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SOME IMPLICIT SUMMATION FORMULAS AND SYMMETRIC IDENTITIES FOR THE GENERALIZED HERMITE-EULER POLYNOMIALS

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Abstract

In this paper, we introduce a new class of generalized Hermite-Euler polynomials and derive some implicit summation formulae and symmetric identities by applying the generating functions. These results extend some known summations and identities of generalized Hermite-Euler polynomials studied by Dattoli et al, Natalini et al,Zhang et al, Yang and Yang et al and Pathan.

1. INTRODUTION

The 2-variable Kampe de Feriet generalization of the Hermite polynomials [4] reads

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{1.1}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} \tag{1.2}$$

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and reduce to the ordinary Hermite polynomials $H_n(x)$ for $y = 1$.
For each integer $k \geq 0$, $S_k(n) = \sum_{i=0}^{n} i^k$ is called sum of integer powers, or simply power sum. The exponential generating function for $S_k(n)$ is

$$\sum_{n=0}^{\infty} S_k(n)\frac{t^k}{k!} = 1 + e^t + e^{2t} + \cdots + e^{nt} = \frac{e^{(n+1)t} - 1}{e^t - 1} \tag{1.3}$$

The classical Bernoulli polynomials $B_n(x)$ and classical Euler polynomials $E_n(x)$ together with their familiar generalizations $B_n^{\alpha}(x)$ and $E_n^{\alpha}(x)$ (of real or complex) of order α are usually defined by means of the following generating functions (see for details [2,3,4,12,15,16] ,see also [10,13,17,18,19,20,21,22,23,24, 25]):

$$\left(\frac{t}{e^t - 1}\right) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{1.4}$$

$$\left(\frac{t}{e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.5}$$

$$\left(\frac{2}{e^t + 1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \tag{1.6}$$

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{\alpha}(x) \frac{t^n}{n!} \qquad (|\, t \,| < 2\pi; 1^{\alpha} = 1) \tag{1.7}$$

and

$$\left(\frac{2}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{\alpha}(x) \frac{t^n}{n!} \qquad (|\, t \,| < 2\pi; 1^{\alpha} = 1) \tag{1.8}$$

So that obviously

$$B_n(x) = B_n^1(x) \; and \; E_n(x) = E_n^1(x)(n \varepsilon N) \tag{1.9}$$

where

$$N_0 = N \cup \{0\} \, (N = 1, 2, 3, \cdots).$$

For the classical Bernoulli numbers B_n and classical Euler numbers E_n, we readily find from (1.9) that

$$B_n^1(0) = B_n(0) = B_n \; and \; E_n^1(0) = E_n(0) = E_n \quad (n \varepsilon N) \tag{1.10}$$

Some generalized forms of Bernoulli polynomials and numbers recently appeared in literature. We recall for example, the generalized Bernoulli polynomials $B_n^{[\alpha, m-1]}(x), m \geq 1$ studied by Natalini and Bernardini [11] defined (in a suitable neighborhood t=0)by means of the generating function

$$G^{[\alpha,m-1]}(x,t)=\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_n^{[\alpha,m-1]}(x)\frac{t^n}{n!} \tag{1.11}$$

For $m=\alpha=1$, (1.11) reduces to the generating function (1.5) of the classical Bernoulli polynomials $B_n(x)$. The Mittag-Leffler function

$$E_{1,m+1}(t)=\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}} \tag{1.12}$$

was considered in the general form by Agarwal [1] (see also [14]).

Lately Kurt [9] presented a new interesting class of generalized Euler polynomials. Explicitly he introduced the next definition.

Definition. For arbitrary real or complex parameter α, the generalized Euler polynomials $E_n^{[m-1,\alpha]}(x), m\epsilon N$, are defined in a suitable neighborhood of $t=0$ by means of the generating function

$$G^{[\alpha,m-1]}(x,t)=\left(\frac{2^m}{e^t+\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}E_n^{[\alpha,m-1]}(x)\frac{t^n}{n!},\quad |\, t\,|<2\pi, 1^{\alpha}=1 \tag{1.13}$$

