt{\varepsilon}}).$$

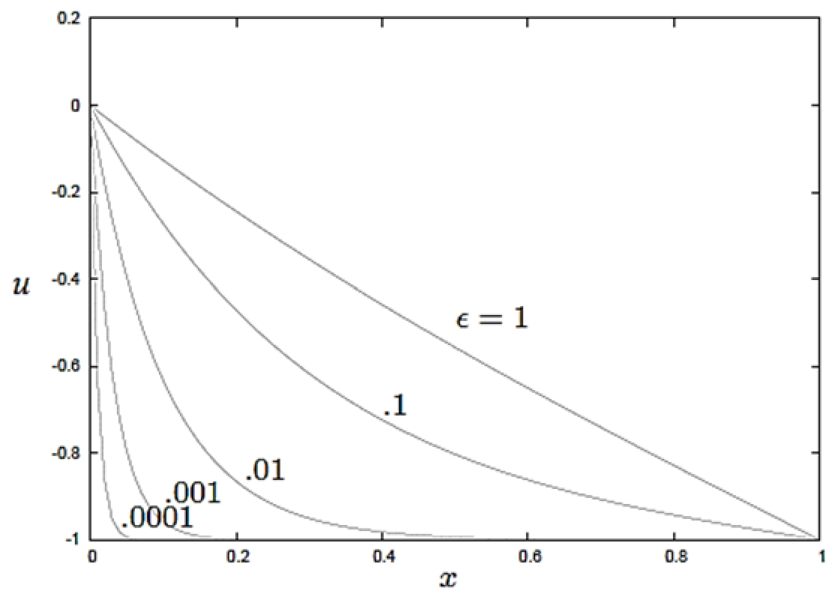

Figure 2: As $\varepsilon \to 0$, $u \approx -1$ everywhere, except close to $x = 0$.

The stiffness of exact solution increase when $\varepsilon \to 0$ as shown in Figure 2. When $\varepsilon = 0.0001$, the small values of perturbation parameter ε cause the exact solution close to -1 as the value x increase. So, we are interested in solving singular perturbation problems numerically for small values of the ε.

Linear shooting method is based on replacement of the singular perturbation problems (1) by two IVPs:

$$\begin{aligned} \varepsilon u_1'' &= P(x)u_1' + Q(x)u_1 + R(x) \quad u_1(a) = \alpha, u_1'(a) = 0 \\ \varepsilon u_2'' &= P(x)u_2' + Q(x)u_2 \quad u_2(a) = 0, u_2'(a) = 1. \end{aligned} \tag{16}$$

A new function will be constructed by solving the two IVPs:

$$u = u_1 + \theta u_2$$

with

$$u(b) = u_1(b) + \theta u_2(b) = \beta \tag{17}$$

hence

$$\theta = \frac{\beta - u_1(b)}{u_2(b)} \tag{18}$$

using the linear shooting method, a series of approximation solutions to the singular perturbation problems will be calculated from the new function.
For detail can refer to Faires & Burden [8].

4 Numerical Examples

We will discuss four numerical examples which widely discussed in the literature. The following notations are used in the tables:

NBVT	New boundary value technique in Kumar et al. [2]
2PD4	Direct block method of order 4

Problem 1:
Consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity.

$$\varepsilon u''(x) + u'(x) = 1 + 2x, 0 \leq x \leq 1.$$

Boundary condition:$u(0) = 0, u(1) = 1$.
Exact solution:

$$u(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{(1 - e^{-x/\varepsilon})}.$$

Problem 2:
Consider the following homogeneous singular perturbation problem.

$$\varepsilon u''(x) + u'(x) - u(x) = 0, 0 \leq x \leq 1.$$

Boundary condition:$u(0) = 1, u(1) = 1$.
Exact solution:

$$\begin{aligned} u(x) &= \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{(e^{m_2} - e^{m_1})}, \\ m_1 &= (-1 + \sqrt{1 + 4\varepsilon})/2\varepsilon, m_2 = (-1 - \sqrt{1 + 4\varepsilon})/2\varepsilon. \end{aligned}$$

