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NUMERICAL STUDY ON THREE-DIMENSIONAL QUADRATIC NONCONFORMING BRICK ELEMENTS

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Abstract

Recently a new nonconforming brick element of fourteen DOFs with quadratic convergence for the energy norm is introduced by Meng, Sheen, Luo, and Kim [23]. The purpose of this paper is to compare this element with the brick elements introduced by Smith and Kidger [31]. The above elements have fourteen degrees of freedom which contain the eight vertex values and the six barycenter values at surfaces. The underlying element are based on P_2. The finite element of Meng-Sheen-Luo-Kim adds the span of four polynomials $\{xyz, x[x^2-\frac{3}{5}(y^2+z^2)], y[y^2-\frac{3}{5}(x^2+z^2)], z[z^2-\frac{3}{5}(x^2+y^2)]\}$, while the Smith-Kidger elements add the span of four other polynomials. In this paper, we particularly consider the two classes of Smith-Kidger elements. The first and fifth types add the span of $\{xyz, x^2y, y^2, z^2x\}$ and the span of $\{xyz, x^2y+xy^2, y^2z+$

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$yz^2, z^2x + zx^2\}$, respectively, while the sixth type adds the span of $\{xyz, xy^2z^2, x^2yz^2, x^2y^2zx\}$. We compare these three elements with the Meng-Sheen-Luo-Kim element numerically and give rates of convergence for Poisson equations.

1 Introduction

It seems that at least for the three-rectangular element, the use of serendipity elements provide a more efficient numerical procedure than the usual trilinear elements since the serendipity elements have only 14 DOFs while the conventional trilinear elements contain 27 DOFs. It goes back to Smith and Kidger [31] who successfully developed three-dimensional brick elements of 14 DOFs. They investigated six most possible 14 DOFs elements systematically considering the Pascal pyramid, and concluded that the type 1 (as well as type 2) and type 6 elements are successful ones. The type 1 element adds the span of four nonsymmetric cubic polynomials $\{xyz, x^2y, y^2z, z^2x\}$ while the type 6 element the span of $\{xyz, xy^2z^2, x^2yz^2, x^2y^2z\}$ to P_2.

Since then, any study of new three dimensional serendipity elements has not been reported. Only recently a new nonconforming brick element of fourteen DOFs with quadratic and cubic convergence in the energy and L^2 norms is introduced by Meng, Sheen, Luo, and Kim [23], which has the same type of DOFs but has only cubic polynomials added to P_2.

The purpose of this paper is to compare these elements numerically.

There are several well-known conforming serendipity elements including the 8 node element whose DOFs are the values at the 8 vertices and the 20 node elements whose DOFs are the values at the 8 vertices and those at the 12 edge midpoints, respectively. A systematic study on serendipity elements in general dimension has been reported by Arnold and Awanou [1].

In the meanwhile nonconforming finite element methods have been developed fast in recent years [16, 17, 26, 32]. The nonconforming finite element methods successfully provide stable numerical solutions of many practical fluid flow and solid mechanics problems, such as, [5, 6, 8, 13, 12, 11, 27], for linear or nonlinear Stokes problems and [2, 4, 10, 20, 22, 19, 24, 38, 15, 37, 3] for elasticity related problems.

For linear nonconforming finite elements for triangles or tetrahedrons, the Crouzeix and Raviart element [8] provides stable finite element pairs for Stokes problems. Wilson defined a 2D nonconforming element on rectangles which may not converge for arbitrary quadrilaterals; see [30]. For three dimensions, Wilson also defined a linear-order nonconforming brick element [7, 36] with 11 DOFs whose basis consists of trilinear polynomials plus $\{1-x^2, 1-y^2, 1-z^2\}$ on $\widehat{\mathbf{K}} = [-1, 1]^3$. This element obtains an $O(h)$ convergence rate in energy norm. (see [7, Page 217, Remark 4.2.3])

Concerning quadrilateral nonconforming elements, Han [14] introduced a rectangular element with local degrees of freedom being five, Rannacher-Turek [27] presented a rotated Q_1 nonconforming element, Douglas-Santos-Sheen-Ye [9] introduced a nonconforming finite element using the four values at the midpoints of edges as degrees of freedom, and Park-Sheen presented a P_1-nonconforming finite element on quadrilateral meshes which has the lowest degrees of freedom [26].

Quadratic nonconforming elements on triangles and simplices have been introduced by Fortin and Soulie [13] and Fortin [12], respectively. Lee and Sheen [21] proposed a quadratic nonconforming element on rectangular meshes, corresponding to the triangular Fortin-Soulie element, and Kim et al. [18] constructed a piecewise P_2-nonconforming finite element for general quadrilateral meshes.

