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THE ADMISSIBLE MONOMIAL BASIS FOR THE POLYNOMIAL ALGEBRA OF FIVE VARIABLES IN DEGREE $2^{s+1} + 2^s - 5$

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Abstract

Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in k variables $x_1, x_2, \ldots, x_k$, each of degree 1. We study the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for P_k as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we explicitly determine all admissible monomials for the case $k = 5$ in degree $2^{s+1} + 2^s - 5$ with s an arbitrary positive integer.

1 Introduction

Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in k variables $x_1, x_2, \ldots, x_k$, each of degree 1. The mod-2 Steenrod algebra $\mathcal{A}$ acts on P_k by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and subject to the Cartan formula $Sq^n(fg) = \sum_{i=0}^{n} Sq^i(f)Sq^{n-i}(g)$, for $f, g \in P_k$ (see Steenrod-Epstein [12]).

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Many authors study the *hit problem* of determination of the minimal set of generators for P_k as a module over the Steenrod algebra, or equivalently, a basis of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

This problem has first been studied by Peterson [8], Wood [16], Singer [11], Priddy [9], who show its relationship to several classical problems in homotopy theory.

The tensor product $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ has explicitly been calculated by Peterson [8] for $k = 2$, Kameko for $k = 3$ in his thesis [5] and Sum [14] for $k = 4$.

The hit problem were then investigated by many authors. (See Bruner-Ha-Hung [1], Carlisle-Wood [2], Crabb-Hubbuck [3], Hung [4], Mothebe [6], Nam [7], Singer [11], Silverman [10], Walker-Wood [15] and others.)

Peterson conjectured in [8] that as a module over the Steenrod algebra $\mathcal{A}$, P_k is generated by monomials in degrees n that satisfy $\alpha(n+k) \leqslant k$, where $\alpha(n)$ denotes the number of ones in dyadic expansion of n, and proved it for $k \leqslant 2$. The conjecture was established in general by Wood [16].

For any nonnegative integer n, set $\mu(n) = \min\{m \in \mathbb{Z} : \alpha(n+m) \leqslant m\}$. Denote by $(P_k)_n$ and $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n$ the subspaces of degree n homogeneous polynomials in the spaces P_k and $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ respectively.

Theorem 1.1 ([Wood [16]). *If* $\mu(n) > k$, *then* $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n = 0$.

From Theorem 1.1, the hit problem is reduced to the cases of degree n with $\mu(n) \leqslant k$.

In this paper, we explicitly determine all the admissible monomials (see Section 2) of P_5 in degree $n = 2^{s+1} + 2^s - 5$ with s an arbitrary positive integer.

For $s = 1$, the problem is easy. There exist exactly 5 admissible monomials of degree 1 in P_5, namely: x_1, x_2, x_3, x_4, x_5. For $s = 2$, we have

Proposition 1.2. *There exist exactly* 110 *admissible monomials of degree* 7 *in* P_5. *Consequently* $\dim((\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7 = 110$.

The main result of the paper is the following:

Theorem 1.3. *For any integer* $s \geqslant 3$, *there exist exactly* 912 *admissible monomials of degree* $2^{s+1}+2^s-5$ *in* P_5. *Consequently* $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{2^{s+1}+2^s-5} = 912$.

In Section 2, we recall some results on the admissible monomials and hit monomials in P_k. We prove Proposition 1.2 in Section 3. Theorem 1.3 will be proved in Section 4.

2 Preliminaries

In this section, we recall some results in Kameko [5], Singer [11] and Sum [13] on the admissible monomials and the hit monomials in P_k.

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a nonnegative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0$ or 1 and $i \geqslant 0$.

Let $x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} \in P_k$. Following Kameko [5], we define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots),$$
$$\sigma(x) = (a_1, a_2, \ldots, a_k),$$

where $\omega_i(x) = \sum_{1 \leqslant j \leqslant k} \alpha_{i-1}(a_j),\ i \geqslant 1$.

Definition 2.1. Let x, y be the monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds

1. $\omega(x) < \omega(y)$,
2. $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Here, the order on the set of sequences of nonnegative integers is the lexicographical one.

Let f, g be homogeneous polynomials of the same degree in P_k. We denote $f \equiv g$ if and only if $f - g \in \mathcal{A}^+ P_k$. If $f \equiv 0$, then f is called hit.

Definition 2.2. A monomial x is said to be inadmissible if there exists the monomials $y_1, y_2, \ldots, y_r$ such that

$$x \equiv y_1 + y_2 + \ldots + y_r \text{ and } y_j < x,\ j = 1, 2, \ldots, r.$$

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all admissible monomials in P_k is a minimal set of $\mathcal{A}$-generators of P_k.

Definition 2.3. A monomial x is said to be strictly inadmissible if and only if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$, for $j = 1, 2, \ldots, t$ and

$$x = y_1 + y_2 + \ldots + y_t + \sum_{1 \leqslant i < 2^s} Sq^i(f_i),$$

with $s \leqslant \max\{i\ ; \omega_i(x) > 0\}$ and $f_i \in P_k$.

The following theorem is one of our main tools.

Theorem 2.4 (Kameko [5], Sum [13]). *Let x, y, w be monomials in P_k such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*

i) *If w is inadmissible, then xw^{2^r} is also inadmissible.*

ii) *If w is strictly inadmissible, then $xw^{2^r} y^{2^{r+s}}$ is inadmissible.*

Now, we recall a result of Singer [11] on the hit monomials in P_k.

Definition 2.5. A monomial $z = x_1^{a_1}x_2^{a_2}\dots x_k^{a_k}$ is called the spike if $b_j = 2^{s_j}-1$ for s_j a nonnegative integer and $j = 1, 2, \dots, k$. If z is the spike with $s_1 > s_2 > \dots > s_{r-1} \geqslant s_r > 0$ and $s_j = 0$ for $j > r$ then it is called the minimal spike.

The following is a criterion for the hit monomials in P_k.

