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ON THE PETERSON HIT PROBLEM OF FIVE VARIABLES AND ITS APPLICATIONS TO THE FIFTH SINGER TRANSFER

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Abstract

We study the Peterson hit problem of finding a minimal set of generators for the polynomial algebra $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we explicitly determine a minimal set of $\mathcal{A}$-generators with $k = 5$ in degree 15. Using this results we show that the fifth Singer transfer is an isomorphism in this degree.

1 Introduction and statement of results

Let V_k be an elementary abelian 2-group of rank k. Denote by BV_k the classifying space of V_k. It may be thought of as the product of k copies of the real projective space $\mathbb{RP}^\infty$. Then

$$P_k := H^*(BV_k) \cong \mathbb{F}_2[x_1, x_2, \ldots, x_k],$$

a polynomial algebra on k generators $x_1, x_2, \ldots, x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements.

Being the cohomology of a space, P_k is a module over the mod 2 Steenrod algebra $\mathcal{A}$. The action of $\mathcal{A}$ on P_k can explicitly be given by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0,\\ x_j^2, & i = 1,\\ 0, & \text{otherwise,}\end{cases}$$

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and subject to the Cartan formula $Sq^n(fg) = \sum_{i=0}^{n} Sq^i(f)Sq^{n-i}(g)$, for $f, g \in P_k$ (see Steenrod-Epstein [22]).

A polynomial f in P_k is called *hit* if it can be written as a finite sum $f = \sum_{i>0} Sq^i(f_i)$ for some polynomials f_i. That means f belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ denotes the augmentation ideal in $\mathcal{A}$. We are interested in the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. In other words, we want to find a basis of the $\mathbb{F}_2$-vector space $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k := QP_k$.

Let $GL_k = GL_k(\mathbb{F}_2)$ be the general linear group over the field $\mathbb{F}_2$. This group acts naturally on P_k by matrix substitution. Since the two actions of $\mathcal{A}$ and GL_k upon P_k commute with each other, there is an action of GL_k on QP_k. The subspace of degree n homogeneous polynomials $(P_k)_n$ and its quotient $(QP_k)_n$ are GL_k-subspaces of the spaces P_k and QP_k respectively.

The hit problem was first studied by Peterson [15], Wood [26], Singer [20], and Priddy [16], who showed its relationship to several classical problems respectively in cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups. The tensor product QP_k was explicitly calculated by Peterson [15] for $k = 1, 2$, by Kameko [10] for $k = 3$, and recently by us [23] for $k = 4$.

Many authors was then investigated the hit problem. (See Boardman [1], Bruner-Hà-Hung [2], Crabb-Hubbuck [5], Hà [6], Hung [7, 8], Kameko [10, 11], Nam [13, 14], Repka-Selick [18], Singer [21], Silverman [19], Wood [26, 27] and others.)

One of our main tools for studying the hit problem is the so-called Kameko squaring operation

$$Sq^0 : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k) \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k).$$

Here $H_*(BV_k)$ is homology with $\mathbb{F}_2$ coefficients, and $PH_*(BV_k)$ denotes the primitive subspace consisting of all elements in the space $H_*(BV_k)$, which are annihilated by every positive-degree operation in the mod 2 Steenrod algebra; therefore, $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k)$ is dual to $QP_k^{GL_k}$. The dual of the Kameko squaring is the homomorphism $Sq^0_* : QP_k^{GL_k} \to QP_k^{GL_k}$. This homomorphism is given by the following GL_k-homomorphism $\widetilde{Sq}^0_* : QP_k \to QP_k$. The latter is given by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_* : P_k \to P_k$, given by

$$\widetilde{Sq}^0_*(x) = \begin{cases} y, & \text{if } x = x_1x_2\dots x_ky^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However, $\widetilde{Sq}^0_*Sq^{2t} = Sq^t\widetilde{Sq}^0_*$ and $\widetilde{Sq}^0_*Sq^{2t+1} = 0$, for any nonnegative integer t.

The Kameko squaring operation commutes with the classical squaring operation on the cohomology of the Steenrod algebra through the Singer transfer

$$\mathrm{Tr}_k : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_d(BV_k) \to \mathrm{Ext}_{\mathcal{A}}^{k,k+d}(\mathbb{F}_2, \mathbb{F}_2).$$

Boardman [1] used this fact to show that Tr_3 is an isomorphism. Bruner-Hà-Hung [2] applied it to prove that Tr_4 does not detect any element in the usual family $\{g_i\}_{i>0}$ of $\mathrm{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$. Recently, Hung and his collaborators have completely determined the image of the fourth Singer transfer Tr_4 (in [2], [8], [6], [14], [9]). Singer showed in [20] that Tr_5 is not an epimorphism in degree 9. In [17], Quỳnh proved that Tr_5 is also not an epimorphism in degree 11. The Singer transfer was also investigated by Chon-Hà [3, 4].

In this paper, we explicitly determine all the admissible monomials (see Section 2) of P_5 in degree 15. Using this results, we prove that the fifth Singer transfer is an isomorphism in this degree. We have

Theorem 1.1. *There exist exactly* 432 *admissible monomials of degree* 15 *in* P_5. *Consequently* $\dim(QP_5)_{15} = 432$.

By using Theorem 1.1, we compute $(QP_5)_{15}^{GL_5}$.

Theorem 1.2. $(QP_5)_{15}^{GL_5}$ *is an* $\mathbb{F}_2$*-vector space of dimension* 2 *with a basis consisting of the* 2 *classes represented by the following polynomials:*

$$\begin{aligned}
p = {} & x_1^{15} + x_2^{15} + x_3^{15} + x_4^{15} + x_5^{15} + x_1x_2^{14} + x_1x_3^{14} + x_1x_4^{14} + x_1x_5^{14} + x_2x_3^{14} \\
& + x_2x_4^{14} + x_2x_5^{14} + x_3x_4^{14} + x_3x_5^{14} + x_4x_5^{14} + x_1x_2^2x_3^{12} + x_1x_2^2x_4^{12} + x_1x_2^2x_5^{12} \\
& + x_1x_3^2x_4^{12} + x_1x_3^2x_5^{12} + x_1x_4^2x_5^{12} + x_2x_3^2x_4^{12} + x_2x_3^2x_5^{12} + x_2x_4^2x_5^{12} + x_3x_4^2x_5^{12} \\
& + x_1x_2^2x_3^4x_4^8 + x_1x_2^2x_3^4x_5^8 + x_1x_2^2x_4^4x_5^8 + x_1x_3^2x_4^4x_5^8 + x_2x_3^2x_4^4x_5^8 + x_1x_2^2x_3^4x_4^4x_5^4, \\
q = {} & x_1x_2x_3x_4^6x_5^6 + x_1x_2x_3^6x_4x_5^6 + x_1x_2x_3^6x_4^6x_5 + x_1x_2^6x_3x_4x_5^6 + x_1x_2^6x_3x_4^6x_5 \\
& + x_1x_2^3x_3^6x_4x_5^4 + x_1x_2^3x_3^6x_4^4x_5 + x_1x_2^6x_3^3x_4x_5^4 + x_1x_2^6x_3^3x_4^4x_5 + x_1^3x_2x_3x_4^4x_5^6 \\
& + x_1^3x_2x_3x_4^6x_5^4 + x_1^3x_2x_3^4x_4x_5^6 + x_1^3x_2x_3^4x_4^6x_5 + x_1^3x_2^4x_3x_4x_5^6 + x_1^3x_2^4x_3x_4^6x_5 \\
& + x_1x_2^3x_3^3x_4^4x_5^4 + x_1^3x_2x_3^3x_4^4x_5^4 + x_1^3x_2^3x_3x_4^4x_5^4 + x_1^3x_2^3x_3^4x_4x_5^4 + x_1^3x_2^3x_3^4x_4^4x_5 \\
& + x_1^3x_2^4x_3^3x_4x_5^4 + x_1^3x_2^4x_3^3x_4^4x_5.
\end{aligned}$$

Using Theorem 1.2, we prove the following which was proved in Hung [8] by using computer computation.

