East-West J. of Mathematics: Vol. 16, No 1 (2014) pp. 34-46

THE ADMISSIBLE MONOMIAL BASIS FOR THE POLYNOMIAL ALGEBRA OF FIVE VARIABLES IN DEGREE $2^{s+1} + 2^s - 5$

Nguyen Khac Tin

Department of Basic Sciences,
University of Technical Education of Ho Chi Minh City,
01, Vo Van Ngan, Thu Duc District, Ho Chi Minh City.
e-mail: tinnk@hcmute.edu.vn

Abstract

Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in k variables $x_1, x_2, \ldots, x_k$, each of degree 1. We study the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for P_k as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we explicitly determine all admissible monomials for the case $k = 5$ in degree $2^{s+1} + 2^s - 5$ with s an arbitrary positive integer.

1 Introduction

Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in k variables $x_1, x_2, \ldots, x_k$, each of degree 1. The mod-2 Steenrod algebra $\mathcal{A}$ acts on P_k by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and subject to the Cartan formula $Sq^n(fg) = \sum_{i=0}^{n} Sq^i(f) Sq^{n-i}(g)$, for $f, g \in P_k$ (see Steenrod-Epstein [12]).

Key words: Steenrod squares, Polynomial algebra, Hit problem.
2010 AMS Mathematics Subject Classification: Primary 55S10; 55S05.

Many authors study the *hit problem* of determination of the minimal set of generators for P_k as a module over the Steenrod algebra, or equivalently, a basis of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

This problem has first been studied by Peterson [8], Wood [16], Singer [11], Priddy [9], who show its relationship to several classical problems in homotopy theory.

The tensor product $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ has explicitly been calculated by Peterson [8] for $k = 2$, Kameko for $k = 3$ in his thesis [5] and Sum [14] for $k = 4$.

The hit problem were then investigated by many authors. (See Bruner-Ha-Hung [1], Carlisle-Wood [2], Crabb-Hubbuck [3], Hung [4], Mothebe [6], Nam [7], Singer [11], Silverman [10], Walker-Wood [15] and others.)

Peterson conjectured in [8] that as a module over the Steenrod algebra $\mathcal{A}$, P_k is generated by monomials in degrees n that satisfy $\alpha(n+k) \leqslant k$, where $\alpha(n)$ denotes the number of ones in dyadic expansion of n, and proved it for $k \leqslant 2$. The conjecture was established in general by Wood [16].

For any nonnegative integer n, set $\mu(n) = \min\{m \in \mathbb{Z} : \alpha(n+m) \leqslant m\}$. Denote by $(P_k)_n$ and $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n$ the subspaces of degree n homogeneous polynomials in the spaces P_k and $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ respectively.

Theorem 1.1 ([Wood [16]). *If* $\mu(n) > k$, *then* $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n = 0$.

From Theorem 1.1, the hit problem is reduced to the cases of degree n with $\mu(n) \leqslant k$.

In this paper, we explicitly determine all the admissible monomials (see Section 2) of P_5 in degree $n = 2^{s+1} + 2^s - 5$ with s an arbitrary positive integer.

For $s = 1$, the problem is easy. There exist exactly 5 admissible monomials of degree 1 in P_5, namely: x_1, x_2, x_3, x_4, x_5. For $s = 2$, we have

Proposition 1.2. *There exist exactly* 110 *admissible monomials of degree* 7 *in* P_5. *Consequently* $\dim((\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7 = 110$.

The main result of the paper is the following:

Theorem 1.3. *For any integer* $s \geqslant 3$, *there exist exactly* 912 *admissible monomials of degree* $2^{s+1}+2^s-5$ *in* P_5. *Consequently* $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{2^{s+1}+2^s-5} = 912$.

In Section 2, we recall some results on the admissible monomials and hit monomials in P_k. We prove Proposition 1.2 in Section 3. Theorem 1.3 will be proved in Section 4.

2 Preliminaries

In this section, we recall some results in Kameko [5], Singer [11] and Sum [13] on the admissible monomials and the hit monomials in P_k.

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a nonnegative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0$ or 1 and $i \geqslant 0$.

Let $x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \in P_k$. Following Kameko [5], we define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots),$$
$$\sigma(x) = (a_1, a_2, \ldots, a_k),$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant k} \alpha_{i-1}(a_j),\ i \geqslant 1$.

Definition 2.1. Let x, y be the monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds

1. $\omega(x) < \omega(y)$,
2. $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Here, the order on the set of sequences of nonnegative integers is the lexicographical one.

Let f, g be homogeneous polynomials of the same degree in P_k. We denote $f \equiv g$ if and only if $f - g \in \mathcal{A}^+P_k$. If $f \equiv 0$, then f is called hit.

Definition 2.2. A monomial x is said to be inadmissible if there exists the monomials $y_1, y_2, \ldots, y_r$ such that

$$x \equiv y_1 + y_2 + \ldots + y_r \text{ and } y_j < x,\ j = 1, 2, \ldots, r.$$

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all admissible monomials in P_k is a minimal set of $\mathcal{A}$-generators of P_k.

Definition 2.3. A monomial x is said to be strictly inadmissible if and only if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$, for $j = 1, 2, \ldots, t$ and

$$x = y_1 + y_2 + \ldots + y_t + \sum_{1\leqslant i<2^s} Sq^i(f_i),$$

with $s \leqslant \max\{i\ ; \omega_i(x) > 0\}$ and $f_i \in P_k$.

The following theorem is one of our main tools.

Theorem 2.4 (Kameko [5], Sum [13]). *Let x, y, w be monomials in P_k such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*

i) *If w is inadmissible, then xw^{2^r} is also inadmissible.*

ii) *If w is strictly inadmissible, then $xw^{2^r}y^{2^{r+s}}$ is inadmissible.*

Now, we recall a result of Singer [11] on the hit monomials in P_k.

Definition 2.5. A monomial $z = x_1^{a_1}x_2^{a_2}\dots x_k^{a_k}$ is called the spike if $b_j = 2^{s_j} - 1$ for s_j a nonnegative integer and $j = 1, 2, \dots, k$. If z is the spike with $s_1 > s_2 > \dots > s_{r-1} \geqslant s_r > 0$ and $s_j = 0$ for $j > r$ then it is called the minimal spike.