For biharmonic problems, the Morley–type element [25, 29] on triangular meshes has been generalized to three and any dimension [34, 35] based on simplicial meshes by Wang and Xu. Similarly, for rectangular meshes, the incomplete biquadratic element [28] has been generalized to three dimensions by Wang, Shi and Xu [33].

The plan of this paper is as follows. In § 2, we define a quadratic nonconforming brick element. In §3, we will introduce an interpolation operator and provide convergence analysis of optimal order. Finally, in §4 we conclude our results.

2 The quadratic nonconforming brick elements

Let $\widehat{\mathbf{K}} = [-1, 1]^3$ and denote the vertices and face-centroids by $V_j, 1 \leq j \leq 8$, and $M_k, 1 \leq k \leq 6$, respectively. (see Fig. 1)

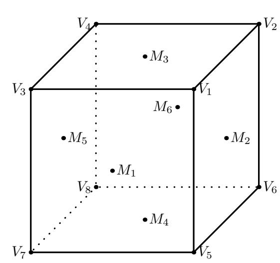

Figure 1: V_j denotes the vertices, $j = 1, 2, \ldots, 8$, and M_k denotes the face-centroid, $k = 1, 2, \ldots, 6$.

2.1 The Smith-Kidger elements

Smith and Kidger [31] defined the following six 14-node elements:

$$\widehat{\mathbb{P}}_{SK}^{(1)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{xyz, x^2y, y^2z, z^2x\}, \tag{1a}$$
$$\widehat{\mathbb{P}}_{SK}^{(2)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{xyz, xy^2, yz^2, zx^2\}, \tag{1b}$$
$$\widehat{\mathbb{P}}_{SK}^{(3)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{xyz, x^3, y^3, z^3\}, \tag{1c}$$
$$\widehat{\mathbb{P}}_{SK}^{(4)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{xyz, x^2yz, xy^2z, xyz^2\}, \tag{1d}$$
$$\widehat{\mathbb{P}}_{SK}^{(5)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{xyz, x^2y + xy^2, y^2z + yz^2, z^2x + zx^2\}, \tag{1e}$$
$$\widehat{\mathbb{P}}_{SK}^{(6)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{xyz, xy^2z^2, x^2yz^2, x^2y^2z\}, \tag{1f}$$

whose DOFs are the function values at the eight vertices and the six face-centroids. They reported that Type 3 element fails and inadmissible, and Type 1 and Type 2 elements produce similar results to Type 3 element but better results than Type 4 element, although Type 1 and Type 2 elements are not symmetric but Type 4 element is symmetric. Type 5 element fails the MacNeal and Harder patch test, while Type 6 element passes it. However, we will present numerical results in \$ 4, where it turns out that Type 6 element does not give better results than Type 4 element. Also, we point out the basis functions given in the paper [31] do not belong to $\widehat{\mathbb{P}}_{SK}^{(6)}$. Indeed, we observe that Type 1 (and 2) and Type 5 elements give optimal convergence results both in L^2 and H^1 norms at least for the second-order elliptic problems, while Type 6 element loses one order of accuracy in each norm.

2.2 Meng-Sheen-Luo-Kim nonconforming brick element

Meng, Sheen, Luo, and Kim [23] introduced the following elements. Set $\theta(x, y, z) = x[x^2 - \frac{3}{5}(y^2 + z^2)]$. Then define the 14-node elements by

$$\widehat{\mathbb{P}}_{MSLK}^{(k)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{xyz, \theta(x, y, z), \theta(y, z, x), \theta(z, x, y)\}, \quad k = 1, 2.$$

with the following two types of DOFs:

1. $\widehat{\Sigma}_{\widehat{\mathbf{K}}}^{(1)}(\widehat{\phi}) = \{\widehat{\phi}(V_i), i = 1, 2, \ldots, 8; \widehat{\phi}(M_j), , j = 1, 2, \ldots, 6\}$ for all $\widehat{\phi} \in \widehat{\mathbb{P}}_{MSLK}^{(1)}$;
2. $\widehat{\Sigma}_{\widehat{\mathbf{K}}}^{(2)}(\widehat{\phi}) = \{\widehat{\phi}(V_i), i = 1, 2, \ldots, 8; \frac{1}{|F_j|}\int_{F_j} \widehat{\phi}\, \mathrm{d}\sigma, j = 1, 2, \ldots, 6\}$ for all $\widehat{\phi} \in \widehat{\mathbb{P}}_{MSLK}^{(2)}$.

Then the following propositions hold, which confirm that the Meng-Sheen-Luo-Kim brick elements fulfill the patch test.

