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APPLICATIONS OF KAPLANSKY-COHEN'S THEOREM

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Abstract

In this note, we prove that for a finitely generated, quasi-projective duo right R-module M which is a self-generator, if every prime submodule of M is M-cyclic, then every submodule of M is M-cyclic.

1 Introduction

In 1949, Irving Kaplansky[5] proved that a commutative ring is a principal domain if and only if every prime ideal is principal. After that, I.S.Cohen (1950) proved that a commutative ring R is noetherian if and only if every prime ideal of R is finitely generated. Since then, many authors have been trying to transfer Kaplansky-Cohen's theorem to non-commutative cases. In 1975, V.R.Chandran [3] proved that this theorem is true for the class of duo rings. B. Zabavsky et al. proved that a ring is right Noetherian if and only if every almost prime right ideal is finitely generated. In 2010, Manuel L. Reyes proved that a right Noetherian ring is a principal right ideal ring if and only if all of its maximal right ideals are principal.

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2 Kaplansky-Cohen's Theorem

A right $R-$module is called duo module if every its submodule is a fully invariant submodule. In particular, a ring is called a right duo ring if every right ideal is two sided. A ring is called a duo ring if it is a left and right duo ring. In 1975, V.R.Chandran (see [3]) proved that in a left duo ring with identity, if every prime ideal is principal, then every ideal is principal (this can be considered as a generalization of Kaplansky - Cohen's theorem for non-commutative rings). Now we will generalize this result for duo modules.

Following Sanh [7], a fully invariant proper submodule X of a right R-module M is called a *prime submodule* of M (we will say that X is prime in M) if for any ideal I of $S = \text{End}(M_R)$, and any fully invariant submodule U of M, $I(U) \subset X$ implies that $I(M) \subset X$ or $U \subset X$. Especially, an ideal P of R is a prime ideal if for any ideals I, J of R, $IJ \subset P$ implies that $I \subset P$ or $J \subset P$. A right $R-$module M is called a prime module if 0 is prime in M.

Lemma 2.1 [8, Theorem 1.10].

Let M be a right R- module, $S = End(M_R)$ and X is a fully invariant submodule of M. If X is prime in M, then I_X is a prime ideal of S. Conversely, if M is a self- generator and I_X is a prime ideal of S, then X is a prime submodule of M.

Lemma 2.2 *Let M be a right $R-$module. Then,*

(1) If M is a finitely generated quasi-projective duo module, then $S = End(M_R)$ is a right duo ring.

(2) If M is a self-generator and S is right duo, then M is a duo module.

Proof (1) Take any right ideal I of S. Then $X = I(M)$ is a fully invariant submodule of M by hypothesis. Since M is finitely generated and quasi-projective, it follows from [11, Theorem 18.4] that $I = I_X$ which a two-sided ideal of S.

(2) For any submodule X of M, we can see that I_X is a two-sided ideal of S by hypothesis. Since M is a self-generator, we get $X = I_X(M)$, proving that X is fully-invariant. □

Recall that a module N is said to be $M-$generated if there is an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite, then N is called a finitely M-generated module. In particular, a module N is called $M-$cyclic if there is an epimorphism from $M \to N$.

Lemma 2.3 *Let M be a quasi-projective module and X, an $M-$cyclic submodule of M. Then I_X is a principal right ideal of S.*

Proof In fact, since X is $M-$cyclic, there exists an epimorphism $\varphi : M \to X$ such that $X = \varphi(M)$. It follows that $\varphi S \subset I_X$. By the quasi-projectivity of M, for any $f \in I_X$, we can find a $\psi \in S$ such that $f = \varphi\psi$, proving that $I_X = \varphi S$.