The following is a criterion for the hit monomials in P_k.

Theorem 2.6 (Singer [11]). *Suppose $x \in P_k$ is a monomial of degree n, where $\alpha(n+k) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$ then x is hit.*

One of the main tools in the study of the hit problem is Kameko's squaring operation $\widetilde{Sq}^0_* : \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \to \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. This homomorphism is induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_* : P_k \to P_k$, given by

$$\widetilde{Sq}^0_*(x) = \begin{cases} y, & \text{if } x = x_1x_2\dots x_ky^2,\\ 0, & \text{otherwise,}\end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However, $\widetilde{Sq}^0_*Sq^{2t} = Sq^t\widetilde{Sq}^0_*$, and $\widetilde{Sq}^0_*Sq^{2t+1} = 0$ for any nonnegative integer t.

Theorem 2.7 (Kameko [5]). *Let m be a positive integer. If $\mu(2m+k) = k$, then $(\widetilde{Sq}^0_*)_m : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{2m+k} \to (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_m$ is an isomorphism of $\mathbb{F}_2$-vector spaces.*

For latter use, we set

$$\begin{aligned} P_k^0 &= \langle\{x = x_1^{a_1}x_2^{a_2}\dots x_k^{a_k}\ ;\ a_1a_2\dots a_k = 0\}\rangle,\\ P_k^+ &= \langle\{x = x_1^{a_1}x_2^{a_2}\dots x_k^{a_k}\ ;\ a_1a_2\dots a_k > 0\}\rangle.\end{aligned}$$

It is easy to see that P_k^0 and P_k^+ are the $\mathcal{A}$-submodules of P_k. Furthermore, we have

$$\mathbb{F}_2 \otimes_{\mathcal{A}} P_k = (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k^0) \oplus (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+).$$

For a polynomial f in P_k, we denote by $[f]$ the class in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ represented by f.

For $1 \leqslant i \leqslant k$, define the homomorphism $f_i = f_{k;i} : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i,\\ x_{j+1}, & \text{if } i \leqslant j < k.\end{cases}$$

It is easy to see that

Proposition 2.8. *If $B_{k-1}(n)$ is the set of all admissible monomials of degree n in P_{k-1}, then $f(B_{k-1}(n)) := \cup_{1\leqslant i\leqslant k} f_i(B_{k-1}(n))$ is the set of all admissible monomials of degree n in P_k^0.*

For any $I = (i_0, i_1, \ldots, i_r)$, $0 < i_0 < i_1 < \ldots < i_r \leqslant k$, $0 \leqslant r < k$, we define the homomorphism $p_I : P_k \to P_{k-1}$ of algebras by substituting

$$p_I(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i_0, \\ \sum_{1 \leqslant s \leqslant r} x_{i_s - 1}, & \text{if } j = i_0, \\ x_{j-1}, & \text{if } i_0 < j \leqslant k. \end{cases}$$

Then p_I is a homomorphism of $\mathcal{A}$-modules. In particular, for $I = (i)$, we have $p_{(i)}(x_i) = 0$.

3 Proof of Proposition 1.2

From now on, denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k.

According to a result in Sum [14], there exist exactly 35 admissible monomials of degree 7 in P_4, namely:

$x_i^7,\ 1 \leqslant i \leqslant 4,\ x_1x_2^2x_3^2x_4^2,$
$x_ix_j^6,\ 1 \leqslant i < j \leqslant 4,$
$x_ix_j^2x_\ell^4,\ 1 \leqslant i < j < \ell \leqslant 4,$
$x_ix_j^3x_\ell^3,\ 1 \leqslant i, j, \ell \leqslant 4,\ i \neq j \neq \ell \neq i,$
$x_ix_j^2x_\ell x_m,\ (i, j, \ell, m)$ is a permutation of (1,2,3,4) such that $i < j$.

By a direct computation we see that $f(B_4(7))$ is the set consisting of 100 admissible monomials in $(P_5^0)_7$. Now we determine all admissible monomials in $(P_5^+)_7$.

Lemma 3.1. *If x is an admissible monomials of degree 7 in P_5 then either $\omega(x) = (3, 2)$ or $\omega(x) = (5, 1)$.*

Proof. Since $\omega_1(x)$ is odd we have $\omega_1(x) = 1$ or $\omega_1(x) = 3$ or $\omega_1(x) = 5$.

If $\omega_1(x) = 1$, then $x = x_iy^2$ with y a monomial of degree 3 in P_5. This contradicts the fact that $x \in P_5^+$.

If $\omega_1(x) = 3$, then $x = x_ix_jx_\ell y^2$ with y a monomial of degree 2 in P_5. Since x is admissible, according to Theorem 2.4, y is admissible. Hence $y = x_mx_n$, where (i, j, ℓ, m, n) is a permutation of $(1, 2, 3, 4, 5)$. So we have $\omega(x) = (3, 2)$.

If $\omega_1(x) = 5$, then $x = x_1x_2x_3x_4x_5x_i^2,\ 1 \leqslant i \leqslant 5$. Hence $\omega(x) = (5, 1)$. $\square$

Lemma 3.2. *The following monomials are strictly inadmissible:*

$$x_1x_2^2x_3^2x_4x_5, x_1^2x_i^2x_jx_\ell x_m,\ (i, j, \ell, m) \text{ is a permutation of } (2, 3, 4, 5).$$

Proof. By a direct computation, we have

$$\begin{aligned} x_1x_2^2x_3^2x_4x_5 &= x_1x_2^2x_3x_4^2x_5 + x_1x_2^2x_3x_4x_5^2 + x_1x_2x_3^2x_4^2x_5 + x_1x_2x_3^2x_4x_5^2 \\ &\quad + x_1x_2x_3x_4^2x_5^2 + Sq^1(x_1^2x_2x_3x_4x_5) + Sq^2(x_1x_2x_3x_4x_5), \\ x_1^2x_i^2x_jx_\ell x_m &= x_1x_i^2x_j^2x_\ell x_m + x_1x_i^2x_jx_\ell^2x_m + x_1x_i^2x_jx_\ell x_m^2 + Sq^1(x_1x_i^2x_jx_\ell x_m). \end{aligned}$$