Table 1: Comparision of NBVT and 2PD4 for solving Problem 1 at $\varepsilon = 10^{-3}, h = 10^{-3}$

x	Local Error	
	NBVT	2PD4
0.000	0	0
0.001	1.26E-03	5.54e-03*
0.010	1.98E-03	2.67e-05
0.020	1.96E-03	2.16e-09
0.030	1.94E-03	7.36e-14
0.040	1.92E-03	2.84e-14
0.050	1.90E-03	2.84e-14
0.100	1.80E-03	2.84e-14
0.300	1.40E-03	2.60e-14
0.500	1.00E-03	2.80e-14
0.700	6.00E-04	2.44e-14
0.900	2.00E-04	7.99e-15
1.000	0	0

Table 2: Comparision of NBVT and 2PD4 for solving Problem 1 at $\varepsilon = 10^{-4}, h = 10^{-4}$

x	Local Error	
	NBVT	2PD4
0.0000	0	0
0.0001	1.26E-04	5.55e-03*
0.0010	2.00E-04	2.67e-05
0.0020	2.00E-04	2.16e-09
0.0030	1.99E-04	1.83e-14
0.0040	1.99E-04	1.21e-13
0.0050	1.99E-04	1.21e-13
0.1000	1.80E-04	1.19e-13
0.3000	1.40E-04	1.09e-13
0.5000	1.00E-04	1.73e-13
0.7000	6.00E-05	1.82e-13
0.9000	2.09E-05	6.11e-14
1.0000	0	0

Table 3: Comparision of NBVT and 2PD4 for solving Problem 2 at $\varepsilon = 10^{-3}, h = 10^{-3}$

x	Local Error	
	NBVT	2PD4
0.000	0	0
0.001	2.00E-07	3.50e-03*
0.010	3.67E-04	1.66e-05
0.020	3.67E-04	1.34e-09
0.030	3.67E-04	8.12e-14
0.040	3.67E-04	1.80e-14
0.050	3.67E-04	1.82e-14
0.100	3.65E-04	1.78e-14
0.300	3.47E-04	1.53e-14
0.500	3.03E-04	1.43e-14
0.700	2.22E-04	1.43e-14
0.900	9.03E-05	4.77e-15
1.000	0	0

Table 4: Comparision of NBVT and 2PD4 for solving Problem 2 at $\varepsilon = 10^{-4}, h = 10^{-4}$

x	Local Error	
	NBVT	2PD4
0.0000	0	0
0.0001	1.00E-08	3.51e-03*
0.0010	3.68E-05	1.69e-05
0.0020	3.68E-05	1.37e-09
0.0030	3.68E-05	2.31e-13
0.0040	3.68E-05	1.66e-13
0.0050	3.68E-05	1.67e-13
0.1000	3.65E-05	1.67e-13
0.3000	3.47E-05	1.53e-13
0.5000	3.02E-05	1.30e-13
0.7000	2.22E-05	1.00e-13
0.9000	9.00E-06	3.52e-14
1.0000	0	0

The code was written in C-language. Table 1 - 4 show the numerical results for the two given linear singular perturbation problems when solved using 2PD4 and compared to NBVT. Through the numerical result, 2PD4 has smaller local error compare to NBVT, therefore this show that 2PD4 is more accurate compare to NBVT. The 2PD4 method will solve the singular perturbation problem directly, while NBVT require reducing to system of first order ordinary differential equation. The advantage of the direct block method in this research is its ability to obtain the solutions at two points simultaneously. Hence, this has shown the efficiency of 2PD4 method in terms of accuracy and also the cost per step is less costly.

5 Conclusion

In this paper, we have shown the proposed direct block method of Adams Moulton type with linear shooting technique using constant step size is suitable for solving second order linear singular perturbation problems.

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