Theorem 1.3 (Hung [8]). *The fifth Singer transfer*

$$\mathrm{Tr}_5 : \mathbb{F}_2 \underset{GL5}{\otimes} PH_{15}(BV_5) \to \mathrm{Ext}_{\mathcal{A}}^{5,20}(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism.

This paper is organized as follows. In Section 2, we recall some needed information on the admissible monomials in P_k and Singer criterion on the hit monomials. We prove Theorem 1.1 in Section 3 by explicitly determine all the admissible monomials of degree 15. Theorems 1.2 and 1.3 will be proved in Sections 4.

2 Preliminaries

In this section, we recall some results in Kameko [10] and Singer [21] which will be used in the next sections.

Notation 2.1. Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a nonnegative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0, 1$ and $i \geqslant 0$.

Let $x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \in P_k$. Set $I_i(x) = \{j \in \mathbb{N}_k : \alpha_i(a_j) = 0\}$, for $i \geqslant 0$. Then we have

$$x = \prod_{i\geqslant 0} X_{I_i(x)}^{2^i}.$$

For a polynomial f in P_k, we denote by $[f]$ the class in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ represented by f. For a subset $S \subset P_k$, we denote

$$[S] = \{[f] : f \in S\} \subset QP_k.$$

Definition 2.2. For a monomial $x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \in P_k$, we define two sequences associated with x by

$$\begin{aligned}\omega(x) &= (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots),\\ \sigma(x) &= (a_1, a_2, \ldots, a_k),\end{aligned}$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant k} \alpha_{i-1}(a_j) = \deg X_{I_{i-1}(x)},\ i \geqslant 1$.

The sequence $\omega(x)$ is called the weight vector of x (see Wood [27]). The weight vectors and the sigma vectors can be ordered by the left lexicographical order.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of nonnegative integers such that $\omega_i = 0$ for $i \gg 0$. Define $\deg\omega = \sum_{i>0} 2^{i-1}\omega_i$. Denote by $P_k(\omega)$ the subspace of P_k spanned by all monomials y such that $\deg y = \deg\omega$, $\omega(y) \leqslant \omega$ and $P_k^-(\omega)$ the subspace of P_k spanned by all monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$. Denote by $\mathcal{A}_s^+$ the subspace of $\mathcal{A}$ spanned by all Sq^j with $1 \leqslant j < 2^s$. Define

$$QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

Then we have

$$(QP_k)_n = \oplus_{\deg\omega=n} QP_k(\omega).$$

Definition 2.3. Let x be a monomial and f, g two homogeneous polynomials of the same degree in P_k. We define $f \equiv g$ if and only if $f - g \in \mathcal{A}^+P_k$. If $f \equiv 0$ then f is called hit.

We recall some relations on the action of the Steenrod squares on P_k.

Proposition 2.4. *Let f be a homogeneous polynomial in P_k.*

i) *If $i > \deg f$ then $Sq^i(f) = 0$. If $i = \deg f$ then $Sq^i(f) = f^2$.*

ii) *If i is not divisible by 2^s then $Sq^i(f^{2^s}) = 0$ while $Sq^{r2^s}(f^{2^s}) = (Sq^r(f))^{2^s}$.*

Definition 2.5. Let x, y be monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds

i) $\omega(x) < \omega(y)$;

ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.6. A monomial x is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$ for $j = 1, 2, \ldots, t$ and $x \equiv y_1 + y_2 + \ldots + y_t$.

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n.

The following theorem is a modification of a result in [10].

Theorem 2.7 (Kameko [10], Sum [24]). *Let x, w be monomials in P_k such that $\omega_i(x) = 0$ for $i > r > 0$. If w is inadmissible, then xw^{2^r} is also inadmissible.*

Proposition 2.8 ([24]). *Let x be an admissible monomial in P_k. Then we have*

i) *If there is an index i_0 such that $\omega_{i_0}(x) = 0$, then $\omega_i(x) = 0$ for all $i > i_0$.*

ii) *If there is an index i_0 such that $\omega_{i_0}(x) < k$, then $\omega_i(x) < k$ for all $i > i_0$.*

Now, we recall a result of Singer [21] on the hit monomials in P_k.

Definition 2.9. A monomial $z = x_1^{b_1}x_2^{b_2}\ldots x_k^{b_k}$ is called a spike if $b_j = 2^{s_j} - 1$ for s_j a nonnegative integer and $j = 1, 2, \ldots, k$. If z is a spike with $s_1 > s_2 > \ldots > s_{r-1} \geqslant s_r > 0$ and $s_j = 0$ for $j > r$, then it is called a minimal spike.

The following is a criterion for the hit monomials in P_k.

Theorem 2.10 (Singer [21]). *Suppose $x \in P_k$ is a monomial of degree n, where $\mu(n) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$ then x is hit.*

For latter use, we set

$$P_k^0 = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}\ ;\ a_1a_2\ldots a_k = 0\}\rangle,$$
$$P_k^+ = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}\ ;\ a_1a_2\ldots a_k > 0\}\rangle.$$

It is easy to see that P_k^0 and P_k^+ are the $\mathcal{A}$-submodules of P_k. Furthermore, we have the following.

Proposition 2.11. *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces*
$$QP_k = QP_k^0 \oplus QP_k^+.$$
Here $QP_k^0 = P_k^0/\mathcal{A}^+.P_k^0$ and $QP_k^+ = P_k^+/\mathcal{A}^+.P_k^+$.

For $1 \leqslant i \leqslant k$, define the homomorphism $f_i = f_{k;i} : P_{k-1} \to P_k$ of algebras by substituting
$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ x_{j+1}, & \text{if } i \leqslant j < k. \end{cases}$$

It is easy to see that

Proposition 2.12. *If $B_{k-1}(n)$ is the set of all admissible monomials of degree n in P_{k-1}, then $f(B_{k-1}(n)) := \cup_{1\leqslant i\leqslant k} f_i(B_{k-1}(n))$ is the set of all admissible monomials of degree n in P_k^0.*

For $1 \leqslant i \leqslant k$, define $\varphi_i : QP_k \to QP_k$, the homomorphism induced by the $\mathcal{A}$-homomorphism $\overline{\varphi}_i : P_k \to P_k$, which is determined by $\overline{\varphi}_1(x_1) = x_1 + x_2$, $\overline{\varphi}_1(x_j) = x_j$ for $j > 1$, and $\overline{\varphi}_i(x_i) = x_{i-1}, \overline{\varphi}_i(x_{i-1}) = x_i$, $\overline{\varphi}_i(x_j) = x_j$ for $j \neq i, i-1,\ 1 < i \leqslant k$. Note that the general linear group GL_k is generated by $\overline{\varphi}_i,\ 0 \leqslant i \leqslant k$ and the symmetric group Σ_k is generated by $\overline{\varphi}_i,\ 1 < i \leqslant k$.