It is easy to see that if we set $m=1$ in (1.13) then it reduces to the generating function (1.8) of the classical Euler polynomials $E_n^{\alpha}(x)$, where α is real or complex parameter.
It may be remarked that since $G^{[\alpha,m-1]}(x,t)=A(t)e^{xt}$, the generalized polynomials $E_n^{[\alpha,m-1]}(x)$ belongs to the class of polynomials. The generalized Euler numbers $E_n^{[\alpha,m-1]}$ are defined by setting $x=0$ in (1.13) and assuming

$$E_n^{[\alpha,m-1]}=E_n^{[\alpha,m-1]}(0)$$

For $\alpha=1$, (1.13) reduces to the generating function

$$G^{[m-1]}(x,t)=\frac{2^m}{e^t+\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}e^{xt}=\sum_{n=0}^{\infty}E_n^{[m-1]}(x)\frac{t^n}{n!},\quad |\, t\,|<2\pi \tag{1.14}$$

and further for $m = 1$ and $x = 0$ in (1.13) gives generalized Euler numbers E_n^{α}. The following connection between Euler and Bernoulli polynomials follows readily from the definitions (1.5) and (1.6)

$$E_n(x) = \frac{2^{n+1}}{n+1}[B_n(\frac{x+1}{2}) - B_n(\frac{x}{2})] \tag{1.15}$$

which exhibits the fact that every Bernoulli polynomials is expressible in terms of Euler polynomials. Dattoli et al [5] introduced a new classes of Hermite-Bernoulli polynomials ${}_HB_n(x,y)$, useful to evaluate partial sums of Hermite polynomials. A generalization of the Bernoulli polynomials $B_n(x)$ and consequently, Bernoulli numbers B_n is defined by Natalini and Bernardini [11] starting from suitable generating functions. Another generalized form of Hermite-Bernoulli polynomials of Natalini and Bernardini [11] can be found in a recent paper of Pathan [13] (see also Khan et al [8]). Recently, Zhang and Yang [23,24,25] also established symmetric identities of the generalized Bernoulli and Apostol-Bernoulli polynomials. Many of these generalized polynomials extend appropriately to generalized Euler polynomials.

Motivated by their importance and potential for applications in certain problems in number theory,combinatorics, classical and numerical analysis and physics, several families of generalized Euler numbers and polynomials and generalized Hermite-Euler polynomials were recently studied by many authors. We introduce in this paper, a new class of generalized Hermite-Euler polynomials, a countable set of polynomials ${}_HE_n(x,y)$ generalizing all the Euler polynomials and their generalizations (1.6),(1.8),(1.13) and (1.14) and Hermite polynomials of 2-variables $H_n(x,y)$ specified by the generating relation (1.2).

The object of this paper is to present a systematic account of these families in a unified and generalized form.We develop some elementary properties and derive the implicit summation formulae for the generalized Hermite-Euler polynomials by using different analytical means on their respective generating functions. The approach given in recent papers of Khan et al [8] and Pathan [13] has indeed allowed the derivation of implicit summation formulae in the two-variable Hermite-Euler Polynomials. In addition to this, some relevant connections between Hermite and Euler polynomials and symmetric identities are given.

2. A NEW CLASS OF GENERALIZED HERMITE-EULER POLYNOMIALS

Definition. The generalized Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x,y)$, $m \geq 1$ for a real or complex parameter α defined by means of the generating function defined in a suitable neighborhood of $t = 0$

$$G^{[\alpha,m-1]}(x,y,t) = e^{yt^2}G^{[\alpha,m-1]}(x,t) = \left(\frac{2^m}{e^t + \sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt+yt^2}$$

$$= \sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x,y)\frac{t^n}{n!} \tag{2.1}$$

Notice that (2.1) contains as its special cases not only generalized Euler polynomials $E_n^{[\alpha,m-1]}(x)$ (c.f.Eq.(1.13)) but also Kampe de Feriet generalization of the Hermite polynomials $H_n(x,y)$(c.f.Eq.(1.2)). For $m = 1$, we obtain from (2.1)

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n^{(\alpha)}(x,y)\frac{t^n}{n!} \tag{2.2}$$

which is a generalization of the generating function of Dattoli et al [5, p.386(1.6)] in the form

$$\left(\frac{2}{e^t+1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n(x,y)\frac{t^n}{n!} \tag{2.3}$$

In particular in terms of generalized Euler numbers $E_{n-s}^{(\alpha)}$ and Hermite polynomials $H_s(x,y)$, Hermite Euler polynomials ${}_HE_n^{(\alpha)}(x,y)$ are represented as

$${}_HE_n^{(\alpha)}(x,y) = \sum_{s=0}^{n} \binom{n}{s} E_{n-s}^{(\alpha)}H_s(x,y)$$