Theorem 2.4 *Let M be a finitely generated, quasi-projective duo right $R-$module which is a self-generator. If every prime submodule of M is M-cyclic, then every ideal in S is principal.*

Proof Let P be a prime ideal of S and $X = P(M)$. Then by Lemma 2.3, I_X is a principal ideal of S. Moreover, from the hypothesis and Lemma 2.2, we can see that S is right duo. It follows from [3, Theorem 1], that every ideal of S is principal and our proof is now complete. □

Corollary 2.5 *Let M be a finitely generated, quasi-projective duo right R-module which is a self-generator. If every prime submodule of M is M-cyclic, then every submodule of M is M-cyclic.*

The following Theorem is due to Manuel. L. Reyes [6].

Theorem 2.6 ([6, Theorem 7.9]) *A Noetherian ring is a principal right ideal ring if and only if its maximal right ideals are principal.*

Lemma 2.7 ([9, Lemma 3.10]) *Let M be a quasi-projective, finitely generated right $R-$module which is a self-generator. Then the following statements hold:*

(1) If X is a maximal submodule of M, then I_X is a maximal right ideal of S;

(2) If P is a maximal right ideal of S, then $X = P(M)$ is a maximal submodule of M and $P = I_X$.

Theorem 2.8 *Let M be a quasi-projective, finitely generated right $R-$module which is a self-generator. If ${}_SM$ and M_R are Noetherian and if every maximal submodule of M is $M-$cyclic, then S is a right principal ring.*

Proof Since M is a quasi-projective, right $R-$module which is a self-generator and ${}_SM$ and M_R are Noetherian, by [9, Theorem 2.1], S is a right and left Noetherian ring. Let P be a maximal right ideal of S and $X = P(M)$. Then X is a maximal submodule of M by Lemma 2.7, and hence X is $M-$ cyclic by hypothesis. It follows from Lemma 2.3 that I_X is a principal right ideal of S. Therefore, by theorem 2.6 we see that S is a principal right ideal ring. □

The following corollary is an immediately consequence.

Corollary 2.9 *Let M be a quasi-projective, finitely generated right $R-$module which is a self-generator. If ${}_SM$ and M_R are Noetherian and if every maximal submodule of M is $M-$cyclic, then every submodule of M is $M-$cyclic.*

Proof By the hypothesis and by the theorem 2.8 above, we see that $S = End_{M_R}$ is a principal right ideal ring. Let U be a submodule of M_R and let

$I_U = \{f \in S/f(M) \subset U\}$. Since S is principal right ideal ring, we have I_U is a principal right ideal of S and $I_U = \varphi S$ for some $\varphi \in S$. It follows from [11, Theorem 18.4], that $U = I_U(M) = \varphi S(M) = \varphi(M)$, proving that U is $M-$cyclic, as required. □

3 Another application of Kaplansky-Cohen's Theorem for fully bounded modules

We recall that (see [10]) a right $R-$module M is called a *bounded module* if every essential submodule contains a fully invariant submodule which is essential as a submodule. In particular, a ring R is right bounded if every essential right ideal of R contains an ideal which is essential as a right ideal. A right $R-$module M is called *fully bounded* if for every prime submodule X of M, the prime factor module M/X is a bounded module. A ring R is right fully bounded if for every prime ideal I of R, the prime factor ring R/I is right bounded.

Theorem 3.1 *Let M be a quasi-projective, finitely generated right $R-$module which is a self-generator. If M is a fully bounded and every prime submodule of M is $M-$cyclic, then S is a principal right ideal ring.*

Proof By [10], we see that S is a right fully bounded ring. By hypothesis, every prime submodule of M is $M-$cyclic, we can see that every prime ideal of S is a principal ideal by Lemma 2.3. By [6, Theorem 7.9], we can conclude that S is a right principal right ideal ring, and this completes our proof. □

The following Theorem is a consequence of Theorem 3.1 and it can be considered as a generalization of Cohen's theorem for fully bounded modules.

Theorem 3.2 *Let M be a quasi-projective, finitely generated right $R-$module which is a self-generator. If M is fully bounded and every prime submodule of M is $M-$cyclic, then every submodule of M is $M-$cyclic.*

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