For any $I = (i_0, i_1, \ldots, i_r),\ 0 < i_0 < i_1 < \ldots < i_r \leqslant k,\ 0 \leqslant r < k$, we define the homomorphism $p_I : P_k \to P_{k-1}$ of algebras by substituting
$$p_I(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i_0, \\ \sum_{1\leqslant s\leqslant r} x_{i_s-1}, & \text{if } j = i_0, \\ x_{j-1}, & \text{if } i_0 < j \leqslant k. \end{cases}$$

Then p_I is a homomorphism of $\mathcal{A}$-modules. In particular, for $I = (i)$, we have $p_{(i)}(x_i) = 0$.

3 Proof of Theorem 1.1

In this section, we explicitly determine all the admissible monomials of degree 15.

Consider the Kameko homomorphism $(\widetilde{Sq}^0_*)^5_5 : (QP_5)_{15} \to (QP_5)_5$. Since this homomorphism is an epimorphism, we have
$$(QP_5)_{15} \cong \mathrm{Ker}(\widetilde{Sq}^0_*)^5_5 \oplus (QP_5)_5 = ((QP_5^0)_{15} \oplus ((QP_5^+)_{15} \cap \mathrm{Ker}(\widetilde{Sq}^0_*)^5_5) \oplus (QP_5)_5.$$

By Proposition 2.12, to compute $(QP_5^0)_{15}$ we need to compute
$$(QP_4)_{15} = (QP_4)^0_{15} \oplus (QP_4)^+_{15}.$$

Using Kameko's results in [10], we have

$$B_3(15) = \{x_1^{15}, x_2^{15}, x_3^{15}, x_1x_2^{14}, x_1x_3^{14}, x_2x_3^{14}, x_1x_2^2x_3^{12},$$
$$x_1x_2^7x_3^7, x_1^7x_2x_3^7, x_1^7x_2^7x_3, x_1^3x_2^5x_3^7, x_1^3x_2^7x_3^5, x_1^7x_2^3x_3^5\}.$$

By a direct computation using Proposition 2.12, we see that $f(B_3(15))$ is the set consisting of 38 admissible monomials in $(P_5^0)_{15}$.

Lemma 3.1. *If x is an admissible monomial of degree 15 in P_4 then either $\omega(x) = (1,1,1,1)$ or $\omega(x) = (3,2,2)$.*

Proof. Since $\deg x$ is odd, we have $\omega_1(x) = 1$ or $\omega_1(x) = 3$.

Suppose $\omega_1(x) = 1$, then $x = x_iy^2$ with y a monomial of degree 7. Since x is admissible, by Theorem 2.7, y is admissible. If $y \notin P_4^+$ then from Kameko [10], $\omega(y) = (1,1,1)$ or $\omega(y) = (3,2)$. A direct computation shows that $x = x_iy^2$ is inadmissible for all monomials y in P_4 with $\omega(y) = (3,2)$. Hence $\omega(x) = (1,1,1,1)$. If $y \in P_4^+$, then y is a permutation of one of the following monomial $x_1x_2x_3x_4^4$, $x_1x_2x_3^2x_4^3$, $x_1x_2^2x_3^2x_4^2$. By a direct computation we see that $x = x_iy^2$ is inadmissible.

If $\omega_1(x) = 3$, then $x = x_ix_jy^2, i < j$ with y a monomial of degree 6 in P_4. By Theorem 2.7, y is admissible. So $\omega_1(y) = 2$ or $\omega_1(y) = 4$. If $\omega_1(y) = 4$, then by Proposition 2.8, x is inadmissible. Hence $\omega_1(y) = 2$ and $\omega(x) = (3,2,2)$. The lemma is proved. $\square$

Proposition 3.2. *$(QP_4^+)_{15}$ is an $\mathbb{F}_2$-vector space of dimension 37 with a basis consisting of all the classes represented by the admissible monomials d_i, $1 \leqslant i \leqslant 37$, which are determined as follows:*

1. $x_1x_2x_3^6x_4^7$	2. $x_1x_2x_3^7x_4^6$	3. $x_1x_2^2x_3^5x_4^7$	4. $x_1x_2^2x_3^7x_4^5$	5. $x_1x_2^3x_3^4x_4^7$
6. $x_1x_2^3x_3^5x_4^6$	7. $x_1x_2^3x_3^6x_4^5$	8. $x_1x_2^3x_3^7x_4^4$	9. $x_1x_2^6x_3x_4^7$	10. $x_1x_2^6x_3^3x_4^5$
11. $x_1x_2^6x_3^7x_4$	12. $x_1x_2^7x_3x_4^6$	13. $x_1x_2^7x_3^2x_4^5$	14. $x_1x_2^7x_3^3x_4^4$	15. $x_1x_2^7x_3^6x_4$
16. $x_1^3x_2x_3^4x_4^7$	17. $x_1^3x_2x_3^5x_4^6$	18. $x_1^3x_2x_3^6x_4^5$	19. $x_1^3x_2x_3^7x_4^4$	20. $x_1^3x_2^3x_3^4x_4^5$
21. $x_1^3x_2^3x_3^5x_4^4$	22. $x_1^3x_2^4x_3x_4^7$	23. $x_1^3x_2^4x_3^3x_4^5$	24. $x_1^3x_2^4x_3^7x_4$	25. $x_1^3x_2^5x_3x_4^6$
26. $x_1^3x_2^5x_3^2x_4^5$	27. $x_1^3x_2^5x_3^3x_4^4$	28. $x_1^3x_2^5x_3^6x_4$	29. $x_1^3x_2^7x_3x_4^4$	30. $x_1^3x_2^7x_3^4x_4$
31. $x_1^7x_2x_3x_4^6$	32. $x_1^7x_2x_3^2x_4^5$	33. $x_1^7x_2x_3^3x_4^4$	34. $x_1^7x_2x_3^6x_4$	35. $x_1^7x_2^3x_3x_4^4$
36. $x_1^7x_2^3x_3^4x_4$	37. $x_1x_2^2x_3^4x_4^8$.			

Proof. From the proof of Lemma 3.1, if x is an admissible monomial of degree 15 in P_4, then x is a permutation of one of the following monomials:

$$x_1x_2x_3^6x_4^7,\ x_1x_2^2x_3^5x_4^7,\ x_1x_2^3x_3^4x_4^7,\ x_1x_2^3x_3^5x_4^6,\ x_1^2x_2^3x_3^5x_4^5,\ x_1^3x_2^3x_3^4x_4^5.$$

By a direct computation we see that if $x \neq d_t$, $1 \leqslant t \leqslant 37$, then x is inadmissible.

Now we prove that the set $\{[d_t] : 1 \leqslant t \leqslant 37\}$ is linearly independent in QP_4^+. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leqslant t \leqslant 37} \gamma_t d_t \equiv 0, \tag{3.1}$$

with $\gamma_t \in \mathbb{F}_2$.