Proposition 1. *[23] By F_k denote the face containing the centroid M_k and by $V_j^{F_k}, j = 1, 2, 3, 4$ denote the vertices on the surface F_k. If $p \in \widehat{\mathbb{P}}^{(1)}_{MSLK}$ and $p(V_j^{F_k}) = 0, j = 1, 2, 3, 4, p(M_k) = 0$, then*

$$\int_{F_k} p(x, y, z)q(x, y, z)\,\mathrm{d}\sigma = 0, \quad k = 1, 2, \ldots, 6, \tag{2}$$

for all $q \in P_1(\mathbb{R}^3)$.

Proposition 2. *[23] If $p \in \widehat{\mathbb{P}}^{(2)}_{MSLK}$ and $p(V_j^{F_k}) = 0, j = 1, 2, 3, 4, \int_{F_k} p\,\mathrm{d}\sigma = 0$, then*

$$\int_{F_k} p(x, y, z)q(x, y, z)\,\mathrm{d}\sigma = 0, \quad k = 1, 2, \ldots, 6, \tag{3}$$

for all $q \in P_1(\mathbb{R}^3)$.

The basis functions for the element $\widehat{\mathbb{P}}^{(1)}_{MSLK}$ whose DOFs are the values at the eight vertices and the six surface-centroids are given as follows:

$$\begin{aligned}
\widehat{\phi}^V_{1,1,1}(x, y, z) = \frac{1}{48}\Big(& -3 + 5(x + y + z) + 3\eta(x, y, z) + 6(xy + xz + yz + xyz) \\
& -5\theta(x, y, z) - 5\theta(y, z, x) - 5\theta(z, x, y)\Big). \\
\widehat{\phi}^V_{1,1,-1}(x, y, z) = \frac{1}{48}\Big(& -3 + 5(x + y - z) + 3\eta(x, y, z) + 6(xy - xz - yz - xyz) \\
& -5\theta(x, y, z) - 5\theta(y, z, x) + 5\theta(z, x, y)\Big), \\
\widehat{\phi}^V_{1,-1,1}(x, y, z) = \frac{1}{48}\Big(& -3 + 5(x - y + z) + 3\eta(x, y, z) + 6(-xy + xz - yz - xyz) \\
& -5\theta(x, y, z) + 5\theta(y, z, x) - 5\theta(z, x, y)\Big), \\
\widehat{\phi}^V_{1,-1,-1}(x, y, z) = \frac{1}{48}\Big(& -3 + 5(x - y - z) + 3\eta(x, y, z) + 6(-xy - xz + yz + xyz) \\
& -5\theta(x, y, z) + 5\theta(y, z, x) + 5\theta(z, x, y)\Big), \\
\widehat{\phi}^V_{-1,1,1}(x, y, z) = \frac{1}{48}\Big(& -3 + 5(-x + y + z) + 3\eta(x, y, z) + 6(-xy - xz + yz - xyz) \\
& +5\theta(x, y, z) - 5\theta(y, z, x) - 5\theta(z, x, y)\Big), \\
\widehat{\phi}^V_{-1,1,-1}(x, y, z) = \frac{1}{48}\Big(& -3 + 5(-x + y - z) + 3\eta(x, y, z) + 6(-xy + xz - yz + xyz) \\
& +5\theta(x, y, z) - 5\theta(y, z, x) + 5\theta(z, x, y)\Big),
\end{aligned}$$

$$\widehat{\phi}^V_{-1,-1,1}(x,y,z) = \frac{1}{48}\Big(-3+5(-x-y+z)+3\eta(x,y,z)+6(xy-xz-yz+xyz)$$
$$+5\theta(x,y,z)+5\theta(y,z,x)-5\theta(z,x,y)\Big),$$
$$\widehat{\phi}^V_{-1,-1,-1}(x,y,z) = \frac{1}{48}\Big(-3+5(-x-y-z)+3\eta(x,y,z)+6(xy+xz+yz-xyz)$$
$$+5\theta(x,y,z)+5\theta(y,z,x)+5\theta(z,x,y)\Big).$$
$$\widehat{\phi}^F_{1,0,0}(x,y,z) = \frac{1}{12}\big(3+x+3\zeta(x,y,z)+5\theta(x,y,z)\big).$$
$$\widehat{\phi}^F_{-1,0,0}(x,y,z) = \frac{1}{12}\big(3-x+3\zeta(x,y,z)-5\theta(x,y,z)\big),$$
$$\widehat{\phi}^F_{0,1,0}(x,y,z) = \frac{1}{12}\big(3+y+3\zeta(y,z,x)+5\theta(y,z,x)\big),$$
$$\widehat{\phi}^F_{0,-1,0}(x,y,z) = \frac{1}{12}\big(3-y+3\zeta(y,z,x)-5\theta(y,z,x)\big),$$
$$\widehat{\phi}^F_{0,0,1}(x,y,z) = \frac{1}{12}\big(3+z+3\zeta(z,x,y)+5\theta(z,x,y)\big),$$
$$\widehat{\phi}^F_{0,0,-1}(x,y,z) = \frac{1}{12}\big(3-z+3\zeta(z,x,u)-5\theta(z,x,y)\big),$$