Theorem 2.6 (Singer [11]). *Suppose $x \in P_k$ is a monomial of degree n, where $\alpha(n+k) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$ then x is hit.*

One of the main tools in the study of the hit problem is Kameko's squaring operation $\widetilde{Sq}^0_* : \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \to \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. This homomorphism is induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_* : P_k \to P_k$, given by

$$\widetilde{Sq}^0_*(x) = \begin{cases} y, & \text{if } x = x_1x_2\dots x_ky^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However, $\widetilde{Sq}^0_*Sq^{2t} = Sq^t\widetilde{Sq}^0_*$, and $\widetilde{Sq}^0_*Sq^{2t+1} = 0$ for any nonnegative integer t.

Theorem 2.7 (Kameko [5]). *Let m be a positive integer. If $\mu(2m+k) = k$, then $(\widetilde{Sq}^0_*)_m : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{2m+k} \to (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_m$ is an isomorphism of $\mathbb{F}_2$-vector spaces.*

For latter use, we set

$$P_k^0 = \langle\{x = x_1^{a_1}x_2^{a_2}\dots x_k^{a_k} \ ; \ a_1a_2\dots a_k = 0\}\rangle,$$
$$P_k^+ = \langle\{x = x_1^{a_1}x_2^{a_2}\dots x_k^{a_k} \ ; \ a_1a_2\dots a_k > 0\}\rangle.$$

It is easy to see that P_k^0 and P_k^+ are the $\mathcal{A}$-submodules of P_k. Furthermore, we have

$$\mathbb{F}_2 \otimes_{\mathcal{A}} P_k = (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k^0) \oplus (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+).$$

For a polynomial f in P_k, we denote by $[f]$ the class in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ represented by f.

For $1 \leqslant i \leqslant k$, define the homomorphism $f_i = f_{k;i} : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ x_{j+1}, & \text{if } i \leqslant j < k. \end{cases}$$

It is easy to see that

Proposition 2.8. *If $B_{k-1}(n)$ is the set of all admissible monomials of degree n in P_{k-1}, then $f(B_{k-1}(n)) := \cup_{1\leqslant i\leqslant k} f_i(B_{k-1}(n))$ is the set of all admissible monomials of degree n in P_k^0.*

For any $I = (i_0, i_1, \ldots, i_r),\ 0 < i_0 < i_1 < \ldots < i_r \leqslant k,\ 0 \leqslant r < k$, we define the homomorphism $p_I : P_k \to P_{k-1}$ of algebras by substituting

$$p_I(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i_0, \\ \sum_{1 \leqslant s \leqslant r} x_{i_s - 1}, & \text{if } j = i_0, \\ x_{j-1}, & \text{if } i_0 < j \leqslant k. \end{cases}$$

Then p_I is a homomorphism of $\mathcal{A}$-modules. In particular, for $I = (i)$, we have $p_{(i)}(x_i) = 0$.

3 Proof of Proposition 1.2

From now on, denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k.

According to a result in Sum [14], there exist exactly 35 admissible monomials of degree 7 in P_4, namely:

$x_i^7,\ 1 \leqslant i \leqslant 4,\ x_1x_2^2x_3^2x_4^2,$
$x_ix_j^6,\ 1 \leqslant i < j \leqslant 4,$
$x_ix_j^2x_\ell^4,\ 1 \leqslant i < j < \ell \leqslant 4,$
$x_ix_j^3x_\ell^3,\ 1 \leqslant i, j, \ell \leqslant 4,\ i \neq j \neq \ell \neq i,$
$x_ix_j^2x_\ell x_m,\ (i, j, \ell, m)$ is a permutation of (1,2,3,4) such that $i < j$.

By a direct computation we see that $f(B_4(7))$ is the set consisting of 100 admissible monomials in $(P_5^0)_7$. Now we determine all admissible monomials in $(P_5^+)_7$.

Lemma 3.1. *If x is an admissible monomials of degree 7 in P_5 then either $\omega(x) = (3, 2)$ or $\omega(x) = (5, 1)$.*

Proof. Since $\omega_1(x)$ is odd we have $\omega_1(x) = 1$ or $\omega_1(x) = 3$ or $\omega_1(x) = 5$.

If $\omega_1(x) = 1$, then $x = x_iy^2$ with y a monomial of degree 3 in P_5. This contradicts the fact that $x \in P_5^+$.

If $\omega_1(x) = 3$, then $x = x_ix_jx_\ell y^2$ with y a monomial of degree 2 in P_5. Since x is admissible, according to Theorem 2.4, y is admissible. Hence $y = x_mx_n$, where (i, j, ℓ, m, n) is a permutation of $(1, 2, 3, 4, 5)$. So we have $\omega(x) = (3, 2)$.

If $\omega_1(x) = 5$, then $x = x_1x_2x_3x_4x_5x_i^2,\ 1 \leqslant i \leqslant 5$. Hence $\omega(x) = (5, 1)$. □

Lemma 3.2. *The following monomials are strictly inadmissible:*

$$x_1x_2^2x_3^2x_4x_5, x_1^2x_i^2x_jx_\ell x_m,\ (i, j, \ell, m) \text{ is a permutation of } (2, 3, 4, 5).$$

Proof. By a direct computation, we have

$$\begin{aligned} x_1x_2^2x_3^2x_4x_5 &= x_1x_2^2x_3x_4^2x_5 + x_1x_2^2x_3x_4x_5^2 + x_1x_2x_3^2x_4^2x_5 + x_1x_2x_3^2x_4x_5^2 \\ &\quad + x_1x_2x_3x_4^2x_5^2 + Sq^1(x_1^2x_2x_3x_4x_5) + Sq^2(x_1x_2x_3x_4x_5), \\ x_1^2x_i^2x_jx_\ell x_m &= x_1x_i^2x_j^2x_\ell x_m + x_1x_i^2x_jx_\ell^2x_m + x_1x_i^2x_jx_\ell x_m^2 + Sq^1(x_1x_i^2x_jx_\ell x_m). \end{aligned}$$