The lemma is proved. $\square$

Let x be a monomial of degee 7 in P_5^+. If $\omega(x) = (5,1)$ then x is a spike. Hence x is admissible. If x is admissible and $\omega(x) = (3,2)$, then by Lemma 3.2, x is one of 5 monomials:

$$\begin{aligned} &a_1 = x_1x_2x_3x_4^2x_5^2,\ a_2 = x_1x_2x_3^2x_4x_5^2,\ a_3 = x_1x_2x_3^2x_4^2x_5, \\ &a_4 = x_1x_2^2x_3x_4x_5^2,\ a_5 = x_1x_2^2x_3x_4^2x_5. \end{aligned}$$

Now we prove that the set $\{[a_i],\ 1 \leqslant i \leqslant 5\}$ is linearly independent in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_5$. Suppose that

$$\mathcal{S} = \gamma_1a_1 + \gamma_2a_2 + \gamma_3a_3 + \gamma_4a_4 + \gamma_5a_5 \equiv 0.$$

By a direct computation, we have

$$\begin{aligned} p_{(1,2)}(\mathcal{S}) &\equiv (\gamma_1+\gamma_2+\gamma_3)x_1x_2^2x_3^2x_4^2 + \gamma_4x_1^3x_2x_3x_4^2 + \gamma_5x_1^3x_2x_3^2x_4 \equiv 0, \\ p_{(1,3)}(\mathcal{S}) &\equiv (\gamma_1+\gamma_4+\gamma_5)x_1x_2^2x_3^2x_4^2 + \gamma_2x_1x_2^3x_3x_4^2 + \gamma_3x_1x_2^3x_3^2x_4 \equiv 0. \end{aligned}$$

From these above equalities, one gets $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 0$. The proposition is proved.

4 Proof of Theorem 1.3

Note that

$$2^{s+1} + 2^s - 5 = 2^{s+1} + 2^{s-1} + 2^{s-2} + 2^{s-3} + 2^{s-3} - 5.$$

If $s > 3$, then $\mu(2^{s+1} + 2^s - 5) = 5$. According to Theorem 2.7, the squaring operation

$$\widetilde{Sq}^0_* : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{2^{s+1}+2^s-5} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{2^s+2^{s-1}-5}$$

is an isomorphism. Hence $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{2^{s+1}+2^s-5} \cong (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{19}$ for any $s > 3$. So, we need only to prove the theorem for $s = 3$.

Since the squaring operation $\widetilde{Sq}^0_* : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{19} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7$ is an epimorphism, we have

$$\begin{aligned} (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{19} &\cong \mathrm{Ker}(\widetilde{Sq}^0_*) \oplus (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7 \\ &= (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^0)_{19} \oplus (\mathrm{Ker}(\widetilde{Sq}^0_*) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{19}) \oplus (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7. \end{aligned}$$

According to a result in Sum [14], there exist exactly 140 admissible monomials of degree 19 in P_4. A direct computation using Proposition 2.8, one gets

Proposition 4.1. *There exist exactly* 550 *admissible monomials of degree* 19 *in* P_5^0. *Consequently* $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^0)_{19} = 550$.

Now we explicitly determine $\operatorname{Ker}(\widetilde{Sq}^0_*) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{19}$.

Lemma 4.2. *Let x be an admissible monomial of degree* 19 *in* P_5. *If* $[x] \in \operatorname{Ker}(\widetilde{Sq}^0_*)$, *then either* $\omega(x) = (3,2,1,1)$ *or* $\omega(x) = (3,2,3)$ *or* $\omega(x) = (3,4,2)$.

Proof. Observe that $z = x_1^{15}x_2^3x_3$ is the minimal spike of degree 9 in P_5 and $\omega(z) = (3,2,1,1)$. Since $\omega_1(x)$ is odd, using Theorem 2.6 and the fact that $[x] \in \operatorname{Ker}\widetilde{Sq}^0_*$, we obtain $\omega_1(x) = 3$. Hence $x = x_ix_jx_\ell y^2$, where y is a monomial of degree 8 and $1 \leqslant i < j < \ell \leqslant 5$. Since x is admissible, by Theorem 2.4, y is also admissible. Applying Theorem 2.6, we see that either $\omega_1(y) = 2$ or $\omega_1(y) = 4$. If $\omega_1(y) = 2$, then $y = x_ix_jy_1^2$ with y_1 a monomial of degree 3 in P_5 and $i < j$. Then either $\omega(y_1) = (1,1)$ or $\omega(y_1) = (3)$. If $\omega_1(y) = 4$, then $y = x_ix_jx_\ell y_2^2$ with y_2 a monomial of degree 2 in P_5 and $i < j < \ell$. Then $\omega(y_2) = (2)$. The lemma is proved. $\square$

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of nonnegative integers such that $\omega_i = 0$ for $i \gg 0$. Define $\deg \omega = \sum_{i>0} 2^{i-1}\omega_i$. We set

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)(\omega) = \operatorname{Span}\{[x] \in \mathbb{F}_2 \otimes_{\mathcal{A}} P_k : x \text{ is an admissible monomial and } \omega(x) = \omega\},$$

and $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+)(\omega) = ((\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)(\omega)) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+)$. It is easy to see that

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n = \bigoplus_{\deg \omega = n} (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)(\omega).$$

Applying Lemma 4.2, one gets

$$\begin{aligned}\operatorname{Ker}(\widetilde{Sq}^0_*) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{19} = ((\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)(3,2,1,1)) \oplus \\ \oplus ((\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)(3,2,3)) \oplus ((\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)(3,4,2)).\end{aligned}$$

Proposition 4.3. $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)(3,2,1,1)$ *is an* $\mathbb{F}_2$*-vector space of dimension* 150 *with a basis consisting of all the classes represented by the following admissible monomials:*