It is possible to define generalized Hermite-Euler numbers ${}_HE_n^{[\alpha,m-1]}$ assuming that

$${}_HE_n^{[\alpha,m-1]}(0,0) = {}_HE_n^{[\alpha,m-1]}$$

Taking $\alpha = m = 1$ and $x = 0$ in (2.1) gives the result

$$\sum_{m=0}^{[\frac{n}{2}]} \binom{n}{m} y^m = {}_HB_n^{(}0,y) \tag{2.4}$$

It is not difficult to prove a theorem which generalizes some results involving implicit summations of Hermite-Euler polynomials and numbers by using $e^{it} = \cos t + i\sin t$ and the result

$$\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n+1) \tag{2.5}$$

Since

$$\left(\frac{2i}{e^{it}+1}\right)^{\alpha}=\left(\frac{2i(\cos t+1-i\sin t)}{(\cos t+1+i\sin t)(\cos t+1-i\sin t)}\right)^{\alpha}=\left(\frac{2i(\cos t+1-i\sin t)}{(\cos t+1)^2+(\sin t)^2}\right)^{\alpha}$$

$$=\left(\frac{(2\sin t)+2i(\cos t+1)}{\Omega}\right)^{\alpha} \tag{2.6}$$

where $\Omega=(\cos t+1)^2+(\sin t)^2$
Therefore using the definition (2.1) and results (2.5) and (2.6), we have

Theorem 2.1 *For a positive integer $\alpha \geq 1$, the following implicit summation formula involving generalized Hermite-Euler polynomials ${}_HE_n^{(\alpha)}(x,y)$ holds true:*

$$e^{ixt-yt^2}\left(\frac{(2\sin t)+2i(\cos t+1)}{\Omega}\right)^{\alpha}$$

$$=\sum_{n=0}^{\infty}{}_HE_{2n}^{(\alpha)}(x,y)\frac{(-1)^n t^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}{}_HE_{2n+1}^{(\alpha)}(x,y)\frac{(-1)^n t^{2n+1}}{(2n+1)!} \tag{2.7}$$

where $\Omega=(\cos t+1)^2+(\sin t)^2$

In case we make use of (2.7) with $\alpha=1$ and then compare real and imaginary parts, we shall get the following result involving Hermite-Euler polynomials ${}_HE_n(x,y)$ of Dattoli et al [5].

Corollary 2.1 *The following implicit summation formulae for Hermite-Euler polynomials ${}_HE_n(x,y)$ holds true:*

$$\sum_{n=0}^{\infty}{}_HE_{2n}(x,y)\frac{(-1)^n t^{2n}}{(2n)!} = \frac{e^{-yt^2}}{\Omega}2[\sin(t-xt)+\sin xt] \tag{2.8}$$

$$\sum_{n=0}^{\infty}{}_HE_{2n+1}(x,y)\frac{(-1)^n t^{2n+1}}{(2n+1)!} = \frac{e^{-yt^2}}{\Omega}2[\cos(t-xt)-\cos xt] \tag{2.9}$$

where $\Omega=(\cos t+1)^2+(\sin t)^2$

Remark 1 On setting $\alpha = 0$ in (2.7) and comparing real and imaginary parts we get the following well-known results of Hermite polynomials (see also Khan et al [8, p.410(1.20)and (1.21)]) in the following form

$$e^{-yt^2}\cos xt=\sum_{n=0}^{\infty}H_{2n}(x,y)\frac{(-1)^n t^{2n}}{(2n)!}$$

$$e^{-yt^2}\sin xt=\sum_{n=0}^{\infty}H_{2n+1}(x,y)\frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

Remark 2 On setting $x = 0$ and $y = 0$ in the above results (2.8) and (2.9), we get the following well known classical results involving Euler numbers

$$\tan(\frac{t}{2}) = \sum_{n=0}^{\infty} E_{2n} \frac{(-1)^n t^{2n}}{(2n)!}, \quad \frac{\cos t - 1}{\cos t + 1} = \sum_{n=0}^{\infty} E_{2n} \frac{t^{2n+1}}{(2n+1)!}$$

Remark 3 When α is a positive integer say equal to p, it is possible to give some of the results for Hermite polynomials $H_n(x, y)$. First we prove the following theorem.