The lemma is proved. □

Let x be a monomial of degee 7 in P_5^+. If $\omega(x) = (5,1)$ then x is a spike. Hence x is admissible. If x is admissible and $\omega(x) = (3,2)$, then by Lemma 3.2, x is one of 5 monomials:

$$\begin{aligned} &a_1 = x_1x_2x_3x_4^2x_5^2,\ a_2 = x_1x_2x_3^2x_4x_5^2,\ a_3 = x_1x_2x_3^2x_4^2x_5, \\ &a_4 = x_1x_2^2x_3x_4x_5^2,\ a_5 = x_1x_2^2x_3x_4^2x_5. \end{aligned}$$

Now we prove that the set $\{[a_i],\ 1 \leqslant i \leqslant 5\}$ is linearly independent in $\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5$. Suppose that

$$\mathcal{S} = \gamma_1a_1 + \gamma_2a_2 + \gamma_3a_3 + \gamma_4a_4 + \gamma_5a_5 \equiv 0.$$

By a direct computation, we have

$$\begin{aligned} p_{(1,2)}(\mathcal{S}) &\equiv (\gamma_1+\gamma_2+\gamma_3)x_1x_2^2x_3^2x_4^2 + \gamma_4x_1^3x_2x_3x_4^2 + \gamma_5x_1^3x_2x_3^2x_4 \equiv 0, \\ p_{(1,3)}(\mathcal{S}) &\equiv (\gamma_1+\gamma_4+\gamma_5)x_1x_2^2x_3^2x_4^2 + \gamma_2x_1x_2^3x_3x_4^2 + \gamma_3x_1x_2^3x_3^2x_4 \equiv 0. \end{aligned}$$

From these above equalities, one gets $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 0$. The proposition is proved.

4 Proof of Theorem 1.3

Note that

$$2^{s+1} + 2^s - 5 = 2^{s+1} + 2^{s-1} + 2^{s-2} + 2^{s-3} + 2^{s-3} - 5.$$

If $s > 3$, then $\mu(2^{s+1} + 2^s - 5) = 5$. According to Theorem 2.7, the squaring operation

$$\widetilde{Sq}^0_* : (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_{2^{s+1}+2^s-5} \longrightarrow (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_{2^s+2^{s-1}-5}$$

is an isomorphism. Hence $(\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_{2^{s+1}+2^s-5} \cong (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_{19}$ for any $s > 3$. So, we need only to prove the theorem for $s = 3$.

Since the squaring operation $\widetilde{Sq}^0_* : (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_{19} \longrightarrow (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_7$ is an epimorphism, we have

$$\begin{aligned} (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_{19} &\cong \mathrm{Ker}(\widetilde{Sq}^0_*) \oplus (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_7 \\ &= (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5^0)_{19} \oplus (\mathrm{Ker}(\widetilde{Sq}^0_*) \cap (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5^+)_{19}) \oplus (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_7. \end{aligned}$$

According to a result in Sum [14], there exist exactly 140 admissible monomials of degree 19 in P_4. A direct computation using Proposition 2.8, one gets

Proposition 4.1. *There exist exactly* 550 *admissible monomials of degree* 19 *in* P_5^0. *Consequently* $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^0)_{19} = 550$.

Now we explicitly determine $\mathrm{Ker}(\widetilde{Sq}^0_*) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{19}$.

Lemma 4.2. *Let x be an admissible monomial of degree* 19 *in* P_5. *If* $[x] \in \mathrm{Ker}(\widetilde{Sq}^0_*)$, *then either* $\omega(x) = (3,2,1,1)$ *or* $\omega(x) = (3,2,3)$ *or* $\omega(x) = (3,4,2)$.

Proof. Observe that $z = x_1^{15}x_2^3x_3$ is the minimal spike of degree 9 in P_5 and $\omega(z) = (3,2,1,1)$. Since $\omega_1(x)$ is odd, using Theorem 2.6 and the fact that $[x] \in \mathrm{Ker}\widetilde{Sq}^0_*$, we obtain $\omega_1(x) = 3$. Hence $x = x_ix_jx_\ell y^2$, where y is a monomial of degree 8 and $1 \leqslant i < j < \ell \leqslant 5$. Since x is admissible, by Theorem 2.4, y is also admissible. Applying Theorem 2.6, we see that either $\omega_1(y) = 2$ or $\omega_1(y) = 4$. If $\omega_1(y) = 2$, then $y = x_ix_jy_1^2$ with y_1 a monomial of degree 3 in P_5 and $i < j$. Then either $\omega(y_1) = (1,1)$ or $\omega(y_1) = (3)$. If $\omega_1(y) = 4$, then $y = x_ix_jx_\ell y_2^2$ with y_2 a monomial of degree 2 in P_5 and $i < j < \ell$. Then $\omega(y_2) = (2)$. The lemma is proved. □

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of nonnegative integers such that $\omega_i = 0$ for $i \gg 0$. Define $\deg\omega = \sum_{i>0} 2^{i-1}\omega_i$. We set

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)(\omega) = \mathrm{Span}\{[x] \in \mathbb{F}_2 \otimes_{\mathcal{A}} P_k : x \text{ is an admissible monomial and } \omega(x) = \omega\},$$

and $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+)(\omega) = ((\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)(\omega)) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+)$. It is easy to see that

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n = \bigoplus_{\deg\omega = n} (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)(\omega).$$

Applying Lemma 4.2, one gets

$$\mathrm{Ker}(\widetilde{Sq}^0_*) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{19} = ((\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)(3,2,1,1)) \oplus \oplus ((\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)(3,2,3)) \oplus ((\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)(3,4,2)).$$

Proposition 4.3. $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)(3,2,1,1)$ *is an $\mathbb{F}_2$-vector space of dimension* 150 *with a basis consisting of all the classes represented by the following admissible monomials:*