where $\eta(x,y,z) = x^2+y^2+z^2$ and $\theta(x,y,z) = x^3-\frac{3}{5}(y^2+z^2)$ and the basis function corresponding to the node $(1,1,1)$ is denoted by $\widehat{\phi}^V_{1,1,1}$, and that corresponding to the surface-centroid by $\widehat{\phi}^F_{1,0,0}$.

The basis functions for the element $\widehat{\mathbb{P}}^{(2)}_{MSLK}$ whose DOFs are the values at the eight vertices and the six average surface integrals are given as follows:

$$\begin{aligned}
\widehat{\psi}^V_{1,1,1}(x,y,z) &= \frac{1}{32}\Big(-5+3x+3y+3z+3\eta(x,y,z)+4xy+4xz+4yz+4xyz \\
&\quad -5\theta(x,y,z)-5\theta(y,z,x)-5\theta(z,x,y)\Big), \\
\widehat{\psi}^V_{1,1,-1}(x,y,z) &= \frac{1}{32}\Big(-5+3x+3y-3z+3\eta(x,y,z)+4xy-4xz-4yz-4xyz \\
&\quad -5\theta(x,y,z)-5\theta(y,z,x)+5\theta(z,x,y)\Big), \\
\widehat{\psi}^V_{1,-1,1}(x,y,z) &= \frac{1}{32}\Big(-5+3x-3y+3z+3\eta(x,y,z)-4xy+4xz-4yz-4xyz \\
&\quad -5\theta(x,y,z)+5\theta(y,z,x)-5\theta(z,x,y)\Big), \\
\widehat{\psi}^V_{1,-1,-1}(x,y,z) &= \frac{1}{32}\Big(-5+3x-3y-3z+3\eta(x,y,z)-4xy-4xz+4yz+4xyz \\
&\quad -5\theta(x,y,z)+5\theta(y,z,x)+5\theta(z,x,y)\Big), \\
\widehat{\psi}^V_{-1,1,1}(x,y,z) &= \frac{1}{32}\Big(-5-3x+3y+3z+3\eta(x,y,z)-4xy-4xz+4yz-4xyz \\
&\quad +5\theta(x,y,z)-5\theta(y,z,x)-5\theta(z,x,y)\Big),
\end{aligned}$$

$$
\begin{aligned}
\widehat{\psi}^{V}_{-1,1,-1}(x,y,z) &= \frac{1}{32}\Big(-5-3x+3y-3z+3\eta(x,y,z)-4xy+4xz-4yz+4xyz \\
&\quad +5\theta(x,y,z)-5\theta(y,z,x)+5\theta(z,x,y)\Big),\\
\widehat{\psi}^{V}_{-1,-1,1}(x,y,z) &= \frac{1}{32}\Big(-5-3x-3y+3z+3\eta(x,y,z)+4xy-4xz-4yz+4xyz \\
&\quad +5\theta(x,y,z)+5\theta(y,z,x)-5\theta(z,x,y)\Big),\\
\widehat{\psi}^{V}_{-1,-1,-1}(x,y,z) &= \frac{1}{32}\Big(-5-3x-3y-3z+3\eta(x,y,z)+4xy+4xz+4yz-4xyz \\
&\quad +5\theta(x,y,z)+5\theta(y,z,x)+5\theta(z,x,y)\Big),\\
\widehat{\psi}^{F}_{1,0,0}(x,y,z) &= \frac{1}{8}\Big(3+x+3\zeta(x,y,z)+5\theta(x,y,z)\Big),\\
\widehat{\psi}^{F}_{-1,0,0}(x,y,z) &= \frac{1}{8}\Big(3-x+3\zeta(x,y,z)-5\theta(x,y,z)\Big),\\
\widehat{\psi}^{F}_{0,1,0}(x,y,z) &= \frac{1}{8}\Big(3+y+3\zeta(y,z,x)+5\theta(y,z,x)\Big),\\
\widehat{\psi}^{F}_{0,-1,0}(x,y,z) &= \frac{1}{8}\Big(3-y+3\zeta(y,z,x)-5\theta(y,z,x)\Big),\\
\widehat{\psi}^{F}_{0,0,1}(x,y,z) &= \frac{1}{8}\Big(3+z+3\zeta(z,x,y)+5\theta(z,x,y)\Big),\\
\widehat{\psi}^{F}_{0,0,-1}(x,y,z) &= \frac{1}{8}\Big(3-z+3\zeta(z,x,y)-5\theta(z,x,y)\Big),
\end{aligned}
$$

where $\widehat{\psi}^{V}_{1,1,1}$ denotes the basis function corresponding to the node $(1,1,1)$, $\widehat{\psi}^{F}_{1,0,0}$ denotes the basis function corresponding to the surface integral containing the centroid $(1,0,0)$, and so on.