$a_1 = x_1x_2x_3x_4^2x_5^{14}$ $a_2 = x_1x_2x_3x_4^{14}x_5^2$ $a_3 = x_1x_2x_3^2x_4x_5^{14}$ $a_4 = x_1x_2x_3^2x_4^{14}x_5$
$a_5 = x_1x_2x_3^{14}x_4x_5^2$ $a_6 = x_1x_2x_3^{14}x_4^2x_5$ $a_7 = x_1x_2^2x_3x_4x_5^{14}$ $a_8 = x_1x_2^2x_3x_4^{14}x_5$
$a_9 = x_1x_2^{14}x_3x_4x_5^2$ $a_{10} = x_1x_2^{14}x_3x_4^2x_5$ $a_{11} = x_1x_2x_3x_4^6x_5^{10}$ $a_{12} = x_1x_2x_3^6x_4x_5^{10}$
$a_{13} = x_1x_2x_3^6x_4^{10}x_5$ $a_{14} = x_1x_2^6x_3x_4x_5^{10}$ $a_{15} = x_1x_2^6x_3x_4^{10}x_5$ $a_{16} = x_1x_2x_3^2x_4^2x_5^{13}$
$a_{17} = x_1x_2x_3^2x_4^{13}x_5^2$ $a_{18} = x_1x_2^2x_3x_4^2x_5^{13}$ $a_{19} = x_1x_2^2x_3x_4^{13}x_5^2$ $a_{20} = x_1x_2^2x_3^{13}x_4x_5^2$
$a_{21} = x_1x_2^2x_3^{13}x_4^2x_5$ $a_{22} = x_1x_2x_3^2x_4^3x_5^{12}$ $a_{23} = x_1x_2x_3^2x_4^{12}x_5^3$ $a_{24} = x_1x_2x_3^3x_4^2x_5^{12}$
$a_{25} = x_1x_2x_3^3x_4^{12}x_5^2$ $a_{26} = x_1x_2^2x_3x_4^3x_5^{12}$ $a_{27} = x_1x_2^2x_3x_4^{12}x_5^3$ $a_{28} = x_1x_2^2x_3^3x_4x_5^{12}$
$a_{29} = x_1x_2^2x_3^3x_4^{12}x_5$ $a_{30} = x_1x_2^2x_3^{12}x_4x_5^3$ $a_{31} = x_1x_2^2x_3^{12}x_4^3x_5$ $a_{32} = x_1x_2^3x_3x_4^2x_5^{12}$
$a_{33} = x_1x_2^3x_3x_4^{12}x_5^2$ $a_{34} = x_1x_2^3x_3^2x_4x_5^{12}$ $a_{35} = x_1x_2^3x_3^2x_4^{12}x_5$ $a_{36} = x_1x_2^3x_3^{12}x_4x_5^2$
$a_{37} = x_1x_2^3x_3^{12}x_4^2x_5$ $a_{38} = x_1^3x_2x_3x_4^2x_5^{12}$ $a_{39} = x_1^3x_2x_3x_4^{12}x_5^2$ $a_{40} = x_1^3x_2x_3^2x_4x_5^{12}$
$a_{41} = x_1^3x_2x_3^2x_4^{12}x_5$ $a_{42} = x_1^3x_2x_3^{12}x_4x_5^2$ $a_{43} = x_1^3x_2x_3^{12}x_4^2x_5$ $a_{44} = x_1^3x_2^{12}x_3x_4x_5^2$
$a_{45} = x_1^3x_2^{12}x_3x_4^2x_5$ $a_{46} = x_1x_2x_3^2x_4^4x_5^{11}$ $a_{47} = x_1x_2^2x_3x_4^4x_5^{11}$ $a_{48} = x_1x_2^2x_3^4x_4x_5^{11}$
$a_{49} = x_1x_2^2x_3^4x_4^{11}x_5$ $a_{50} = x_1x_2x_3^2x_4^5x_5^{10}$ $a_{51} = x_1x_2^2x_3x_4^5x_5^{10}$ $a_{52} = x_1x_2^2x_3^5x_4x_5^{10}$
$a_{53} = x_1x_2^2x_3^5x_4^{10}x_5$ $a_{54} = x_1x_2x_3^2x_4^6x_5^9$ $a_{55} = x_1x_2x_3^6x_4^2x_5^9$ $a_{56} = x_1x_2x_3^6x_4^9x_5^2$
$a_{57} = x_1x_2^2x_3x_4^6x_5^9$ $a_{58} = x_1x_2^6x_3x_4^2x_5^9$ $a_{59} = x_1x_2^6x_3x_4^9x_5^2$ $a_{60} = x_1x_2^6x_3^9x_4x_5^2$
$a_{61} = x_1x_2^6x_3^9x_4^2x_5$ $a_{62} = x_1x_2x_3^2x_4^7x_5^8$ $a_{63} = x_1x_2x_3^7x_4^2x_5^8$ $a_{64} = x_1x_2x_3^7x_4^8x_5^2$
$a_{65} = x_1x_2^2x_3x_4^7x_5^8$ $a_{66} = x_1x_2^2x_3^7x_4x_5^8$ $a_{67} = x_1x_2^2x_3^7x_4^8x_5$ $a_{68} = x_1x_2^7x_3x_4^2x_5^8$
$a_{69} = x_1x_2^7x_3x_4^8x_5^2$ $a_{70} = x_1x_2^7x_3^2x_4x_5^8$ $a_{71} = x_1x_2^7x_3^2x_4^8x_5$ $a_{72} = x_1x_2^7x_3^8x_4x_5^2$
$a_{73} = x_1x_2^7x_3^8x_4^2x_5$ $a_{74} = x_1^7x_2x_3x_4^2x_5^8$ $a_{75} = x_1^7x_2x_3x_4^8x_5^2$ $a_{76} = x_1^7x_2x_3^2x_4x_5^8$