$a_1 = x_1x_2x_3x_4^2x_5^{14}$ $a_2 = x_1x_2x_3x_4^{14}x_5^2$ $a_3 = x_1x_2x_3^2x_4x_5^{14}$ $a_4 = x_1x_2x_3^2x_4^{14}x_5$
$a_5 = x_1x_2x_3^{14}x_4x_5^2$ $a_6 = x_1x_2x_3^{14}x_4^2x_5$ $a_7 = x_1x_2^2x_3x_4x_5^{14}$ $a_8 = x_1x_2^2x_3x_4^{14}x_5$
$a_9 = x_1x_2^{14}x_3x_4x_5^2$ $a_{10} = x_1x_2^{14}x_3x_4^2x_5$ $a_{11} = x_1x_2x_3x_4^6x_5^{10}$ $a_{12} = x_1x_2x_3^6x_4x_5^{10}$
$a_{13} = x_1x_2x_3^6x_4^{10}x_5$ $a_{14} = x_1x_2^6x_3x_4x_5^{10}$ $a_{15} = x_1x_2^6x_3x_4^{10}x_5$ $a_{16} = x_1x_2x_3^2x_4^2x_5^{13}$
$a_{17} = x_1x_2x_3^2x_4^{13}x_5^2$ $a_{18} = x_1x_2^2x_3x_4^2x_5^{13}$ $a_{19} = x_1x_2^2x_3x_4^{13}x_5^2$ $a_{20} = x_1x_2^2x_3^{13}x_4x_5^2$
$a_{21} = x_1x_2^2x_3^{13}x_4^2x_5$ $a_{22} = x_1x_2x_3^2x_4^3x_5^{12}$ $a_{23} = x_1x_2x_3^2x_4^{12}x_5^3$ $a_{24} = x_1x_2x_3^3x_4^2x_5^{12}$
$a_{25} = x_1x_2x_3^3x_4^{12}x_5^2$ $a_{26} = x_1x_2^2x_3x_4^3x_5^{12}$ $a_{27} = x_1x_2^2x_3x_4^{12}x_5^3$ $a_{28} = x_1x_2^2x_3^3x_4x_5^{12}$
$a_{29} = x_1x_2^2x_3^3x_4^{12}x_5$ $a_{30} = x_1x_2^2x_3^{12}x_4x_5^3$ $a_{31} = x_1x_2^2x_3^{12}x_4^3x_5$ $a_{32} = x_1x_2^3x_3x_4^2x_5^{12}$
$a_{33} = x_1x_2^3x_3x_4^{12}x_5^2$ $a_{34} = x_1x_2^3x_3^2x_4x_5^{12}$ $a_{35} = x_1x_2^3x_3^2x_4^{12}x_5$ $a_{36} = x_1x_2^3x_3^{12}x_4x_5^2$
$a_{37} = x_1x_2^3x_3^{12}x_4^2x_5$ $a_{38} = x_1^3x_2x_3x_4^2x_5^{12}$ $a_{39} = x_1^3x_2x_3x_4^{12}x_5^2$ $a_{40} = x_1^3x_2x_3^2x_4x_5^{12}$
$a_{41} = x_1^3x_2x_3^2x_4^{12}x_5$ $a_{42} = x_1^3x_2x_3^{12}x_4x_5^2$ $a_{43} = x_1^3x_2x_3^{12}x_4^2x_5$ $a_{44} = x_1^3x_2^{12}x_3x_4x_5^2$
$a_{45} = x_1^3x_2^{12}x_3x_4^2x_5$ $a_{46} = x_1x_2x_3^2x_4^4x_5^{11}$ $a_{47} = x_1x_2^2x_3x_4^4x_5^{11}$ $a_{48} = x_1x_2^2x_3^4x_4x_5^{11}$
$a_{49} = x_1x_2^2x_3^4x_4^{11}x_5$ $a_{50} = x_1x_2x_3^2x_4^5x_5^{10}$ $a_{51} = x_1x_2^2x_3x_4^5x_5^{10}$ $a_{52} = x_1x_2^2x_3^5x_4x_5^{10}$
$a_{53} = x_1x_2^2x_3^5x_4^{10}x_5$ $a_{54} = x_1x_2x_3^2x_4^6x_5^9$ $a_{55} = x_1x_2x_3^6x_4^2x_5^9$ $a_{56} = x_1x_2x_3^6x_4^9x_5^2$
$a_{57} = x_1x_2^2x_3x_4^6x_5^9$ $a_{58} = x_1x_2^6x_3x_4^2x_5^9$ $a_{59} = x_1x_2^6x_3x_4^9x_5^2$ $a_{60} = x_1x_2^6x_3^9x_4x_5^2$
$a_{61} = x_1x_2^6x_3^9x_4^2x_5$ $a_{62} = x_1x_2x_3^2x_4^7x_5^8$ $a_{63} = x_1x_2x_3^7x_4^2x_5^8$ $a_{64} = x_1x_2x_3^7x_4^8x_5^2$
$a_{65} = x_1x_2^2x_3x_4^7x_5^8$ $a_{66} = x_1x_2^2x_3^7x_4x_5^8$ $a_{67} = x_1x_2^2x_3^7x_4^8x_5$ $a_{68} = x_1x_2^7x_3x_4^2x_5^8$
$a_{69} = x_1x_2^7x_3x_4^8x_5^2$ $a_{70} = x_1x_2^7x_3^2x_4x_5^8$ $a_{71} = x_1x_2^7x_3^2x_4^8x_5$ $a_{72} = x_1x_2^7x_3^8x_4x_5^2$
$a_{73} = x_1x_2^7x_3^8x_4^2x_5$ $a_{74} = x_1^7x_2x_3x_4^2x_5^8$ $a_{75} = x_1^7x_2x_3x_4^8x_5^2$ $a_{76} = x_1^7x_2x_3^2x_4x_5^8$