By Kameko [10], $B_3(15)$ is the set consisting of 7 monomials:

$$v_1 = x_1x_2^7x_3^7,\ v_2 = x_1^3x_2^5x_3^7, v_3 = x_1^3x_2^7x_3^5,$$
$$v_4 = x_1^7x_2x_3^7, v_5 = x_1^7x_2^3x_3^5, v_6 = x_1^7x_2^7x_3, v_7 = x_1x_2^2x_3^{12}.$$

By a direct computation, we explicitly compute $p_I(\mathcal{S})$ in terms of $v_1, v_2, \dots v_7$. From the relations $p_I(\mathcal{S}) \equiv 0$ for $I = (i,j)$ with $1 \leqslant i < j \leqslant 4$ and for $I = (1,i,j)$ with $2 \leqslant i < j \leqslant 4$, one gets $\gamma_t = 0$ for $t \neq 1$, 2 , 9, 11, 12, 15, 16, 19, 22, 24, 29, 30, 31 and $\gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22}$, $\gamma_2 = \gamma_{11} = \gamma_{19} = \gamma_{24}$, $\gamma_{12} = \gamma_{15} = \gamma_{29} = \gamma_{30}$, $\gamma_{31} = \gamma_{34} = \gamma_{35} = \gamma_{36}$. Hence the relation (3.1) becomes

$$\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_3 + \gamma_{31}\theta_4 + \gamma_{37}d_{37} \equiv 0, \tag{3.2}$$

where

$$\theta_1 = d_1 + d_9 + d_{16} + d_{22},\quad \theta_2 = d_2 + d_{11} + d_{19} + d_{24},$$
$$\theta_3 = d_{12} + d_{15} + d_{29} + d_{30},\quad \theta_4 = d_{31} + d_{34} + d_{35} + d_{36}.$$

Now, we prove that $\gamma_1 = \gamma_2 = \gamma_{12} = \gamma_{31} = 0$.

The proof is divided into 4 steps.

Step 1. Under the homomorphism φ_1, the image of (3.2) is

$$\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_3 + \gamma_{31}(\theta_4 + \theta_3) + \gamma_{37}(d_{37} + v_7) \equiv 0. \tag{3.3}$$

Since $v_7 \in P_4^0$, $\gamma_{37} = 0$. Combining (3.2) and (3.3), we get

$$\gamma_{31}\theta_3 \equiv 0. \tag{3.4}$$

If the polynomial θ_3 is hit, then we have

$$\theta_3 = Sq^1(A) + Sq^2(B) + Sq^4(C),$$

for some polynomials $A \in (P_4^+)_{14}, B \in (P_4^+)_{13}, C \in (P_4^+)_{11}$. Let $(Sq^2)^3$ act on the both sides of this equality. We get

$$(Sq^2)^3(\theta_3) = (Sq^2)^3Sq^4(C),$$

By a direct calculation, we see that the monomial $x = x_1^8x_2^7x_3^4x_4^2$ is a term of $(Sq^2)^3(\theta_3)$. If this monomial is a term of $(Sq^2)^3Sq^4(y)$ for a monomial

$y \in (P_4^+)_{11}$, then $y = x_2^7 f_2(z)$ with $z \in P_3$ and $\deg z = 4$. Using the Cartan formula, we see that x is a term of $x_2^7(Sq^2)^3Sq^4(z) = x_2^7(Sq^2)^3(z^2) = 0$. Hence

$$(Sq^2)^3(\theta_3) \neq (Sq^2)^3Sq^4(C),$$

for all $C \in (P_4^+)_{11}$ and we have a contradiction. So $[\theta_3] \neq 0$ and $\gamma_{31} = 0$.

Step 2. Since $\gamma_{31} = 0$, the homomorphism φ_2 sends (3.2) to

$$\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_4 \equiv 0. \tag{3.5}$$

Using the relation (3.5) and by the same argument as given in Step 1, we get $\gamma_{12} = 0$.

Step 3. Since $\gamma_{31} = \gamma_{12} = 0$, the homomorphism φ_3 sends (3.2) to

$$\gamma_1[\theta_1] + \gamma_2[\theta_3] = 0. \tag{3.6}$$

Using the relation (3.6) and by the same argument as given in Step 2, we obtain $\gamma_3 = 0$.

Step 4. Since $\gamma_{31} = \gamma_{12} = \gamma_2 = 0$, the homomorphism φ_4 sends (3.2) to

$$\gamma_1\theta_2 = 0.$$

Using this relation and by the same argument as given in Step 3, we obtain $\gamma_1 = 0$. The proposition is proved. $\square$

Corollary 3.3. *The set* $[f(B_4(15))]$ *is a basis of the* $\mathbb{F}_2$*-vector space* $(QP_5^0)_{15}$*. Consequently* $\dim(QP_5^0)_{15} = 270$.

Now we compute $(QP_5)_5 = (QP_5^0)_5 \oplus (QP_5^+)_5$. Using Kameko's results in [10], we have $B_3(5) = \{x_1x_2x_3^3, x_1x_2^3x_3, x_1^3x_2x_3\}$. A direct computation, we easily obtain

$$B_4(5) = f(B_3(5)) \cup \{x_1x_2^2x_3x_4, x_1x_2x_3^2x_4, x_1x_2x_3x_4^2\}.$$

This implies $\dim(QP_4)_5 = 15$. It is easy to see that $(QP_5^+)_5 = \langle [x_1x_2x_3x_4x_5] \rangle$. So we get

$$B_5(5) = f(B_4(5)) \cup \{x_1x_2x_3x_4x_5\}.$$

Combining this with Proposition 2.12 we obtain

Proposition 3.4. *The set* $[B_5(5)]$ *is a basis of the* $\mathbb{F}_2$*-vector space* $(QP_5)_5$*. Consequently* $\dim(QP_5)_5 = 46$.

Now we compute $(QP_5^+)_{15} \cap \mathrm{Ker}(\widetilde{Sq}^0_*)^5_5$.

Lemma 3.5. *If x is an admissible monomial of degree 15 in P_5^+ and $[x] \in$ Ker$(\widetilde{Sq}^0_*)$, then $\omega(x)$ is one of the sequences:* $(1,1,3)$, $(3,2,2)$, $(3,4,1)$.

Proof. Since $x \in P_5^+$ and $[x] \in \mathrm{Ker}(\widetilde{Sq}^0_*)$, using Proposition 2.8, we see that x is a permutation of one of the following monomials:

$$\begin{array}{lllll} x_1x_2x_3^2x_4^4x_5^7, & x_1x_2^2x_3^2x_4^3x_5^7, & x_1x_2x_3x_4^6x_5^6, & x_1x_2x_3^2x_4^5x_5^6, & x_1x_2x_3^3x_4^4x_5^6, \\ x_1x_2^2x_3^2x_4^4x_5^6, & x_1x_2^2x_3^3x_4^3x_5^6, & x_1^2x_2^2x_3^2x_4^3x_5^6, & x_1x_2^2x_3^2x_4^5x_5^5, & x_1x_2^2x_3^3x_4^4x_5^5, \\ x_1^2x_2^2x_3^3x_4^3x_5^5, & x_1x_2^2x_3^4x_4^4x_5^4, & x_1^2x_2^2x_3^3x_4^4x_5^4 & x_1x_2^3x_3^3x_4^4x_5^4. & \end{array}$$

We have

$$\begin{aligned} x_1x_2^2x_3^2x_4^4x_5^6 &= x_1x_2^2x_3^2x_4^2x_5^8 + Sq^1(x_1^2x_2x_3x_4^4x_5^6 + x_2^2x_3x_4^2x_5^8) \\ &\qquad + Sq^2(x_1x_2x_3x_4^4x_5^6 + x_1x_2x_3x_4^2x_5^8) \\ x_1^2x_2^2x_3^3x_4^4x_5^4 &= x_1x_2^2x_3^4x_4^4x_5^4 + Sq^1(x_1x_2^2x_3^3x_4^4x_5^4) \\ x_1^2x_2^2x_3^2x_4^3x_5^6 &= x_1x_2^2x_3^2x_4^4x_5^6 + Sq^1(x_1x_2^2x_3^2x_4^3x_5^6). \end{aligned}$$