$a_{77} = x_1^7x_2x_3^2x_4^8x_5$ $a_{78} = x_1^7x_2x_3^8x_4x_5^2$ $a_{79} = x_1^7x_2x_3^8x_4^2x_5$ $a_{80} = x_1^7x_2^8x_3x_4x_5^2$
$a_{81} = x_1^7x_2^8x_3x_4^2x_5$ $a_{82} = x_1x_2x_3^3x_4^4x_5^{10}$ $a_{83} = x_1x_2^3x_3x_4^4x_5^{10}$ $a_{84} = x_1x_2^3x_3^4x_4x_5^{10}$
$a_{85} = x_1x_2^3x_3^4x_4^{10}x_5$ $a_{86} = x_1^3x_2x_3x_4^4x_5^{10}$ $a_{87} = x_1^3x_2x_3^4x_4x_5^{10}$ $a_{88} = x_1^3x_2x_3^4x_4^{10}x_5$
$a_{89} = x_1^3x_2^4x_3x_4x_5^{10}$ $a_{90} = x_1^3x_2^4x_3x_4^{10}x_5$ $a_{91} = x_1x_2x_3^3x_4^6x_5^8$ $a_{92} = x_1x_2x_3^6x_4^3x_5^8$
$a_{93} = x_1x_2x_3^6x_4^8x_5^3$ $a_{94} = x_1x_2^3x_3x_4^6x_5^8$ $a_{95} = x_1x_2^3x_3^6x_4x_5^8$ $a_{96} = x_1x_2^3x_3^6x_4^8x_5$
$a_{97} = x_1x_2^6x_3x_4^3x_5^8$ $a_{98} = x_1x_2^6x_3x_4^8x_5^3$ $a_{99} = x_1x_2^6x_3^3x_4x_5^8$ $a_{100} = x_1x_2^6x_3^3x_4^8x_5$
$a_{101} = x_1x_2^6x_3^8x_4x_5^3$ $a_{102} = x_1x_2^6x_3^8x_4^3x_5$ $a_{103} = x_1^3x_2x_3x_4^6x_5^8$ $a_{104} = x_1^3x_2x_3^6x_4x_5^8$
$a_{105} = x_1^3x_2x_3^6x_4^8x_5$ $a_{106} = x_1x_2^2x_3^5x_4^2x_5^9$ $a_{107} = x_1x_2^2x_3^5x_4^9x_5^2$ $a_{108} = x_1x_2^2x_3^3x_4^4x_5^9$
$a_{109} = x_1x_2^2x_3^4x_4^3x_5^9$ $a_{110} = x_1x_2^2x_3^4x_4^9x_5^3$ $a_{111} = x_1x_2^3x_3^2x_4^4x_5^9$ $a_{112} = x_1x_2^3x_3^4x_4^2x_5^9$
$a_{113} = x_1x_2^3x_3^4x_4^9x_5^2$ $a_{114} = x_1^3x_2x_3^2x_4^4x_5^9$ $a_{115} = x_1^3x_2x_3^4x_4^2x_5^9$ $a_{116} = x_1^3x_2x_3^4x_4^9x_5^2$
$a_{117} = x_1^3x_2^4x_3x_4^2x_5^9$ $a_{118} = x_1^3x_2^4x_3x_4^9x_5^2$ $a_{119} = x_1^3x_2^4x_3^9x_4x_5^2$ $a_{120} = x_1^3x_2^4x_3^9x_4^2x_5$
$a_{121} = x_1x_2^2x_3^3x_4^5x_5^8$ $a_{122} = x_1x_2^2x_3^5x_4^3x_5^8$ $a_{123} = x_1x_2^2x_3^5x_4^8x_5^3$ $a_{124} = x_1x_2^3x_3^2x_4^5x_5^8$
$a_{125} = x_1x_2^3x_3^5x_4^2x_5^8$ $a_{126} = x_1x_2^3x_3^5x_4^8x_5^2$ $a_{127} = x_1^3x_2x_3^2x_4^5x_5^8$ $a_{128} = x_1^3x_2x_3^5x_4^2x_5^8$
$a_{129} = x_1^3x_2x_3^5x_4^8x_5^2$ $a_{130} = x_1^3x_2^5x_3x_4^2x_5^8$ $a_{131} = x_1^3x_2^5x_3x_4^8x_5^2$ $a_{132} = x_1^3x_2^5x_3^2x_4x_5^8$
$a_{133} = x_1^3x_2^5x_3^2x_4^8x_5$ $a_{134} = x_1^3x_2^5x_3^8x_4x_5^2$ $a_{135} = x_1^3x_2^5x_3^8x_4^2x_5$ $a_{136} = x_1x_2^3x_3^3x_4^4x_5^8$
$a_{137} = x_1x_2^3x_3^4x_4^3x_5^8$ $a_{138} = x_1x_2^3x_3^4x_4^8x_5^3$ $a_{139} = x_1^3x_2x_3^3x_4^4x_5^8$ $a_{140} = x_1^3x_2x_3^4x_4^3x_5^8$
$a_{141} = x_1^3x_2x_3^4x_4^8x_5^3$ $a_{142} = x_1^3x_2^3x_3x_4^4x_5^8$ $a_{143} = x_1^3x_2^3x_3^4x_4x_5^8$ $a_{144} = x_1^3x_2^3x_3^4x_4^8x_5$
$a_{145} = x_1^3x_2^4x_3x_4^3x_5^8$ $a_{146} = x_1^3x_2^4x_3x_4^8x_5^3$ $a_{147} = x_1^3x_2^4x_3^3x_4x_5^8$ $a_{148} = x_1^3x_2^4x_3^3x_4^8x_5$
$a_{149} = x_1^3x_2^4x_3^8x_4x_5^3$ $a_{150} = x_1^3x_2^4x_3^8x_4^3x_5$.