$a_{77} = x_1^7x_2x_3^2x_4^8x_5$ $a_{78} = x_1^7x_2x_3^8x_4x_5^2$ $a_{79} = x_1^7x_2x_3^8x_4^2x_5$ $a_{80} = x_1^7x_2^8x_3x_4x_5^2$
$a_{81} = x_1^7x_2^8x_3x_4^2x_5$ $a_{82} = x_1x_2x_3^3x_4^4x_5^{10}$ $a_{83} = x_1x_2^3x_3x_4^4x_5^{10}$ $a_{84} = x_1x_2^3x_3^4x_4x_5^{10}$
$a_{85} = x_1x_2^3x_3^4x_4^{10}x_5$ $a_{86} = x_1^3x_2x_3x_4^4x_5^{10}$ $a_{87} = x_1^3x_2x_3^4x_4x_5^{10}$ $a_{88} = x_1^3x_2x_3^4x_4^{10}x_5$
$a_{89} = x_1^3x_2^4x_3x_4x_5^{10}$ $a_{90} = x_1^3x_2^4x_3x_4^{10}x_5$ $a_{91} = x_1x_2x_3^3x_4^6x_5^8$ $a_{92} = x_1x_2x_3^6x_4^3x_5^8$
$a_{93} = x_1x_2x_3^6x_4^8x_5^3$ $a_{94} = x_1x_2^3x_3x_4^6x_5^8$ $a_{95} = x_1x_2^3x_3^6x_4x_5^8$ $a_{96} = x_1x_2^3x_3^6x_4^8x_5$
$a_{97} = x_1x_2^6x_3x_4^3x_5^8$ $a_{98} = x_1x_2^6x_3x_4^8x_5^3$ $a_{99} = x_1x_2^6x_3^3x_4x_5^8$ $a_{100} = x_1x_2^6x_3^3x_4^8x_5$
$a_{101} = x_1x_2^6x_3^8x_4x_5^3$ $a_{102} = x_1x_2^6x_3^8x_4^3x_5$ $a_{103} = x_1^3x_2x_3x_4^6x_5^8$ $a_{104} = x_1^3x_2x_3^6x_4x_5^8$
$a_{105} = x_1^3x_2x_3^6x_4^8x_5$ $a_{106} = x_1x_2^2x_3^5x_4^2x_5^9$ $a_{107} = x_1x_2^2x_3^5x_4^9x_5^2$ $a_{108} = x_1x_2^2x_3^3x_4^4x_5^9$
$a_{109} = x_1x_2^2x_3^4x_4^3x_5^9$ $a_{110} = x_1x_2^2x_3^4x_4^9x_5^3$ $a_{111} = x_1x_2^3x_3^2x_4^4x_5^9$ $a_{112} = x_1x_2^3x_3^4x_4^2x_5^9$
$a_{113} = x_1x_2^3x_3^4x_4^9x_5^2$ $a_{114} = x_1^3x_2x_3^2x_4^4x_5^9$ $a_{115} = x_1^3x_2x_3^4x_4^2x_5^9$ $a_{116} = x_1^3x_2x_3^4x_4^9x_5^2$
$a_{117} = x_1^3x_2^4x_3x_4^2x_5^9$ $a_{118} = x_1^3x_2^4x_3x_4^9x_5^2$ $a_{119} = x_1^3x_2^4x_3^9x_4x_5^2$ $a_{120} = x_1^3x_2^4x_3^9x_4^2x_5$
$a_{121} = x_1x_2^2x_3^3x_4^5x_5^8$ $a_{122} = x_1x_2^2x_3^5x_4^3x_5^8$ $a_{123} = x_1x_2^2x_3^5x_4^8x_5^3$ $a_{124} = x_1x_2^3x_3^2x_4^5x_5^8$
$a_{125} = x_1x_2^3x_3^5x_4^2x_5^8$ $a_{126} = x_1x_2^3x_3^5x_4^8x_5^2$ $a_{127} = x_1^3x_2x_3^2x_4^5x_5^8$ $a_{128} = x_1^3x_2x_3^5x_4^2x_5^8$
$a_{129} = x_1^3x_2x_3^5x_4^8x_5^2$ $a_{130} = x_1^3x_2^5x_3x_4^2x_5^8$ $a_{131} = x_1^3x_2^5x_3x_4^8x_5^2$ $a_{132} = x_1^3x_2^5x_3^2x_4x_5^8$
$a_{133} = x_1^3x_2^5x_3^2x_4^8x_5$ $a_{134} = x_1^3x_2^5x_3^8x_4x_5^2$ $a_{135} = x_1^3x_2^5x_3^8x_4^2x_5$ $a_{136} = x_1x_2^3x_3^3x_4^4x_5^8$
$a_{137} = x_1x_2^3x_3^4x_4^3x_5^8$ $a_{138} = x_1x_2^3x_3^4x_4^8x_5^3$ $a_{139} = x_1^3x_2x_3^3x_4^4x_5^8$ $a_{140} = x_1^3x_2x_3^4x_4^3x_5^8$
$a_{141} = x_1^3x_2x_3^4x_4^8x_5^3$ $a_{142} = x_1^3x_2^3x_3x_4^4x_5^8$ $a_{143} = x_1^3x_2^3x_3^4x_4x_5^8$ $a_{144} = x_1^3x_2^3x_3^4x_4^8x_5$
$a_{145} = x_1^3x_2^4x_3x_4^3x_5^8$ $a_{146} = x_1^3x_2^4x_3x_4^8x_5^3$ $a_{147} = x_1^3x_2^4x_3^3x_4x_5^8$ $a_{148} = x_1^3x_2^4x_3^3x_4^8x_5$
$a_{149} = x_1^3x_2^4x_3^8x_4x_5^3$ $a_{150} = x_1^3x_2^4x_3^8x_4^3x_5.$