Theorem 2.2 *For a positive integer* $p \geq 1$ *the following implicit summation formula involving generalized Hermite-Euler polynomials* ${}_HE_n^{[p,m-1]}(x, y)$ *holds true:*

$$H_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k!}{(k+m)!}\right)^p {}_HE_{n-pk}^{[p,m-1]}(x, y) \tag{2.10}$$

Proof. By exploiting the generating function (2.1), we can write

$$e^{xt+yt^2} = \left(\frac{\sum\limits_{h=m}^{\infty} \frac{t^h}{h!} + 2\sum\limits_{h=0}^{m-1} \frac{t^h}{h!}}{2^m}\right)^p \sum_{n=0}^{\infty} {}_HE_n^{[p,m-1]}(x, y) \frac{t^n}{n!} \tag{2.11}$$

Setting h by h+m in the sum $\sum\limits_{h=m}^{\infty} \frac{t^h}{h!}$ to get r.h.s, we get

$$e^{xt+yt^2} = \left(\frac{\sum\limits_{h=m}^{\infty} \frac{t^{h+m}}{(h+m)!} + 2\sum\limits_{h=0}^{m-1} \frac{t^h}{h!}}{2^m}\right)^p \sum_{n=0}^{\infty} {}_HE_n^{[p,m-1]}(x, y) \frac{t^n}{n!} \tag{2.12}$$

which on using (1.2) yields

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = \left(\frac{\sum\limits_{h=m}^{\infty} \frac{t^h}{h!} + 2\sum\limits_{h=0}^{m-1} \frac{t^h}{h!}}{2^m}\right)^p \sum_{n=0}^{\infty} {}_HE_n^{[p,m-1]}(x, y) \frac{t^n}{n!} \tag{2.13}$$

Now replacing h by k and n by n-kp-mp, using (2.1) and the lemma [3, p.101(1)] in the right hand side of above equation (2.13) and then comparing the coefficients t^n, we get (2.10).

Remark 1. From equation (2.10), since $H_n(x,0) = x^n$, we get the following result in the special case when y=0 and p=1.

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+m)!} E_{n-k}^{[m-1]}(x) \tag{2.14}$$

Remark 2. Taking $p = 1$ in Theorem 2.2 gives the representation of Hermite polynomials in terms of finite sums of generalized Euler polynomials $E_n^{[m-1]}(x)$ of Natalini and Bernardini [11]

$$H_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+m)!} {}_HE_{n-k}^{[m-1]}(x,y) \tag{2.15}$$

Theorem 2.3. *The following implicit summation formulae for Hermite-Euler polynomials* ${}_HE_n^{[\alpha,m-1]}(x,y)$ *holds true:*

$${}_HE_n^{[\alpha,m-1]}(x,y) = \sum_{r=0}^{n} \binom{n}{r} \frac{k!}{(k+m)!} E_{n-r}^{[\alpha,m-1]}(x-z) H_r(z,y) \tag{2.16}$$

Proof. By exploiting the generating function (1.2), we can write equation (2.1) as

$$\left(\frac{2^m}{e^t + \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} \right)^{\alpha} e^{(x-z)t} e^{zt+yt^2} = \sum_{n=0}^{\infty} E_n^{[\alpha,m-1]}(x-z) \frac{t^n}{n!} \sum_{r=0}^{\infty} H_r(z,y) \frac{t^r}{r!} \tag{2.17}$$

Now replacing n by n-r, using (2.1) and the lemma [3,p.101(1)] in the right hand side of equation (2.17), we get

$$\sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x,y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} E_{n-r}^{[\alpha,m-1]}(x-z) H_r(z,y) \frac{t^n}{(n-r)!r!}$$

On equating the coefficients of the like powers of t, we get (2.16)

Remark 1. Letting $z = x$ in (2.16) gives

$${}_HE_n^{[\alpha,m-1]}(x,y) = \sum_{r=0}^{n} \binom{n}{r} E_{n-r}^{[\alpha,m-1]}(x-z) H_r(x,y) \tag{2.18}$$

For $\alpha = 1$, compare this result with a known result of Pathan [13] for Bernoulli polynomials which further for $m = 1$ gives a known result of Dattoli [5,p.386(1.7)]. Further taking $y = 0$ in formula (2.16), we obtain

$$ {}_HE_n^{[\alpha,m-1]}(x) = \sum_{r=0}^{n} \binom{n}{r} E_{n-r}^{[\alpha,m-1]}(x-z) \tag{2.19}$$