2.3 Nonconforming brick element spaces

Assume that $\Omega \in \mathbb{R}^3$ is a parallelepiped domain with boundary Γ. Let $(\mathrm{T}_h)_{h>0}$ be a regular family of triangulation of Ω into parallelepipeds $\mathbf{K}_j, j = 1, 2, \ldots, N_{\mathbf{K}}$, where $h = \max_{\mathbf{K}\in\mathrm{T}_h} h_{\mathbf{K}}$ with $h_{\mathbf{K}} = \mathrm{diam}(\mathbf{K})$. For each $\mathbf{K} \in \mathrm{T}_h$, let $F_{\mathbf{K}} : \widehat{\mathbf{K}} \to \mathbb{R}^3$ be an invertible affine mapping such that

$$\mathbf{K} = F_{\mathbf{K}}(\widehat{\mathbf{K}}),$$

and denote $\phi_{\mathbf{K}} = \widehat{\phi} \circ F_{\mathbf{K}}^{-1} : \mathbf{K} \to \mathbb{R}$ for all $\widehat{\phi} \in \widehat{\mathbb{P}}$, whose collection will be designated by

$$\mathbb{P}_{\mathbf{K}} = \mathrm{Span}\{\phi_{\mathbf{K}},\, \widehat{\phi} \in \widehat{\mathbb{P}}\},$$

where $\widehat{\mathbb{P}}$ can be $\widehat{\mathbb{P}}^{(1)}_{SK}, \widehat{\mathbb{P}}^{(5)}_{SK}, \widehat{\mathbb{P}}^{(6)}_{SK}, \widehat{\mathbb{P}}^{(1)}_{MSLK}$, or $\widehat{\mathbb{P}}^{(2)}_{MSLK}$.

Let N_V and N_F denote the numbers of vertices and faces, respectively. Then set

$$
\begin{aligned}
\mathrm{V}_h &= \{V_1, V_2, \ldots, V_{N_V} : \quad \text{the set of all vertices of } \mathbf{K} \in \mathrm{T}_h\},\\
\mathrm{F}_h &= \{F_1, F_2, \ldots, F_{N_F} : \quad \text{the set of all faces of } \mathbf{K} \in \mathrm{T}_h\},\\
\mathrm{M}_h &= \{M_1, M_2, \ldots, M_{N_F} : \quad \text{the set of all face-centroids on } \mathrm{F}_h\}.
\end{aligned}
$$

We are now in a position to define the following nonconforming finite element spaces:

$$\mathrm{NC}^h = \{\phi : \Omega \to \mathbb{R} |\ \phi|_{\mathbf{K}} \in \mathbb{P}_{\mathbf{K}} \forall \mathbf{K} \in \mathrm{T}_h, \phi \text{ is continuous at all } V_j \in \mathrm{V}_h, M_k \in \mathrm{M}_h\},$$
$$\mathrm{NC}_0^h = \{\phi \in \mathrm{NC}^h |\ \phi(V) = 0\ \forall V_j \in \mathrm{V}_h \cap \Gamma \text{ and } \phi(M) = 0\ \forall M_k \in \mathrm{M}_h \cap \Gamma\}.$$

Obviously,

$$\dim(\mathrm{NC}^h) = N_V + N_F, \text{ and } \dim(\mathrm{NC}_0^h) = N_V^i + N_F^i,$$

where N_V^i and N_F^i mean the numbers of interior vertices and faces, respectively. NC^h and NC_0^h will serve as the nonconforming spaces for Neumann and Dirichlet problems, respectively.