We need the following for the proof of the proposition.

Lemma 4.4. *The following monomials are strictly inadmissible:*
$x_i^2x_j,\ i<j;\ \ x_i^2x_jx_\ell,\ i<j<\ell;\ \ x_i^2x_jx_\ell x_m^3,\ i<j<\ell,\ m\neq i,j,\ell.$

Proof. We have

$$\begin{aligned} x_i^2x_j &= x_ix_j^2 + Sq^1(x_ix_j),\\ x_i^2x_jx_\ell &= x_ix_j^2x_\ell + x_ix_jx_\ell^2 + Sq^1(x_ix_jx_\ell),\\ x_i^2x_jx_\ell x_m^3 &= x_ix_j^2x_\ell x_m^3 + x_ix_jx_\ell^2x_m^3 + x_ix_jx_\ell x_m^4 + Sq^1(x_ix_jx_\ell x_m^3). \end{aligned}$$

The lemma follows. ☐

Proof of Proposition 4.3. Let x be a monomial in P_5 and $\omega(x) = (3,2,1,1)$. Then x is a permutation of one of the following monomials:

$$\begin{array}{lllll} x_1x_2x_3x_4^2x_5^{14} & x_1x_2x_3x_4^6x_5^{10} & x_1x_2x_3^2x_4^2x_5^{13} & x_1x_2x_3^2x_4^3x_5^{12} & x_1x_2x_3^2x_4^4x_5^{11}\\ x_1x_2x_3^2x_4^5x_5^{10} & x_1x_2x_3^2x_4^6x_5^9 & x_1x_2x_3^2x_4^7x_5^8 & x_1x_2x_3^3x_4^4x_5^{10} & x_1x_2x_3^3x_4^6x_5^8\\ x_1x_2^2x_3^2x_4^5x_5^9 & x_1x_2^2x_3^3x_4^4x_5^9 & x_1x_2^2x_3^3x_4^5x_5^8 & x_1x_2^3x_3^3x_4^4x_5^8. & \end{array}$$

A direct computation using Theorem 2.4 and Lemmas 3.2, 4.4 shows that if $x \neq a_t,\ 1 \leqslant t \leqslant 150$, then x is inadmissible.

Now, we prove that the set $\{[a_t] : 1 \leqslant t \leqslant 150\}$ is linearly independent in $\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1\leqslant t\leqslant 150}\gamma_t a_t \equiv 0,$$

with $\gamma_t \in \mathbb{F}_2$.

By a direct computation from the relations $p_{(i,j)}(\mathcal{S}) \equiv 0, 1 \leqslant i < j \leqslant 5$, and $p_{(1,2,3)}(\mathcal{S}) \equiv 0$, one gets $\gamma_t = 0$ for all t. The proposition is proved. ☐

Proposition 4.5. $(\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5^+)(3,2,3)$ *is an* $\mathbb{F}_2$*-vector space of dimension* 47 *with a basis consisting of all the classes represented by the following admissible monomials:*

$$\begin{array}{llll} b_1 = x_1x_2^2x_3^4x_4^5x_5^7 & b_2 = x_1x_2^2x_3^4x_4^7x_5^5 & b_3 = x_1x_2^2x_3^5x_4^4x_5^7 & b_4 = x_1x_2^2x_3^5x_4^7x_5^4\\ b_5 = x_1x_2^2x_3^7x_4^4x_5^5 & b_6 = x_1x_2^2x_3^7x_4^5x_5^4 & b_7 = x_1x_2^7x_3^2x_4^4x_5^5 & b_8 = x_1x_2^7x_3^2x_4^5x_5^4\\ b_9 = x_1^7x_2x_3^2x_4^4x_5^5 & b_{10} = x_1^7x_2x_3^2x_4^5x_5^4 & b_{11} = x_1x_2^2x_3^5x_4^5x_5^6 & b_{12} = x_1x_2^2x_3^5x_4^6x_5^5\\ b_{13} = x_1x_2^3x_3^4x_4^4x_5^7 & b_{14} = x_1x_2^3x_3^4x_4^7x_5^4 & b_{15} = x_1x_2^3x_3^7x_4^4x_5^4 & b_{16} = x_1x_2^7x_3^3x_4^4x_5^4\\ b_{17} = x_1^3x_2x_3^4x_4^4x_5^7 & b_{18} = x_1^3x_2x_3^4x_4^7x_5^4 & b_{19} = x_1^3x_2x_3^7x_4^4x_5^4 & b_{20} = x_1^3x_2^7x_3x_4^4x_5^4\\ b_{21} = x_1^7x_2x_3^3x_4^4x_5^4 & b_{22} = x_1^7x_2^3x_3x_4^4x_5^4 & b_{23} = x_1x_2^3x_3^4x_4^5x_5^6 & b_{24} = x_1x_2^3x_3^4x_4^6x_5^5\\ b_{25} = x_1x_2^3x_3^5x_4^4x_5^6 & b_{26} = x_1x_2^3x_3^5x_4^6x_5^4 & b_{27} = x_1x_2^3x_3^6x_4^4x_5^5 & b_{28} = x_1x_2^3x_3^6x_4^5x_5^4\\ b_{29} = x_1x_2^6x_3^3x_4^4x_5^5 & b_{30} = x_1x_2^6x_3^3x_4^5x_5^4 & b_{31} = x_1^3x_2x_3^4x_4^5x_5^6 & b_{32} = x_1^3x_2x_3^4x_4^6x_5^5\\ b_{33} = x_1^3x_2x_3^5x_4^4x_5^6 & b_{34} = x_1^3x_2x_3^5x_4^6x_5^4 & b_{35} = x_1^3x_2x_3^6x_4^4x_5^5 & b_{36} = x_1^3x_2x_3^6x_4^5x_5^4\\ b_{37} = x_1^3x_2^5x_3x_4^4x_5^6 & b_{38} = x_1^3x_2^5x_3x_4^6x_5^4 & b_{39} = x_1^3x_2^5x_3^6x_4x_5^4 & b_{40} = x_1^3x_2^5x_3^2x_4^4x_5^5\\ b_{41} = x_1^3x_2^5x_3^2x_4^5x_5^4 & b_{42} = x_1^3x_2^3x_3^4x_4^4x_5^5 & b_{43} = x_1^3x_2^3x_3^4x_4^5x_5^4 & b_{44} = x_1^3x_2^3x_3^5x_4^4x_5^4\\ b_{45} = x_1^3x_2^4x_3^3x_4^4x_5^5 & b_{46} = x_1^3x_2^4x_3^3x_4^5x_5^4 & b_{47} = x_1^3x_2^5x_3^3x_4^4x_5^4. & \end{array}$$

Lemma 4.6. *The following monomials are strictly inadmissible:*

(i) $x_i^2x_j^3x_\ell^3,\ i \neq j \neq \ell \neq i,$

(ii) $x_i^2x_jx_\ell^2x_m^3,\ i<j,\ i,j\neq \ell,m,\ \ell\neq m$,

(iii) $x_i^2x_jx_\ell x_m^2x_n^2,\ (i,j,\ell,m,n)$ is a permutation of $(1,2,3,4,5)$ and $i<j<\ell$.