Theorem 2.4. *The following implicit summation formulae for Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x,y)$ holds true:*

$$ {}_HE_n^{[\alpha,m-1]}(x+1,y) - {}_HE_n^{[\alpha,m-1]}(x,y) = \sum_{m=0}^{n-1} \binom{n}{m} {}_HE_{n-m}^{[\alpha,m-1]}(x,y) \tag{2.20}$$

Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x+1,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x,y)\frac{t^n}{n!} = \left(\frac{2^m}{e^t + \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} \right)^{\alpha} (e^t-1)e^{zt+yt^2}$$

$$= \sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x,y)\frac{t^n}{n!} \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} - 1 \right)$$

$$= \sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x,y)\frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{t^m}{m!} - \sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x,y)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} {}_HE_{n-m}^{[\alpha,m-1]}(x,y)\frac{t^n}{(n-m)!} - \sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x,y)\frac{t^n}{n!}$$

Finally, equating the coefficients of the like powers of t, we get (2.20).

Theorem 2.5. *The following implicit summation formula for Hermite-Euler polynomials ${}_HE_n^{(\alpha)}(x,y)$ holds true:*

$$ {}_HE_n^{(\alpha)}(\alpha - x, y) = (-1)^n {}_HE_n^{(\alpha)}(x,y) \tag{2.21}$$

Proof. We replace -t by t in (2.1) and then subtract the result from (2.1) itself finding

$$e^{yt^2} \left[\left(\frac{2}{e^t+1} \right)^{\alpha} e^{xt} - \left(\frac{2}{e^{-t}+1} \right)^{\alpha} e^{-xt} \right] = \sum_{n=0}^{\infty} [1-(-1)^n] {}_HE_n^{(\alpha)}(x,y)\frac{t^n}{n!} \tag{2.22}$$

which is equivalent to

$$\sum_{n=0}^{\infty} {}_HE_n^{(\alpha)}(x,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HE_n^{(\alpha)}(\alpha - x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} [1-(-1)^n] {}_HE_n^{(\alpha)}(x,y)\frac{t^n}{n!}$$

and thus by equating coefficients of like powers of t we get (2.21).

3. IMPLICIT FORMULAE INVOLVING HERMITE-EULER POLYNOMIALS

This section of the paper is devoted to employing the definition of the Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x,y)$ to obtain some finite summations. For the derivation of implicit formulae involving the Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x,y)$ the same considerations as developed for the ordinary Hermite and related polynomials in Khan et al [8] and Pathan [13] hold as well. First we prove the following results involving Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x,y)$.

Theorem 3.1 *The following implicit summation formulae for Hermite-Euler polynomials* ${}_HE_n^{[\alpha,m-1]}(x,y)$ *holds true:*

$$ {}_HE_{k+l}^{[\alpha,m-1]}(z,y) = \sum_{n,p=0}^{k,l} \frac{k!l!(z-x)^{n+p}\,{}_HE_{k+l-p-n}^{[\alpha,m-1]}(x,y)}{(k-n)!(l-p)!n!p!} \tag{3.1} $$

Proof. We replace t by $t + u$ and rewrite the generating function (2.1) as

$$ \left(\frac{(2)^m}{e^{t+u}+\sum\limits_{h=0}^{m-1}\frac{(t+u)^h}{h!}}\right)^{\alpha} e^{y(t+u)^2} = e^{-x(t+u)} \sum_{k,l=0}^{\infty} {}_HE_{k+l}^{[\alpha,m-1]}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} \tag{3.2} $$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$ e^{(z-x)(t+u)} \sum_{k,l=0}^{\infty} {}_HE_{k+l}^{[\alpha,m-1]}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_HE_{k+l}^{[\alpha,m-1]}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} \tag{3.3} $$

On expanding exponential function (3.3) gives

$$ \sum_{N=0}^{\infty} \frac{[(z-x)(t+u)]^N}{N!} \sum_{k,l=0}^{\infty} {}_HE_{k+l}^{[\alpha,m-1]}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_HE_{k+l}^{[\alpha,m-1]}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} \tag{3.4} $$

which on using formula [12,p.52(2)]

$$ \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{3.5} $$

in the left hand side becomes

$$\sum_{n,p=0}^{\infty}\frac{(z-x)^{n+p}}{n!p!}t^n u^p\sum_{k,l=0}^{\infty}{}_HE_{k+l}^{[\alpha,m-1]}(x,y)\frac{t^k}{k!}\frac{u^l}{l!}=\sum_{k,l=0}^{\infty}{}_HE_{k+l}^{[\alpha,m-1]}(z,y)\frac{t^k}{k!}\frac{u^l}{l!} \tag{3.6}$$