We need the following for the proof of the proposition.

Lemma 4.4. *The following monomials are strictly inadmissible:*
$x_i^2x_j,\ i<j;\ \ x_i^2x_jx_\ell,\ i<j<\ell;\ \ x_i^2x_jx_\ell x_m^3,\ i<j<\ell,\ m\neq i,j,\ell.$

Proof. We have

$$\begin{aligned}
x_i^2x_j &= x_ix_j^2 + Sq^1(x_ix_j),\\
x_i^2x_jx_\ell &= x_ix_j^2x_\ell + x_ix_jx_\ell^2 + Sq^1(x_ix_jx_\ell),\\
x_i^2x_jx_\ell x_m^3 &= x_ix_j^2x_\ell x_m^3 + x_ix_jx_\ell^2x_m^3 + x_ix_jx_\ell x_m^4 + Sq^1(x_ix_jx_\ell x_m^3).
\end{aligned}$$

The lemma follows. □

Proof of Proposition 4.3. Let x be a monomial in P_5 and $\omega(x) = (3,2,1,1)$. Then x is a permutation of one of the following monomials:

$$\begin{array}{lllll}
x_1x_2x_3x_4^2x_5^{14} & x_1x_2x_3x_4^6x_5^{10} & x_1x_2x_3^2x_4^2x_5^{13} & x_1x_2x_3^2x_4^3x_5^{12} & x_1x_2x_3^2x_4^4x_5^{11}\\
x_1x_2x_3^2x_4^5x_5^{10} & x_1x_2x_3^2x_4^6x_5^{9} & x_1x_2x_3^2x_4^7x_5^{8} & x_1x_2x_3^3x_4^4x_5^{10} & x_1x_2x_3^3x_4^6x_5^{8}\\
x_1x_2^2x_3^2x_4^5x_5^{9} & x_1x_2^2x_3^3x_4^4x_5^{9} & x_1x_2^2x_3^3x_4^5x_5^{8} & x_1x_2^3x_3^3x_4^4x_5^{8}. &
\end{array}$$

A direct computation using Theorem 2.4 and Lemmas 3.2, 4.4 shows that if $x \neq a_t,\ 1 \leqslant t \leqslant 150$, then x is inadmissible.

Now, we prove that the set $\{[a_t] : 1 \leqslant t \leqslant 150\}$ is linearly independent in $\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1\leqslant t\leqslant 150} \gamma_t a_t \equiv 0,$$

with $\gamma_t \in \mathbb{F}_2$.

By a direct computation from the relations $p_{(i,j)}(\mathcal{S}) \equiv 0, 1 \leqslant i < j \leqslant 5$, and $p_{(1,2,3)}(\mathcal{S}) \equiv 0$, one gets $\gamma_t = 0$ for all t. The proposition is proved. □

Proposition 4.5. $(\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5^+)(3,2,3)$ *is an* $\mathbb{F}_2$*-vector space of dimension* 47 *with a basis consisting of all the classes represented by the following admissible monomials:*

$$\begin{array}{llll}
b_1 = x_1x_2^2x_3^4x_4^5x_5^7 & b_2 = x_1x_2^2x_3^4x_4^7x_5^5 & b_3 = x_1x_2^2x_3^5x_4^4x_5^7 & b_4 = x_1x_2^2x_3^5x_4^7x_5^4\\
b_5 = x_1x_2^2x_3^7x_4^4x_5^5 & b_6 = x_1x_2^2x_3^7x_4^5x_5^4 & b_7 = x_1x_2^7x_3^2x_4^4x_5^5 & b_8 = x_1x_2^7x_3^2x_4^5x_5^4\\
b_9 = x_1^7x_2x_3^2x_4^4x_5^5 & b_{10} = x_1^7x_2x_3^2x_4^5x_5^4 & b_{11} = x_1x_2^2x_3^5x_4^5x_5^6 & b_{12} = x_1x_2^2x_3^5x_4^6x_5^5\\
b_{13} = x_1x_2^3x_3^4x_4^4x_5^7 & b_{14} = x_1x_2^3x_3^4x_4^7x_5^4 & b_{15} = x_1x_2^3x_3^7x_4^4x_5^4 & b_{16} = x_1x_2^7x_3^3x_4^4x_5^4\\
b_{17} = x_1^3x_2x_3^4x_4^4x_5^7 & b_{18} = x_1^3x_2x_3^4x_4^7x_5^4 & b_{19} = x_1^3x_2x_3^7x_4^4x_5^4 & b_{20} = x_1^3x_2^7x_3x_4^4x_5^4\\
b_{21} = x_1^7x_2x_3^3x_4^4x_5^4 & b_{22} = x_1^7x_2^3x_3x_4^4x_5^4 & b_{23} = x_1x_2^3x_3^4x_4^5x_5^6 & b_{24} = x_1x_2^3x_3^4x_4^6x_5^5\\
b_{25} = x_1x_2^3x_3^5x_4^4x_5^6 & b_{26} = x_1x_2^3x_3^5x_4^6x_5^4 & b_{27} = x_1x_2^3x_3^6x_4^4x_5^5 & b_{28} = x_1x_2^3x_3^6x_4^5x_5^4\\
b_{29} = x_1x_2^6x_3^3x_4^4x_5^5 & b_{30} = x_1x_2^6x_3^3x_4^5x_5^4 & b_{31} = x_1^3x_2x_3^4x_4^5x_5^6 & b_{32} = x_1^3x_2x_3^4x_4^6x_5^5\\
b_{33} = x_1^3x_2x_3^5x_4^4x_5^6 & b_{34} = x_1^3x_2x_3^5x_4^6x_5^4 & b_{35} = x_1^3x_2x_3^6x_4^4x_5^5 & b_{36} = x_1^3x_2x_3^6x_4^5x_5^4\\
b_{37} = x_1^3x_2^5x_3x_4^4x_5^6 & b_{38} = x_1^3x_2^5x_3x_4^6x_5^4 & b_{39} = x_1^3x_2^5x_3^6x_4x_5^4 & b_{40} = x_1^3x_2^5x_3^2x_4^4x_5^5\\
b_{41} = x_1^3x_2^5x_3^2x_4^5x_5^4 & b_{42} = x_1^3x_2^3x_3^4x_4^4x_5^5 & b_{43} = x_1^3x_2^3x_3^4x_4^5x_5^4 & b_{44} = x_1^3x_2^3x_3^5x_4^4x_5^4\\
b_{45} = x_1^3x_2^4x_3^3x_4^4x_5^5 & b_{46} = x_1^3x_2^4x_3^3x_4^5x_5^4 & b_{47} = x_1^3x_2^5x_3^3x_4^4x_5^4. &
\end{array}$$

Lemma 4.6. *The following monomials are strictly inadmissible:*

(i) $x_i^2x_j^3x_\ell^3,\ i \neq j \neq \ell \neq i,$

(ii) $x_i^2x_jx_\ell^2x_m^3,\ i<j,\ i,j\neq \ell,m,\ \ell\neq m,$

(iii) $x_i^2x_jx_\ell x_m^2x_n^2,\ (i,j,\ell,m,n)$ is a permutation of $(1,2,3,4,5)$ and $i<j<\ell$.