Since $\omega(x_1x_2^2x_3^2x_4^2x_5^8) = (1,3,0,1) < (1,3,2,0) = \omega(x_1x_2^2x_3^2x_4^4x_5^6)$, $\omega(x_1x_2^2x_3^4x_4^4x_5^4) = (1,1,3) < (1,3,2) = \omega(x_1^2x_2^2x_3^3x_4^4x_5^4)$, $\omega(x_1x_2^2x_3^2x_4^4x_5^6) = (1,3,2) < (1,5,1) = \omega(x_1^2x_2^2x_3^2x_4^3x_5^6)$, if the monomial x is a permutation of one of the monomials $x_1x_2^2x_3^2x_4^4x_5^6$, $x_1^2x_2^2x_3^3x_4^4x_5^4$, $x_1^2x_2^2x_3^2x_4^3x_5^6$, then x is inadmissible. The lemma follows. $\square$

From Lemma 3.5, we have

$$\begin{aligned} (QP_5^+)_{15} \cap \mathrm{Ker}(\widetilde{Sq}^0_*)^5_5 = ((QP_5^+) \cap QP_5(1,1,3)) \oplus \\ \oplus ((QP_5^+) \cap QP_5(3,4,1)) \oplus ((QP_5^+) \cap QP_5(3,2,2)). \end{aligned}$$

Proposition 3.6. $QP_5^+ \cap QP_5(1,1,3) = \langle [x_1x_2^2x_3^4x_4^4x_5^4] \rangle$.

Proof. From the proof of Lemma 3.5, if x is a monomial of degree 15 in P_5 and $\omega(x) = (1,1,3)$ then x is a permutation of the monomial $x_1x_2^2x_3^4x_4^4x_5^4$. By a direct computation, we have $x \equiv x_1x_2^2x_3^4x_4^4x_5^4$, completing the proof. $\square$

Proposition 3.7. $QP_5^+ \cap QP_5(3,4,1)$ *is an* $\mathbb{F}_2$*-vector space of dimension* 40 *with a basis consisting of all the classes represented by the admissible monomials* a_i, $1 \leqslant i \leqslant 40$, *which are determined as follows:*

1. $x_1x_2^2x_3^2x_4^3x_5^7$	2. $x_1x_2^2x_3^2x_4^7x_5^3$	3. $x_1x_2^2x_3^3x_4^2x_5^7$	4. $x_1x_2^2x_3^3x_4^3x_5^6$
5. $x_1x_2^2x_3^3x_4^6x_5^3$	6. $x_1x_2^2x_3^3x_4^7x_5^2$	7. $x_1x_2^2x_3^7x_4^2x_5^3$	8. $x_1x_2^2x_3^7x_4^3x_5^2$
9. $x_1x_2^3x_3^2x_4^2x_5^7$	10. $x_1x_2^3x_3^2x_4^3x_5^6$	11. $x_1x_2^3x_3^2x_4^6x_5^3$	12. $x_1x_2^3x_3^2x_4^7x_5^2$
13. $x_1x_2^3x_3^3x_4^2x_5^6$	14. $x_1x_2^3x_3^3x_4^6x_5^2$	15. $x_1x_2^3x_3^6x_4^2x_5^3$	16. $x_1x_2^3x_3^6x_4^3x_5^2$
17. $x_1x_2^3x_3^7x_4^2x_5^2$	18. $x_1x_2^7x_3^2x_4^2x_5^3$	19. $x_1x_2^7x_3^2x_4^3x_5^2$	20. $x_1x_2^7x_3^3x_4^2x_5^2$
21. $x_1^3x_2x_3^2x_4^2x_5^7$	22. $x_1^3x_2x_3^2x_4^3x_5^6$	23. $x_1^3x_2x_3^2x_4^6x_5^3$	24. $x_1^3x_2x_3^2x_4^7x_5^2$
25. $x_1^3x_2x_3^3x_4^2x_5^6$	26. $x_1^3x_2x_3^3x_4^6x_5^2$	27. $x_1^3x_2x_3^6x_4^2x_5^3$	28. $x_1^3x_2x_3^6x_4^3x_5^2$
29. $x_1^3x_2x_3^7x_4^2x_5^2$	30. $x_1^3x_2^3x_3x_4^2x_5^6$	31. $x_1^3x_2^3x_3x_4^6x_5^2$	32. $x_1^3x_2^3x_3^5x_4^2x_5^2$
33. $x_1^3x_2^5x_3^2x_4^2x_5^3$	34. $x_1^3x_2^5x_3^2x_4^3x_5^2$	35. $x_1^3x_2^5x_3^3x_4^2x_5^2$	36. $x_1^3x_2^7x_3x_4^2x_5^2$
37. $x_1^7x_2x_3^2x_4^2x_5^3$	38. $x_1^7x_2x_3^2x_4^3x_5^2$	39. $x_1^7x_2x_3^3x_4^2x_5^2$	40. $x_1^7x_2^3x_3x_4^2x_5^2$.

Proof. Let x be an admissible monomial of degree 15 in P_5 and $\omega(x) = (3,4,1)$. From the proof of Lemma 3.5, x is a permutation of one of the monomials $x_1x_2^2x_3^2x_4^3x_5^7,\ x_1x_2^2x_3^3x_4^3x_5^6,\ x_1^2x_2^2x_3^3x_4^3x_5^5$. A direct computation shows that if $x \neq a_t,\ 1 \leqslant t \leqslant 40$, then x is inadmissible.

Now, we prove that the set $\{[a_t] : 1 \leqslant t \leqslant 40\}$ is linearly independent in QP_5. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leqslant t \leqslant 40} \gamma_t a_t \equiv 0,$$

with $\gamma_t \in \mathbb{F}_2$. By a direct computation, we explicitly compute $p_{(1,j)}(\mathcal{S})$ in terms of $d_i,\ 1 \leqslant j \leqslant 37$. From the relations $p_{(1,j)}(\mathcal{S}) \equiv 0$ for $1 \leqslant j \leqslant 5$, we obtain $\gamma_t = 0$ for $1 \leqslant t \leqslant 40$. The proposition is proved. $\square$

Proposition 3.8. *$QP_5^+ \cap QP_5(3,2,2)$ is an $\mathbb{F}_2$-vector space of dimension 75 with a basis consisting of all the classes represented by the admissible monomials $b_t,\ 1 \leqslant t \leqslant 75$, which are determined as follows:*