(iv) $x_i^3x_j^5x_\ell^6x_m^5,\ i<j<\ell,\ m\neq i,j,\ell$,

(v) $x_i^3x_j^4x_\ell^5x_m^7,\ i\neq j\neq \ell\neq m\neq i$,

(vi) $x_i^3x_j^5x_\ell^5x_m^6,\ i<j<\ell<m$,

Lemma 4.7. *Let* (i,j,ℓ,m,n) *be a permutation of* $(1,2,3,4,5)$. *Then the following monomials are strictly inadmissible:*

(i) $x_i^4x_jx_\ell^3x_m^4x_n^7,\ x_i^4x_jx_\ell^3x_m^5x_n^6,\quad i<j$,

(ii) $x_i^3x_j^4x_\ell^5x_m^2x_n^5,\ \ x_i^3x_j^4x_\ell^4x_m^3x_n^5\ i=1,m>3$.

The above lemmas are proved by a direct computation.

Proof of Proposition 4.5. Let x be a monomial in P_5 and $\omega(x)=(3,2,3)$. Then x is a permutation of one of the following monomials:

$$x_1x_2^2x_3^4x_4^5x_5^7,\ x_1x_2^2x_3^5x_4^5x_5^6,\ x_1x_2^3x_3^4x_4^4x_5^7,\ x_1x_2^3x_3^4x_4^5x_5^6,\ x_1^2x_2^3x_3^4x_4^5x_5^5,\ x_1^3x_2^3x_3^4x_4^4x_5^5.$$

By a direct computation using Theorem 2.4 and Lemmas 3.2, 4.4, 4.6, 4.7 we see that if $x\neq b_t, \forall t, 1\leqslant t\leqslant 47$, then x is inadmissible.

Now, we prove that the set $\{[b_t]:1\leqslant i\leqslant 47\}$ is linearly independent in $\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5$. Suppose there is a linear relation

$$\mathcal{S}=\sum_{1\leqslant t\leqslant 47}\gamma_tb_t\equiv 0,$$

with $\gamma_t\in\mathbb{F}_2, 1\leqslant t\leqslant 47$.

Computing directly from the relations $p_{(i,j)}(\mathcal{S})\equiv 0, 1\leqslant i<j\leqslant 5$, gives $\gamma_t=0$ for all $1\leqslant t\leqslant 47$. The proposition is proved. $\square$

Proposition 4.8. $(\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5^+)(3,4,2)$ *is an* $\mathbb{F}_2$*-vector space of dimension* 55 *with a basis consisting of all the classes represented by the following admissible monomials:*

$c_1=x_1x_2^2x_3^2x_4^7x_5^7$	$c_2=x_1x_2^2x_3^7x_4^2x_5^7$	$c_3=x_1x_2^2x_3^7x_4^7x_5^2$	$c_4=x_1x_2^7x_3^2x_4^2x_5^7$
$c_5=x_1x_2^7x_3^2x_4^7x_5^2$	$c_6=x_1x_2^7x_3^7x_4^2x_5^2$	$c_7=x_1^7x_2x_3^2x_4^2x_5^7$	$c_8=x_1^7x_2x_3^2x_4^7x_5^2$
$c_9=x_1^7x_2x_3^7x_4^2x_5^2$	$c_{10}=x_1^7x_2^7x_3x_4^2x_5^2$	$c_{11}=x_1x_2^2x_3^3x_4^6x_5^7$	$c_{12}=x_1x_2^2x_3^3x_4^7x_5^6$
$c_{13}=x_1x_2^2x_3^7x_4^3x_5^6$	$c_{14}=x_1x_2^3x_3^2x_4^6x_5^7$	$c_{15}=x_1x_2^3x_3^2x_4^7x_5^6$	$c_{16}=x_1x_2^3x_3^6x_4^2x_5^7$
$c_{17}=x_1x_2^3x_3^6x_4^7x_5^2$	$c_{18}=x_1x_2^3x_3^7x_4^2x_5^6$	$c_{19}=x_1x_2^3x_3^7x_4^6x_5^2$	$c_{20}=x_1x_2^7x_3^2x_4^3x_5^6$
$c_{21}=x_1x_2^7x_3^3x_4^2x_5^6$	$c_{22}=x_1x_2^7x_3^3x_4^6x_5^2$	$c_{23}=x_1^3x_2x_3^2x_4^6x_5^7$	$c_{24}=x_1^3x_2x_3^2x_4^7x_5^6$
$c_{25}=x_1^3x_2x_3^6x_4^2x_5^7$	$c_{26}=x_1^3x_2x_3^6x_4^7x_5^2$	$c_{27}=x_1^3x_2x_3^7x_4^2x_5^6$	$c_{28}=x_1^3x_2x_3^7x_4^6x_5^2$