2.4 Nonconforming brick element methods

Consider the following Dirichlet problem:

$$-\nabla \cdot (\boldsymbol{\alpha} \nabla u) + \beta u = f, \quad \Omega, \tag{4a}$$
$$u = 0, \quad \Gamma, \tag{4b}$$

with $\boldsymbol{\alpha} = (\alpha_{jk}), \alpha_{jk}, \beta \in L^\infty(\Omega), j, k = 1, 2, 3, 0 < \alpha_* |\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi}^t \boldsymbol{\alpha}(x) \boldsymbol{\xi} \leq \alpha^* |\boldsymbol{\xi}|^2 < \infty, \boldsymbol{\xi} \in \mathbb{R}^3, \beta(x) \geq 0, x \in \Omega$, and $f \in H^1(\Omega)$. We will assume that the coefficients are sufficiently smooth and that the elliptic problem (4) has an $H^3(\Omega)$-regular solution so that u with $\|u\|_3 \leq C \|f\|_1$. The weak problem is then given as usual: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad v \in H_0^1(\Omega), \tag{5}$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is the bilinear form defined by $a(u, v) = (\boldsymbol{\alpha} \nabla u, \nabla v) + (\beta u, v)$ for all $u, v \in H_0^1(\Omega)$. Our nonconforming method for Problem (4) states as follows: find $u_h \in \mathrm{NC}_0^h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad v_h \in \mathrm{NC}_0^h, \tag{6}$$

where

$$a_h(u, v) = \sum_{\mathbf{K} \in \mathrm{T}_h} a_{\mathbf{K}}(u, v),$$

with $a_{\mathbf{K}}$ being the restriction of a to $\mathbf{K}$.

Then we have the following optimal order of convergence theorem.

Theorem 1. *[23] Let $u \in H^3(\Omega) \cap H_0^1(\Omega)$ and $u_h \in \mathrm{NC}_0^h$ satisfy (5) and (6), respectively. Then we have the energy- and L^2-norm error estimates:*

$$||u - u_h||_h \leq Ch^2 ||u||_{H^3(\Omega)},$$
$$||u - u_h||_0 \leq Ch^3 ||u||_{H^3(\Omega)}.$$

Instead of the Dirichlet problem, if the following Neumann problem

$$-\nabla \cdot (\boldsymbol{\alpha}\nabla u) + \beta u = f, \quad \Omega, \tag{7a}$$
$$\nu \cdot (\boldsymbol{\alpha}\nabla u) + \gamma u = g, \quad \Gamma. \tag{7b}$$

is considered, the weak problem (5) is then replaced by finding $u \in H^1(\Omega)$ such that

$$a^n(u, v) = (f, v) + \langle g, v\rangle, \quad v \in H^1(\Omega), \tag{8}$$

where a^n is the bilinear form defined by $a^n(u, v) = (\boldsymbol{\alpha}\nabla u, \nabla v)+(\beta u, v)+\langle \gamma u, v\rangle$ for all $u, v \in H^1(\Omega)$, and $\langle \cdot, \cdot\rangle$ is the paring between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Thus, the nonconforming method for Problem (7) states as follows: find $u_h \in \mathrm{NC}^h$ such that

$$a^n_h(u_h, v_h) = (f, v_h) + \langle g, v_h\rangle, \quad v_h \in \mathrm{NC}^h. \tag{9}$$

Then all the arguments given above for Dirichlet case hold analogously, which are omitted here, and therefore we have the results as follows.

Theorem 2. *[23] Let $u \in H^3(\Omega)$ and $u_h \in \mathrm{NC}^h$ satisfy* (8) *and* (9)*, respectively. Then we have the energy- and L^2-norm error estimates:*

$$||u - u_h||_h \le Ch^2||u||_{H^3(\Omega)},$$
$$||u - u_h||_0 \le Ch^3||u||_{H^3(\Omega)}.$$

3 Numerical comparison of brick elements

In this section, we present numerical results to confirm the theoretical results obtained in the previous section. In our numerical experiments, we consider two types of meshes: first type of uniform cubic meshes and the second type of hexahedral meshes such that each $h \times 2h \times h$ rectangular parallelpiped is divided into two hexahedrals whose shape are shown in Fig 2. Notice that the hexahedrals in second type of meshes are not parallelepipes.

Consider the following Dirichlet problem:

$$-\Delta u = f, \quad \text{in} \quad \Omega, \tag{10a}$$
$$u = 0, \quad \text{on} \quad \Gamma. \tag{10b}$$

where $\Omega = (0, 1)^3$. The source term f is calculated from the exact solution

$$u(x, y, z) = e^{(x+y+z)} \sin(\pi x) \sin(\pi y) \sin(\pi z)$$

3.1 Numerical results for Meng-Sheen-Luo-Kim brick element

The following Tables 1–2 show the numerical results on the uniform cubic meshes using the Meng-Sheen-Luo-Kim nonconforming brick finite element with two types of degrees of freedom, addressing the error reduction ratios in L^2- and energy-norm.