1. $x_1x_2x_3x_4^6x_5^6$	2. $x_1x_2x_3^2x_4^4x_5^7$	3. $x_1x_2x_3^2x_4^5x_5^6$	4. $x_1x_2x_3^2x_4^6x_5^5$
5. $x_1x_2x_3^2x_4^7x_5^4$	6. $x_1x_2x_3^3x_4^4x_5^6$	7. $x_1x_2x_3^3x_4^6x_5^4$	8. $x_1x_2x_3^6x_4x_5^6$
9. $x_1x_2x_3^6x_4^2x_5^5$	10. $x_1x_2x_3^6x_4^3x_5^4$	11. $x_1x_2x_3^6x_4^6x_5$	12. $x_1x_2x_3^7x_4^2x_5^4$
13. $x_1x_2^2x_3x_4^4x_5^7$	14. $x_1x_2^2x_3x_4^5x_5^6$	15. $x_1x_2^2x_3x_4^6x_5^5$	16. $x_1x_2^2x_3x_4^7x_5^4$
17. $x_1x_2^2x_3^3x_4^4x_5^5$	18. $x_1x_2^2x_3^3x_4^5x_5^4$	19. $x_1x_2^2x_3^4x_4x_5^7$	20. $x_1x_2^2x_3^4x_4^3x_5^5$
21. $x_1x_2^2x_3^4x_4^7x_5$	22. $x_1x_2^2x_3^5x_4x_5^6$	23. $x_1x_2^2x_3^5x_4^2x_5^5$	24. $x_1x_2^2x_3^5x_4^3x_5^4$
25. $x_1x_2^2x_3^5x_4^6x_5$	26. $x_1x_2^2x_3^7x_4x_5^4$	27. $x_1x_2^2x_3^7x_4^4x_5$	28. $x_1x_2^3x_3x_4^4x_5^6$
29. $x_1x_2^3x_3x_4^6x_5^4$	30. $x_1x_2^3x_3^2x_4^4x_5^5$	31. $x_1x_2^3x_3^2x_4^5x_5^4$	32. $x_1x_2^3x_3^3x_4^4x_5^4$
33. $x_1x_2^3x_3^4x_4x_5^6$	34. $x_1x_2^3x_3^4x_4^2x_5^5$	35. $x_1x_2^3x_3^4x_4^3x_5^4$	36. $x_1x_2^3x_3^4x_4^6x_5$
37. $x_1x_2^3x_3^5x_4^2x_5^4$	38. $x_1x_2^3x_3^6x_4x_5^4$	39. $x_1x_2^3x_3^6x_4^4x_5$	40. $x_1x_2^6x_3x_4x_5^6$
41. $x_1x_2^6x_3x_4^2x_5^5$	42. $x_1x_2^6x_3x_4^3x_5^4$	43. $x_1x_2^6x_3x_4^6x_5$	44. $x_1x_2^6x_3^3x_4x_5^4$
45. $x_1x_2^6x_3^3x_4^4x_5$	46. $x_1x_2^7x_3x_4^2x_5^4$	47. $x_1x_2^7x_3^2x_4x_5^4$	48. $x_1x_2^7x_3^2x_4^4x_5$
49. $x_1^3x_2x_3x_4^4x_5^6$	50. $x_1^3x_2x_3x_4^6x_5^4$	51. $x_1^3x_2x_3^2x_4^4x_5^5$	52. $x_1^3x_2x_3^2x_4^5x_5^4$
53. $x_1^3x_2x_3^3x_4^4x_5^4$	54. $x_1^3x_2x_3^4x_4x_5^6$	55. $x_1^3x_2x_3^4x_4^2x_5^5$	56. $x_1^3x_2x_3^4x_4^3x_5^4$
57. $x_1^3x_2x_3^4x_4^6x_5$	58. $x_1^3x_2x_3^5x_4^2x_5^4$	59. $x_1^3x_2x_3^6x_4x_5^4$	60. $x_1^3x_2x_3^6x_4^4x_5$
61. $x_1^3x_2^3x_3x_4^4x_5^4$	62. $x_1^3x_2^3x_3^4x_4x_5^4$	63. $x_1^3x_2^3x_3^4x_4^4x_5$	64. $x_1^3x_2^4x_3x_4x_5^6$
65. $x_1^3x_2^4x_3x_4^2x_5^5$	66. $x_1^3x_2^4x_3x_4^3x_5^4$	67. $x_1^3x_2^4x_3x_4^6x_5$	68. $x_1^3x_2^4x_3^3x_4x_5^4$
69. $x_1^3x_2^4x_3^3x_4^4x_5$	70. $x_1^3x_2^5x_3x_4^2x_5^4$	71. $x_1^3x_2^5x_3^2x_4x_5^4$	72. $x_1^3x_2^5x_3^2x_4^4x_5$
73. $x_1^7x_2x_3x_4^2x_5^4$	74. $x_1^7x_2x_3^2x_4x_5^4$	75. $x_1^7x_2x_3^2x_4^4x_5$.	

Proof. Let x be an admissible monomial of degree 15 in P_5 and $\omega(x) = (3,2,2)$. From the proof of Lemma 3.5, x is a permutation of one of the monomials:

$$\begin{aligned} &x_1x_2x_3^2x_4^4x_5^7, \quad x_1x_2x_3x_4^6x_5^6, \quad x_1x_2x_3^2x_4^5x_5^6, \quad x_1x_2x_3^3x_4^4x_5^6, \\ &x_1x_2^2x_3^2x_4^5x_5^5, \quad x_1x_2^2x_3^3x_4^4x_5^5 \quad x_1x_2^3x_3^3x_4^4x_5^4. \end{aligned}$$

By a direct computation, we see that if $x \neq b_t,\ 1 \leqslant t \leqslant 75$, then x is inadmissible.

Now, we prove that the set $\{[b_t] : 1 \leqslant t \leqslant 75\}$ is linearly independent in QP_5. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1\leqslant t\leqslant 75} \gamma_t b_t \equiv 0, \tag{3.7}$$

with $\gamma_t \in \mathbb{F}_2$. By a direct computation, we explicitly compute $p_{(i,j)}(\mathcal{S})$ in terms of d_t, $1 \leqslant t \leqslant 37$. From the relations $p_{(i,j)}(\mathcal{S}) \equiv 0$ for $1 \leqslant i < j \leqslant 5$, one gets $\gamma_t = 0$ for $t \notin J$ with

$$J = \{1, 8, 11, 32, 38, 39, 40, 43, 44, 45, 49, 50, 53, 54, 57, 61, 62, 63, 64, 67, 68, 69\}$$

and $\gamma_t = \gamma_1$ for $t \in J$. Hence the relation (3.7) becomes

$$\gamma_1 q \equiv 0,$$

where $q = b_1 + b_8 + b_{11} + b_{32} + b_{38} + b_{39} + b_{40} + b_{43} + b_{44} + b_{45} + b_{49} + b_{50} + b_{53} + b_{54} + b_{57} + b_{61} + b_{62} + b_{63} + b_{64} + b_{67} + b_{68} + b_{69}$.

If the polynomial q is hit, then we have

$$q = Sq^1(A) + Sq^2(B) + Sq^4(C),$$

for some polynomials $A \in (P_5^+)_{14}, B \in (P_5^+)_{13}, C \in (P_5^+)_{11}$. Let $(Sq^2)^3$ act on the both sides of this equality. Since $(Sq^2)^3Sq^1 = 0$ and $(Sq^2)^3Sq^2 = 0$ we get

$$(Sq^2)^3(q) = (Sq^2)^3Sq^4(C).$$

By a direct calculation, we have

$$(Sq^2)^3(q) = D + \text{other terms},$$

where $D = x_1^3(x_2^2x_3^8x_4^4x_5^4 + x_2^8x_3^2x_4^4x_5^4 + x_2^8x_3^4x_4^2x_5^4 + x_2^8x_3^4x_4^4x_5^2 + x_2^4x_3^8x_4^2x_5^4 + x_2^8x_3^4x_4^4x_5^2 + x_2^6x_3^4x_4^4x_5^4 + x_2^4x_3^6x_4^4x_5^4)$. Hence there is a polynomial $C' \in (P_4)_8$ such that D is a term of $(Sq^2)^3Sq^4(x_1^3f_1(C'))$. Using the Cartan formula we see that D is a term of $x_1^3f_1((Sq^2)^3Sq^4(C'))$. A direct computation shows that D is not a term of $x_1^3f_1((Sq^2)^3Sq^4(C'))$ for any $C' \in (P_4)_8$. Hence

$$(Sq^2)^3(q) \neq (Sq^2)^3Sq^4(C),$$

for all $C \in (P_5^+)_{11}$ and we have a contradiction. So $[q] \neq 0$ and $\gamma_1 = 0$. The proposition is proved. $\square$

4 Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Since $\widetilde{Sq}^0_* = (\widetilde{Sq}^0_*)^5_{15} : (QP_5)_{15} \to (QP_5)_5$ is a homomorphism of GL_5-modules, we have a direct summand decomposition of the GL_5-modules: $(QP_5)_{15} = \mathrm{Ker}(\widetilde{Sq}^0_*)^5_5 \oplus (QP_5)_5$. Hence

$$(QP_5)_{15}^{GL_5} = (\mathrm{Ker}(\widetilde{Sq}^0_*)^5_5)^{GL_5} \oplus (QP_5)_5^{GL_5}.$$

By a direct computation using Proposition 3.4 we easily obtain $(QP_5)_5^{GL_5} = 0$.