(iv) $x_i^3x_j^5x_\ell^6x_m^5,\ i<j<\ell,\ m\neq i,j,\ell,$

(v) $x_i^3x_j^4x_\ell^5x_m^7,\ i\neq j\neq \ell\neq m\neq i,$

(vi) $x_i^3x_j^5x_\ell^5x_m^6,\ i<j<\ell<m,$

Lemma 4.7. *Let* (i,j,ℓ,m,n) *be a permutation of* $(1,2,3,4,5)$. *Then the following monomials are strictly inadmissible:*

(i) $x_i^4x_jx_\ell^3x_m^4x_n^7,\ x_i^4x_jx_\ell^3x_m^5x_n^6,\ \ i<j,$

(ii) $x_i^3x_j^4x_\ell^5x_m^2x_n^5,\ x_i^3x_j^4x_\ell^4x_m^3x_n^5\ i=1, m>3.$

The above lemmas are proved by a direct computation.

Proof of Proposition 4.5. Let x be a monomial in P_5 and $\omega(x)=(3,2,3)$. Then x is a permutation of one of the following monomials:

$$x_1x_2^2x_3^4x_4^5x_5^7,\ x_1x_2^2x_3^5x_4^5x_5^6,\ x_1x_2^3x_3^4x_4^4x_5^7,\ x_1x_2^3x_3^4x_4^5x_5^6,\ x_1^2x_2^3x_3^4x_4^5x_5^5,\ x_1^3x_2^3x_3^4x_4^4x_5^5.$$

By a direct computation using Theorem 2.4 and Lemmas 3.2, 4.4, 4.6, 4.7 we see that if $x\neq b_t, \forall t, 1\leqslant t\leqslant 47$, then x is inadmissible.

Now, we prove that the set $\{[b_t]: 1\leqslant i\leqslant 47\}$ is linearly independent in $\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5$. Suppose there is a linear relation

$$\mathcal{S}=\sum_{1\leqslant t\leqslant 47}\gamma_t b_t\equiv 0,$$

with $\gamma_t\in\mathbb{F}_2, 1\leqslant t\leqslant 47$.

Computing directly from the relations $p_{(i,j)}(\mathcal{S})\equiv 0, 1\leqslant i<j\leqslant 5$, gives $\gamma_t=0$ for all $1\leqslant t\leqslant 47$. The proposition is proved. $\square$

Proposition 4.8. $(\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5^+)(3,4,2)$ *is an* $\mathbb{F}_2$*-vector space of dimension* 55 *with a basis consisting of all the classes represented by the following admissible monomials:*

$c_1=x_1x_2^2x_3^2x_4^7x_5^7$	$c_2=x_1x_2^2x_3^7x_4^2x_5^7$	$c_3=x_1x_2^2x_3^7x_4^7x_5^2$	$c_4=x_1x_2^7x_3^2x_4^2x_5^7$
$c_5=x_1x_2^7x_3^2x_4^7x_5^2$	$c_6=x_1x_2^7x_3^7x_4^2x_5^2$	$c_7=x_1^7x_2x_3^2x_4^2x_5^7$	$c_8=x_1^7x_2x_3^2x_4^7x_5^2$
$c_9=x_1^7x_2x_3^7x_4^2x_5^2$	$c_{10}=x_1^7x_2^7x_3x_4^2x_5^2$	$c_{11}=x_1x_2^2x_3^3x_4^6x_5^7$	$c_{12}=x_1x_2^2x_3^3x_4^7x_5^6$
$c_{13}=x_1x_2^2x_3^7x_4^3x_5^6$	$c_{14}=x_1x_2^3x_3^2x_4^6x_5^7$	$c_{15}=x_1x_2^3x_3^2x_4^7x_5^6$	$c_{16}=x_1x_2^3x_3^6x_4^2x_5^7$
$c_{17}=x_1x_2^3x_3^6x_4^7x_5^2$	$c_{18}=x_1x_2^3x_3^7x_4^2x_5^6$	$c_{19}=x_1x_2^3x_3^7x_4^6x_5^2$	$c_{20}=x_1x_2^7x_3^2x_4^3x_5^6$
$c_{21}=x_1x_2^7x_3^3x_4^2x_5^6$	$c_{22}=x_1x_2^7x_3^3x_4^6x_5^2$	$c_{23}=x_1^3x_2x_3^2x_4^6x_5^7$	$c_{24}=x_1^3x_2x_3^2x_4^7x_5^6$
$c_{25}=x_1^3x_2x_3^6x_4^2x_5^7$	$c_{26}=x_1^3x_2x_3^6x_4^7x_5^2$	$c_{27}=x_1^3x_2x_3^7x_4^2x_5^6$	$c_{28}=x_1^3x_2x_3^7x_4^6x_5^2$