$c_{29} = x_1^3x_2^7x_3x_4^2x_5^6$	$c_{30} = x_1^3x_2^7x_3x_4^6x_5^2$	$c_{31} = x_1^7x_2x_3^2x_4^3x_5^6$	$c_{32} = x_1^7x_2x_3^3x_4^2x_5^6$
$c_{33} = x_1^7x_2x_3^3x_4^6x_5^2$	$c_{34} = x_1^7x_2^3x_3x_4^2x_5^6$	$c_{35} = x_1^7x_2^3x_3x_4^6x_5^2$	$c_{36} = x_1x_2^3x_3^3x_4^6x_5^6$
$c_{37} = x_1x_2^3x_3^6x_4^3x_5^6$	$c_{38} = x_1x_2^3x_3^6x_4^6x_5^3$	$c_{39} = x_1^3x_2x_3^3x_4^6x_5^6$	$c_{40} = x_1^3x_2x_3^6x_4^3x_5^6$
$c_{41} = x_1^3x_2x_3^6x_4^6x_5^3$	$c_{42} = x_1^3x_2^3x_3x_4^6x_5^6$	$c_{43} = x_1^3x_2^5x_3^2x_4^2x_5^7$	$c_{44} = x_1^3x_2^5x_3^2x_4^7x_5^2$
$c_{45} = x_1^3x_2^5x_3^7x_4^2x_5^2$	$c_{46} = x_1^3x_2^7x_3^5x_4^2x_5^2$	$c_{47} = x_1^7x_2^3x_3^5x_4^2x_5^2$	$c_{48} = x_1^3x_2^3x_3^5x_4^2x_5^6$
$c_{49} = x_1^3x_2^3x_3^5x_4^6x_5^2$	$c_{50} = x_1^3x_2^5x_3^2x_4^3x_5^6$	$c_{51} = x_1^3x_2^5x_3^2x_4^6x_5^3$	$c_{52} = x_1^3x_2^5x_3^3x_4^2x_5^6$
$c_{53} = x_1^3x_2^5x_3^3x_4^6x_5^2$	$c_{54} = x_1^3x_2^5x_3^6x_4^2x_5^3$	$c_{55} = x_1^3x_2^5x_3^6x_4^3x_5^2.$	

Lemma 4.9. *Let (i,j,ℓ,m,n) be a permutation of $(1,2,3,4,5)$. Then the following monomials are strictly inadmissible:*

(i) $x_i^2x_jx_\ell x_mx_n^3,\ i<j<\ell<m,$

(ii) $x_ix_j^6x_\ell^3x_m^2x_n^7,\ i<j<\ell,\ x_ix_j^6x_\ell^3x_m^3x_n^6,\ i=1, j=2.$

Proof. We prove the lemma for $x = x_1x_2^6x_3^3x_4^2x_5^7$. The others can be proved by a similar computation. A direct computation shows

$$\begin{aligned}x &= x_1x_2^5x_3^4x_4^2x_5^7 + x_1x_2^5x_3^3x_4^2x_5^8 + x_1^2x_2^3x_3^3x_4^2x_5^9 + x_1^2x_2^4x_3^4x_4^2x_5^7 + x_1^2x_2^4x_3^3x_4^2x_5^8\\ &+ x_1^2x_2^3x_3^4x_4^2x_5^8 + x_1^4x_2^4x_3^3x_4x_5^7 + x_1^4x_2^3x_3^4x_4x_5^7 + x_1^4x_2^3x_3^3x_4x_5^8 + x_1x_2^4x_3^3x_4^4x_5^7\\ &+ x_1x_2^3x_3^4x_4^4x_5^7 + x_1x_2^3x_3^3x_4^4x_5^8 + x_1x_2^4x_3^5x_4^2x_5^7 + x_1x_2^3x_3^5x_4^2x_5^8 + x_1x_2^3x_3^6x_4^2x_5^7\\ &+ Sq^1(x_1x_2^5x_3^3x_4^2x_5^7 + x_1^4x_2^3x_3^3x_4x_5^7 + x_1x_2^3x_3^3x_4^4x_5^7 + x_1x_2^3x_3^5x_4^2x_5^7)\\ &+ Sq^2(x_1x_2^3x_3^3x_4^2x_5^7).\end{aligned}$$

Hence x is strictly inadmissible. ∎

Proof of Proposition 4.8. Let x be a monomial in P_5 and $\omega(x) = (3,4,2)$. Then x is a permutation of one of the following monomials:

$$x_1x_2^2x_3^2x_4^7x_5^7,\ \ x_1x_2^2x_3^3x_4^6x_5^7,\ \ x_1x_2^3x_3^3x_4^6x_5^6,\ \ x_1^2x_2^2x_3^3x_4^5x_5^7,\ \ x_1^2x_2^3x_3^3x_4^5x_5^6.$$

By a direct computation using Theorem 2.4 and Lemma 4.9, we see that if $x \neq c_t,\ 1 \leqslant t \leqslant 55$, then x is inadmissible.

Now, we prove that the set $\{[c_t] : 1 \leqslant t \leqslant 55\}$ is linearly independent in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_5$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1\leqslant t\leqslant 55} \gamma_t c_t \equiv 0,$$

with $\gamma_t \in \mathbb{F}_2,\ 1 \leqslant t \leqslant 55$.

By a direct computation from the relations $p_{(i,j)}(\mathcal{S}) \equiv 0, 1 \leqslant i < j \leqslant 5$, one gets $\gamma_t = 0$ for all t. The proposition is proved. ∎

Combining Propositions 1.2, 4.1, 4.3, 4.5, 4.8, we obtain

$$\begin{aligned}B_5(19) = f(B_4(19)) &\cup \{a_i : 1 \leqslant i \leqslant 150\} \cup\\ &\cup \{b_j : 1 \leqslant j \leqslant 47\} \cup \{c_\ell : 1 \leqslant \ell \leqslant 55\} \cup \psi(B_5(7)),\end{aligned}$$

where $\psi : P_5 \to P_5$ is the homomorphism determined by $\psi(x) = x_1x_2x_3x_4x_5x^2$ for all $x \in P_5$. For $s > 3$, we have

$$B_5(2^{s+1} + 2^s - 5) = \psi^{s-3}(B_5(19)).$$

Theorem 1.3 is proved.

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