It is easy to see that

$$\mathrm{Ker}(\widetilde{Sq}^0_*)_5^5 = QP_5(1,1,1,1) \oplus QP_5(1,1,3) \oplus QP_5(3,2,2) \oplus QP_5(3,4,1),$$

where $QP_5(1,1,1,1) \oplus QP_5(1,1,3)$, $QP_5(3,2,2)$ and $QP_5(3,4,1)$ are the GL_5-submodules of $\mathrm{Ker}(\widetilde{Sq}^0_*)_5^5$. By a direct computation using Theorem 1.1 and the homomorphisms $\varphi_i : QP_5 \to QP_5,\ 1 \leqslant i \leqslant 5$, one gets

$$(QP_5(1,1,1,1) \oplus QP_5(1,1,3))^{GL_5} = \langle [p] \rangle,$$
$$QP_5(3,2,2)^{GL_5} = \langle [q] \rangle,\ QP_5(3,4,1)^{GL_5} = 0.$$

The theorem is proved. $\square$

Proof of Theorem 1.3. First of all, we briefy recall the definition of the Singer transfer. Let $\widehat{P}_1$ be the submodule of $\mathbb{F}_2[x_1, x_1^{-1}]$ spanned by all powers x_1^i with $i \geqslant -1$. The usual $\mathcal{A}$-action on $P_1 = \mathbb{F}_2[x_1]$ is canonically extended to an $\mathcal{A}$-action on $\mathbb{F}_2[x_1, x_1^{-1}]$ (see Singer [20]). $\widehat{P}_1$ is an $\mathcal{A}$-submodule of $\mathbb{F}_2[x_1, x_1^{-1}]$. The inclusion $P_1 \subset \widehat{P}_1$ gives rise to a short exact sequence of $\mathcal{A}$-modules:

$$0 \longrightarrow P_1 \longrightarrow \widehat{P}_1 \longrightarrow \Sigma^{-1}\mathbb{F}_2 \longrightarrow 0.$$

Let e_1 be the corresponding element in $\mathrm{Ext}^1_{\mathcal{A}}(\Sigma^{-1}\mathbb{F}_2, P_1)$. Singer set $e_k = e_1 \otimes \ldots \otimes e_1 \in \mathrm{Ext}^k_{\mathcal{A}}(\Sigma^{-k}\mathbb{F}_2, P_k)$. Then, he defined $\mathrm{Tr}_k^* : \mathrm{Tor}_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2) \to \mathrm{Tor}_0^{\mathcal{A}}(\mathbb{F}_2, P_k) = QP_k$ by $\mathrm{Tr}_k^*(z) = e_k \cap z$. Its image is a submodule of $(QP_k)^{GL_k}$. The k-th Singer transfer is defined to be the dual of Tr_k^*.

The algebra $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ is described in terms of the mod-2 lambda algebra Λ (see Lin [12]). Recall that Λ is a bigraded differential algebra over $\mathbb{F}_2$ generated by $\lambda_j \in \Lambda^{1,j}$, $j \geqslant 0$, with the relations

$$\lambda_j \lambda_{2j+1+m} = \sum_{\nu \geqslant 0} \binom{m-\nu-1}{\nu} \lambda_{j+m-\nu}\lambda_{2j+1+\nu},$$

for $m \geqslant 0$ and the differential

$$\delta(\lambda_k) = \sum_{\nu \geqslant 0} \binom{k-\nu-1}{\nu+1} \lambda_{k-\nu-1}\lambda_\nu,$$

for $k > 0$ and that $H^{s,t}(\Lambda, \delta) = \mathrm{Ext}_{\mathcal{A}}^{s,t+s}(\mathbb{F}_2, \mathbb{F}_2)$. It is easy to see that $\lambda_{2^i-1} \in \Lambda^{1,2^i-1},\ i \geqslant 0$, and $\bar{d}_0 = \lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3 + \lambda_1\lambda_5\lambda_1\lambda_7 \in \Lambda^{4,14}$ are the cycles in the lambda algebra Λ.

Proposition 4.1 (See Lin [12]). $\mathrm{Ext}_{\mathcal{A}}^{5,20}(\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Span}\{h_0^4 h_4, h_1 d_0\}$, *with* $h_i = [\lambda_{2^i-1}] \in \mathrm{Ext}_{\mathcal{A}}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ *and* $d_0 = [\bar{d}_0] \in \mathrm{Ext}_{\mathcal{A}}^{4,18}(\mathbb{F}_2, \mathbb{F}_2)$.

It is well known that $H_*(B\mathbb{V}_k)$ is the dual of $H^*(B\mathbb{V}_k) = P_k$. So

$$H_*(B\mathbb{V}_k) = \Gamma(a_1, a_2, \ldots, a_k)$$

is the divided power algebra generated by $a_1, a_2, \ldots, a_k$, where a_i is dual to $x_i \in P_k$ with respect to the basis of P_k consisting of all monomials in $x_1, x_2, \ldots, x_k$. In [3], Chon and Hà defined a homomorphism of algebras

$$\phi = \underset{k\geqslant 1}{\oplus}\phi_k : \underset{k\geqslant 1}{\oplus} H_*(B\mathbb{V}_k) \to \underset{k\geqslant 1}{\oplus}\Lambda_k = \Lambda,$$

which induces the Singer transfer. Here the homomorphism $\phi_k : H_*(B\mathbb{V}_k) \to \Lambda_k$ is defined by the following inductive formula:

$$\phi_k(a^{(I,t)}) = \begin{cases} \lambda_t, & \text{if } k-1 = \ell(I) = 0,\\ \sum_{i\geqslant t}\phi_{k-1}(Sq^{i-t}a^I)\lambda_i, & \text{if } k-1 = \ell(I) > 0, \end{cases}$$

for any $a^{(I,t)} = a_1^{(i_1)}a_1^{(i_2)}\ldots a_{k-1}^{(i_{k-1})}a_k^{(t)} \in H_*(B\mathbb{V}_k)$ and $I = (i_1, i_2, \ldots, i_{k-1})$.

Proposition 4.2 (See Chon and Hà [3]). *If $b \in PH_*(BV_k)$, then $\phi_k(b)$ is a cycle in the lambda algebra Λ and $\mathrm{Tr}_k([b]) = [\phi_k(b)]$.*

Now we are ready to prove Theorem 1.3.