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.4732E−01	-	0.1384E+01	-
$8\times8\times8$	1687	0.6145E−02	2.94	0.3791E+00	1.87
$16\times16\times16$	14895	0.7835E−03	2.97	0.9744E−01	1.96
$32\times32\times32$	125023	0.9857E−04	2.99	0.2454E−01	1.99
$64\times64\times64$	1024191	0.1234E−04	3.00	0.6148E−02	2.00

Table 1: L^2- and broken energy-norm errors and their reduction ratios on the uniform cubic meshes with $\widehat{\mathbb{P}}^{(1)}_{MSLK}$

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.4732E−01	-	0.1384E+01	-
$8\times8\times8$	1687	0.6145E−02	2.94	0.3791E+00	1.87
$16\times16\times16$	14895	0.7835E−03	2.97	0.9744E−01	1.96
$32\times32\times32$	125023	0.9857E−04	2.99	0.2454E−01	1.99
$64\times64\times64$	1024191	0.1234E−04	3.00	0.6148E−02	2.00

Table 2: L^2-and broken energy-norm errors and their reduction ratios on the uniform cubic meshes with $\widehat{\mathbb{P}}^{(2)}_{MSLK}$

Next, the following Tables 3–4 show the numerical results on the non-uniform meshes (see Fig 2) using the Meng-Sheen-Luo-Kim nonconforming brick finite element with two types of degrees of freedom, addressing the error reduction ratios in L^2- and energy norms.

Observe that both degrees of freedom for Meng-Sheen-Luo-Kim brick element gives optimal convergence results both in L^2 and H^1 norms at least for uniform cubic meshes. But both elements lose one order of accuracy in each norm if the meshes are not of parallelepiped shape. We do not see any differences in numerical values using either $\widehat{\mathbb{P}}^{(1)}_{MSLK}$ or $\widehat{\mathbb{P}}^{(2)}_{MSLK}$. This implies that one can use either face-average values or surface-centroid values associated with six faces DOFs.

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.1099E+00	-	0.2129E+01	-
$8\times8\times8$	1687	0.2293E−01	2.26	0.8453E+00	1.33
$16\times16\times16$	14895	0.5286E−02	2.12	0.3745E+00	1.17
$32\times32\times32$	125023	0.1289E−02	2.04	0.1794E+00	1.06
$64\times64\times64$	1024191	0.3200E−03	2.01	0.8848E−01	1.02

Table 3: L^2- and broken energy-norm errors and their reduction ratios on the non-uniform meshes ($\theta = 0.5$) with $\widehat{\mathbb{P}}^{(1)}_{MSLK}$

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.1099E+00	-	0.2129E+01	-
$8\times8\times8$	1687	0.2293E−01	2.26	0.8453E+00	1.33
$16\times16\times16$	14895	0.5286E−02	2.12	0.3745E+00	1.17
$32\times32\times32$	125023	0.1289E−02	2.04	0.1794E+00	1.06
$64\times64\times64$	1024191	0.3200E−03	2.01	0.8848E−01	1.02

Table 4: L^2- and broken energy-norm errors and their reduction ratios on the non-uniform meshes ($\theta = 0.5$) with $\widehat{\mathbb{P}}^{(2)}_{MSLK}$

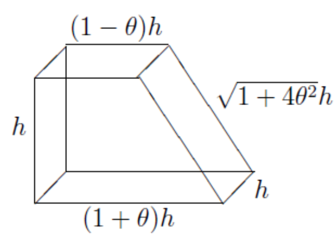

Figure 2: The shape of hexahedral element with θ, which is not a parallelepipe.

3.2 Numerical results for Smith-Kidger brick elements

The following Tables 5–6 show the numerical results for the uniform cubic meshes and non–uniform meshes using the $\widehat{\mathbb{P}}^{(1)}_{SK}$ brick element, addressing the error reduction ratios in L^2 and energy norms. The following Tables 7–8 show the numerical results for the uniform cubic meshes and non–uniform meshes using $\widehat{\mathbb{P}}^{(5)}_{SK}$ brick element, addressing the error reduction ratios in L^2 and energy norms.