$c_{29} = x_1^3x_2^7x_3x_4^2x_5^6$ $c_{30} = x_1^3x_2^7x_3x_4^6x_5^2$ $c_{31} = x_1^7x_2x_3^2x_4^3x_5^6$ $c_{32} = x_1^7x_2x_3^3x_4^2x_5^6$
$c_{33} = x_1^7x_2x_3^3x_4^6x_5^2$ $c_{34} = x_1^7x_2^3x_3x_4^2x_5^6$ $c_{35} = x_1^7x_2^3x_3x_4^6x_5^2$ $c_{36} = x_1x_2^3x_3^3x_4^6x_5^6$
$c_{37} = x_1x_2^3x_3^6x_4^3x_5^6$ $c_{38} = x_1x_2^3x_3^6x_4^6x_5^3$ $c_{39} = x_1^3x_2x_3^3x_4^6x_5^6$ $c_{40} = x_1^3x_2x_3^6x_4^3x_5^6$
$c_{41} = x_1^3x_2x_3^6x_4^6x_5^3$ $c_{42} = x_1^3x_2^3x_3x_4^6x_5^6$ $c_{43} = x_1^3x_2^5x_3^2x_4^2x_5^7$ $c_{44} = x_1^3x_2^5x_3^2x_4^7x_5^2$
$c_{45} = x_1^3x_2^5x_3^7x_4^2x_5^2$ $c_{46} = x_1^3x_2^7x_3^5x_4^2x_5^2$ $c_{47} = x_1^7x_2^3x_3^5x_4^2x_5^2$ $c_{48} = x_1^3x_2^3x_3^5x_4^2x_5^6$
$c_{49} = x_1^3x_2^3x_3^5x_4^6x_5^2$ $c_{50} = x_1^3x_2^5x_3^2x_4^3x_5^6$ $c_{51} = x_1^3x_2^5x_3^2x_4^6x_5^3$ $c_{52} = x_1^3x_2^5x_3^3x_4^2x_5^6$
$c_{53} = x_1^3x_2^5x_3^3x_4^6x_5^2$ $c_{54} = x_1^3x_2^5x_3^6x_4^2x_5^3$ $c_{55} = x_1^3x_2^5x_3^6x_4^3x_5^2.$

Lemma 4.9. *Let (i, j, ℓ, m, n) be a permutation of $(1, 2, 3, 4, 5)$. Then the following monomials are strictly inadmissible:*
(i) $x_i^2x_jx_\ell x_mx_n^3,\ i < j < \ell < m,$
(ii) $x_ix_j^6x_\ell^3x_m^2x_n^7,\ i < j < \ell,\ x_ix_j^6x_\ell^3x_m^3x_n^6,\ i = 1, j = 2.$

Proof. We prove the lemma for $x = x_1x_2^6x_3^3x_4^2x_5^7$. The others can be proved by a similar computation. A direct computation shows

$$\begin{aligned} x = {} & x_1x_2^5x_3^4x_4^2x_5^7 + x_1x_2^5x_3^3x_4^2x_5^8 + x_1^2x_2^3x_3^3x_4^2x_5^9 + x_1^2x_2^4x_3^4x_4^2x_5^7 + x_1^2x_2^4x_3^3x_4^2x_5^8 \\ & + x_1^2x_2^3x_3^4x_4^2x_5^8 + x_1^4x_2^4x_3^3x_4x_5^7 + x_1^4x_2^3x_3^4x_4x_5^7 + x_1^4x_2^3x_3^3x_4x_5^8 + x_1x_2^4x_3^3x_4^4x_5^7 \\ & + x_1x_2^3x_3^4x_4^4x_5^7 + x_1x_2^3x_3^3x_4^4x_5^8 + x_1x_2^4x_3^5x_4^2x_5^7 + x_1x_2^3x_3^5x_4^2x_5^8 + x_1x_2^3x_3^6x_4^2x_5^7 \\ & + Sq^1(x_1x_2^5x_3^3x_4^2x_5^7 + x_1^4x_2^3x_3^3x_4x_5^7 + x_1x_2^3x_3^3x_4^4x_5^7 + x_1x_2^3x_3^5x_4^2x_5^7) \\ & + Sq^2(x_1x_2^3x_3^3x_4^2x_5^7). \end{aligned}$$

Hence x is strictly inadmissible. $\square$

Proof of Proposition 4.8. Let x be a monomial in P_5 and $\omega(x) = (3, 4, 2)$. Then x is a permutation of one of the following monomials:

$$x_1x_2^2x_3^2x_4^7x_5^7,\ \ x_1x_2^2x_3^3x_4^6x_5^7,\ \ x_1x_2^3x_3^3x_4^6x_5^6,\ \ x_1^2x_2^2x_3^3x_4^5x_5^7,\ \ x_1^2x_2^3x_3^3x_4^5x_5^6.$$

By a direct computation using Theorem 2.4 and Lemma 4.9, we see that if $x \neq c_t,\ 1 \leqslant t \leqslant 55$, then x is inadmissible.

Now, we prove that the set $\{[c_t] : 1 \leqslant t \leqslant 55\}$ is linearly independent in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_5$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leqslant t \leqslant 55} \gamma_t c_t \equiv 0,$$

with $\gamma_t \in \mathbb{F}_2,\ 1 \leqslant t \leqslant 55$.

By a direct computation from the relations $p_{(i,j)}(\mathcal{S}) \equiv 0, 1 \leqslant i < j \leqslant 5$, one gets $\gamma_t = 0$ for all t. The proposition is proved. $\square$

Combining Propositions 1.2, 4.1, 4.3, 4.5, 4.8, we obtain

$$\begin{aligned} B_5(19) = f(B_4(19)) \cup \{a_i : 1 \leqslant i \leqslant 150\} \cup \\ \cup \{b_j : 1 \leqslant j \leqslant 47\} \cup \{c_\ell : 1 \leqslant \ell \leqslant 55\} \cup \psi(B_5(7)), \end{aligned}$$

where $\psi : P_5 \to P_5$ is the homomorphism determined by $\psi(x) = x_1x_2x_3x_4x_5x^2$ for all $x \in P_5$. For $s > 3$, we have

$$B_5(2^{s+1} + 2^s - 5) = \psi^{s-3}(B_5(19)).$$

Theorem 1.3 is proved.

Acknowledgment. I would like to express my warmest thanks to my adviser, Ass. Prof. Nguyen Sum, for his inspiring guidance and generous help in finding the proofs as good as in describing the results.

I would like to thank all colleagues at the Foundation Sciences Faculty, University of Technical Education of Ho Chi Minh city for helpful conversations.

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