According to Theorem 1.2, $\{[p],[q]\}$ is a basis of $(QP_5)_{15}^{GL_5}$. Let $\{p^*, q^*\}$ be the basis of $\mathbb{F}_2 \underset{GL_5}{\otimes} PH_{15}(B\mathbb{V}_5)$ which is dual to $\{[p],[q]\}$. It is easy to see that $a_5^{(15)} \in PH_{15}(B\mathbb{V}_5)$ and $\langle a_5^{(15)}, p\rangle = 1, \langle a_5^{(15)}, q\rangle = 0$. Consider the element $b = \sum_{I\in\mathcal{J}} a^I \in H_{15}(B\mathbb{V}_5)$, where $\mathcal{J}$ is the set of all the following sequences:

$(1,1,1,6,6),\ (1,2,2,5,5),\ (1,2,1,6,5),\ (1,1,2,5,6),\ (1,4,2,5,3),\ (1,4,1,6,3),$
$(1,3,2,6,3),\ (1,2,4,3,5),\ (1,1,4,3,6),\ (1,4,4,3,3),\ (1,6,1,1,6),\ (1,5,2,2,5),$
$(1,6,1,2,5),\ (1,5,2,1,6),\ (1,5,2,4,3),\ (1,6,1,4,3),\ (1,6,2,3,3),\ (1,3,4,2,5),$
$(1,3,4,1,6),\ (1,3,3,2,6),\ (1,3,4,4,3),\ (1,1,6,1,6),\ (1,2,5,2,5),\ (1,2,6,1,5),$
$(1,1,5,2,6),\ (1,4,5,2,3),\ (1,4,6,1,3),\ (1,3,6,2,3),\ (1,2,3,4,5),\ (1,1,3,4,6),$
$(1,4,3,4,3),\ (1,3,1,5,5),\ (1,5,5,1,3),\ (1,5,1,3,5),\ (1,5,3,1,5),\ (1,5,3,3,3).$

By a direct computation we see that $b \in PH_{15}(B\mathbb{V}_5)$ and $\langle b, p\rangle = 0,\ \langle b, q\rangle = 1$. Hence we obtain $[a_5^{(15)}] = p^*$ and $[b] = q^*$. A direct computation shows

$$\begin{aligned} \phi_5(a_5^{(15)}) &= \lambda_0^4\lambda_{15},\\ \phi_5(b) &= \lambda_1\bar{d}_0 + \delta(\lambda_1\lambda_9\lambda_3^2 + \lambda_1\lambda_3\lambda_9\lambda_3). \end{aligned}$$

Using Proposition 4.2, one gets $\mathrm{Tr}_5(p^*) = \mathrm{Tr}_5([a_5^{(15)}]) = h_0^4h_4$ and $\mathrm{Tr}_5(q^*) = \mathrm{Tr}_5([b]) = h_1d_0$. The theorem follows. $\square$

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References

[1] J. M. Boardman, *Modular representations on the homology of power of real projective space*, in: M. C. Tangora (Ed.), Algebraic Topology, Oaxtepec, 1991, in: Contemp. Math., vol. 146, 1993, pp. 49-70, MR1224907.

[2] R. R. Bruner, L. M. Hà and N. H. V. Hung, *On behavior of the algebraic transfer*, Trans. Amer. Math. Soc. 357 (2005), 473-487, MR2095619.

[3] P. H. Chon and L. M. Hà, *Lambda algebra and the Singer transfer*, C. R. Math. Acad. Sci. Paris 349 (2011), 21-23, MR2755689.

[4] P. H. Chon and L. M. Hà, *On May spectral sequence and the algebraic transfer*, Manuscripta Math. 138 (2012), 141-160, MR2898751.

[5] M. C. Crabb and J. R. Hubbuck, *Representations of the homology of BV and the Steenrod algebra II*, Algebraic Topology: new trend in localization and periodicity, Progr. Math. 136 (1996), 143-154, MR1397726.

[6] L. M. Hà, *Sub-Hopf algebras of the Steenrod algebra and the Singer transfer*, "Proceedings of the International School and Conference in Algebraic Topology, Hà Nˆọi 2004", Geom. Topol. Monogr., Geom. Topol. Publ., Coventry, vol. 11 (2007), 81-105, MR2402802.

[7] N. H. V. Hung, *Spherical classes and the algebraic transfer*, Trans. Amer. Math. Soc. 349 (1997), 3893-3910, MR1433119.

[8] N. H. V. Hung, *The cohomology of the Steenrod algebra and representations of the general linear groups*, Trans. Amer. Math. Soc. 357 (2005), 4065-4089, MR2159700.

[9] N. H. V. Hung and V. T. N. Quỳnh, *The image of Singer's fourth transfer,* C. R. Math. Acad. Sci. Paris 347 (2009), 1415-1418, MR2588792.

[10] M. Kameko, *Products of projective spaces as Steenrod modules*, PhD. Thesis, Johns Hopkins University, 1990.

[11] M. Kameko, *Generators of the cohomology of BV_3*, J. Math. Kyoto Univ. 38 (1998), 587-593, MR1661173.

[12] W. H. Lin, $\mathrm{Ext}_{\mathcal{A}}^{4,*}(\mathbb{Z}/2,\mathbb{Z}/2)$ *and* $\mathrm{Ext}_{\mathcal{A}}^{5,*}(\mathbb{Z}/2,\mathbb{Z}/2)$, Topology Appl., 155 (2008), 459-496, MR2380930.

[13] T. N. Nam, *$\mathcal{A}$-générateurs génériquess pour l'algèbre polynomiale*, Adv. Math. 186 (2004), 334-362, MR2073910.

[14] T. N. Nam, *Transfert algébrique et action du groupe linéaire sur les puissances divisées modulo 2*, Ann. Inst. Fourier (Grenoble) 58 (2008), 1785-1837, MR2445834.

[15] F. P. Peterson, *Generators of $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$ as a module over the Steenrod algebra*, Abstracts Amer. Math. Soc. No. **833** April 1987.

[16] S. Priddy, *On characterizing summands in the classifying space of a group, I*, Amer. Jour. Math. 112 (1990), 737-748, MR1073007.

[17] V. T. N. Quỳnh, *On behavior of the fifth algebraic transfer*, "Proceedings of the International School and Conference in Algebraic Topology, Hà Nˆọi 2004", Geom. Topol. Monogr., Geom. Topol. Publ., Coventry, vol. 11 (2007), 309-326, MR2402811.

[18] J. Repka and P. Selick, *On the subalgebra of $H_*((\mathbb{R}P^\infty)^n;\mathbb{F}_2)$ annihilated by Steenrod operations*, J. Pure Appl. Algebra 127 (1998), 273-288, MR1617199.

[19] J. H. Silverman, *Hit polynomials and the canonical antiautomorphism of the Steenrod algebra*, Proc. Amer. Math. Soc. 123 (1995), 627-637, MR1254854.

[20] W. M. Singer, *The transfer in homological algebra*, Math. Zeit. 202 (1989), 493-523, MR1022818.

[21] W. M. Singer, *On the action of the Steenrod squares on polynomial algebras*, Proc. Amer. Math. Soc. 111 (1991), 577-583, MR1045150.

[22] N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press, Princeton N.J (1962), MR0145525.

[23] N. Sum, *The negative answer to Kameko's conjecture on the hit problem*, C. R. Math. Acad. Sci. Paris, Ser. I 348 (2010), 669-672.

[24] N. Sum, *The negative answer to Kameko's conjecture on the hit problem*, Adv. Math. 225 (2010), 2365-2390.

[25] N. Sum, *On the hit problem for the polynomial algebra*, C. R. Math. Acad. Sci. Paris, Ser. I, 351 (2013), 565-568.

[26] R. M. W. Wood, *Steenrod squares of polynomials and the Peterson conjecture*, Math. Proc. Cambriges Phil. Soc. 105 (1989), 307-309, MR0974986.

[27] R. M. W. Wood, *Problems in the Steenrod algebra*, Bull. London Math. Soc. 30 (1998) 449-517, MR1643834.