The following Tables 9–10 show the numerical results for the uniform cubic

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.3287E−01	-	0.1047E+01	-
$8\times8\times8$	1687	0.3960E−02	3.05	0.2625E+00	2.00
$16\times16\times16$	14895	0.4871E−03	3.02	0.6549E−01	2.00
$32\times32\times32$	125023	0.6060E−04	3.01	0.1636E−01	2.00
$64\times64\times64$	1024191	0.7566E−05	3.00	0.4089E−02	2.00

Table 5: L^2- and energy-norm errors and their reduction ratios on the uniform cubic meshes using $\widehat{\mathbb{P}}_{SK}^{(1)}$

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.7601E−01	-	0.1806E+01	-
$8\times8\times8$	1687	0.1640E−01	2.21	0.7051E+00	1.36
$16\times16\times16$	14895	0.3901E−02	2.07	0.3191E+00	1.14
$32\times32\times32$	125023	0.9631E−03	2.02	0.1550E+00	1.04
$64\times64\times64$	1024191	0.2401E−03	2.00	0.7693E−01	1.01

Table 6: L^2- and energy-norm errors and their reduction ratios on the non-uniform meshes($\theta = 0.5$) using $\widehat{\mathbb{P}}_{SK}^{(1)}$

meshes and non–uniform meshes using $\widehat{\mathbb{P}}_{SK}^{(6)}$ brick element, addressing the error reduction ratios in L^2 and energy norms.

Observe that Type 1 (and 2) and Type 5 elements give optimal convergence results both in L^2 and H^1 norms at least for the second-order elliptic problems for uniform cubic meshes. But Type 6 element loses one order of accuracy in each norm in this case. This element may fail the patch test given in Propositions 1 and 2, contrary to the claim of passing McNeal and Harder patch test stated in the paper [31]. The other three elements $\widehat{\mathbb{P}}_{SK}^{(1)}$, $\widehat{\mathbb{P}}_{SK}^{(2)}$, and $\widehat{\mathbb{P}}_{SK}^{(1)}$ may pass the patch test given in Proposition 1, which needs to be investigated. We also notice that all the elements presented in this subsection lose one order of accuracy if the meshes are not of parallelepiped shape as the Meng-Sheen-Luo-Kim brick elements.

4 Conclusions

In this paper we examined four types of nonconforming brick elements with fourteen degrees of freedom. All the elements perform well to solve Dirichlet problem with optimal convergence properties. They are readily applicable to approximating three–dimensional solid mechanics. To compare the Meng-

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.3574E−01	-	0.1139E+01	-
$8\times8\times8$	1687	0.4618E−02	2.95	0.3052E+00	1.90
$16\times16\times16$	14895	0.5800E−03	2.99	0.7793E−01	1.97
$32\times32\times32$	125023	0.7254E−04	3.00	0.1960E−01	1.99
$64\times64\times64$	1024191	0.9067E−05	3.00	0.4906E−02	2.00

Table 7: L^2- and energy-norm errors and their reduction ratios on the uniform cubic meshes using $\widehat{\mathbb{P}}_{SK}^{(5)}$

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.7827E−01	-	0.1913E+01	-
$8\times8\times8$	1687	0.1666E−01	2.23	0.7892E+00	1.28
$16\times16\times16$	14895	0.3920E−02	2.09	0.3592E+00	1.14
$32\times32\times32$	125023	0.9615E−03	2.03	0.1735E+00	1.05
$64\times64\times64$	1024191	0.2388E−03	2.00	0.8569E−01	1.02

Table 8: L^2- and energy-norm errors and their reduction ratios on the non-uniform meshes($\theta = 0.5$) using $\widehat{\mathbb{P}}_{SK}^{(5)}$

Sheen-Luo-Kim elements with Smith-Kidger elements, the former are symmetric and contain cubic polynomials only, while the latter are either nonsymmetric or contain quintic polynomials. Hence potentially the Meng-Sheen-Luo-Kim elements may behave better for some nonsymmetric problems, but this point is still under investigation.

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meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.7507E−01	-	0.1232E+01	-
$8\times8\times8$	1687	0.2315E−01	1.70	0.7671E+00	0.68
$16\times16\times16$	14895	0.6065E−02	1.93	0.4165E+00	0.88
$32\times32\times32$	125023	0.1536E−02	1.98	0.2131E+00	0.97
$64\times64\times64$	1024191	0.3867E−03	1.99	0.1073E+00	0.99

Table 9: L^2- and energy-norm errors and their reduction ratios on the uniform cubic meshes using $\widehat{\mathbb{P}}_{SK}^{(6)}$

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\|u-u_h\|_{1,h}$	ratio
$4\times4\times4$	171	0.9388E−01	-	0.1878E+01	-
$8\times8\times8$	1687	0.2587E−01	1.86	0.9904E+00	0.92
$16\times16\times16$	14895	0.6607E−02	1.97	0.5102E+00	0.96
$32\times32\times32$	125023	0.1660E−02	1.99	0.2573E+00	0.99
$64\times64\times64$	1024191	0.4156E−03	2.00	0.1288E+00	1.00

Table 10: L^2- and energy-norm errors and their reduction ratios on the non-uniform meshes($\theta = 0.5$) using $\widehat{\mathbb{P}}_{SK}^{(6)}$

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