Now replacing k by k-n, l by l-p and using the lemma [12,p.100(1)] in the left hand side of (3.6), we get

$$\sum_{n,p=0}^{\infty}\sum_{k,l=0}^{\infty}\frac{(z-x)^{n+p}}{n!p!}{}_HE_{k+l-n-p}^{[\alpha,m-1]}(x,y)\frac{t^k}{(k-n)!}\frac{u^l}{(l-p)!}=\sum_{k,l=0}^{\infty}{}_HE_{k+l}^{[\alpha,m-1]}(z,y)\frac{t^k}{k!}\frac{u^l}{l!} \tag{3.7}$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result.

Remark 1. By taking $l = 0$ in equation (3.1), we immediately deduce the following result.

Corollary 3.1. The following implicit summation formula for Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x,y)$ holds true:

$${}_HE_k^{[\alpha,m-1]}(z,y)=\sum_{n=0}^{k}\binom{k}{n}(z-x)^n\,{}_HE_{k-n}^{[\alpha,m-1]}(x,y) \tag{3.8}$$

Remark 2. On replacing z by z+x and setting $y = 0$ in Theorem (3.1), we get the following result involving Hermite-Euler polynomials of one variable

$${}_HE_{k+l}^{[\alpha,m-1]}(z+x)=\sum_{n,m=0}^{k,l}\frac{k!l!(z)^{n+m}\,{}_HE_{k+l-m-n}^{[\alpha,m-1]}(x)}{(k-n)!(l-m)!n!m!} \tag{3.9}$$

whereas by setting z=0 in Theorem 3.1, we get another result involving Hermite-Euler polynomials of one and two variables

$${}_HE_{k+l}^{[\alpha,m-1]}(y)=\sum_{n,m=0}^{k,l}\frac{k!l!(-x)^{n+m}\,{}_HE_{k+l-m-n}^{[\alpha,m-1]}(x,y)}{(k-n)!(l-m)!n!m!} \tag{3.10}$$

Remark 3. Along with the above results we will exploit extended forms of Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x)$ by setting y=0 in the Theorem (3.1) to get

$${}_HE_{k+l}^{[\alpha,m-1]}(y)=\sum_{n,m=0}^{k,l}\frac{k!l!(z-x)^{n+m}\,{}_HE_{k+l-m-n}^{[\alpha,m-1]}(x)}{(k-n)!(l-m)!n!m!} \tag{3.11}$$

Remark 4. A straightforward expression of the ${}_HE_n(x,y)$ is suggested by a special case of the Theorem (3.1) for $\alpha, m = 1$ in the following form

$$ {}_HE_{k+l}(z,y) = \sum_{n,m=0}^{k,l} \binom{k}{n} \binom{l}{m} (z-x)^{n+m} {}_HE_{k+l-m-n}(x,y) \tag{3.12} $$

where ${}_HE_{k+l}(x,y)$ denotes the Hermite-Euler polynomials defined by (2.3).

Theorem 3.2 *The following implicit summation formula involving Hermite-Euler polynomials* ${}_HE_n^{[\alpha,m-1]}(x,y)$ *holds true:*

$$ {}_HE_{k+l}^{[\alpha,m-1]}(z+x,u+y) = \sum_{m=0}^{n} \binom{n}{m} {}_HE_{n-m}(x,y)H_m(z,u) \tag{3.13} $$

Proof. We replace x by x+z and y by y+u in (2.1), use (1.2) and rewrite the generating function as

$$ \left(\frac{2^m}{e^t + \sum_{h=0}^{m-1} \frac{t^h}{h!}} \right)^{\alpha} e^{(xt+yt^2} e^{zt+ut^2} = \sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x,y) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(z,u) \frac{t^m}{m!} $$

$$ = \sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(x+z,y+u) \frac{t^n}{n!} \tag{3.14} $$

Now replacing n by n-m and comparing the coefficients of t, we get the result (3.13).

Theorem 3.3 *The following implicit summation formula involving Hermite-Euler polynomials* ${}_HE_n^{[\alpha,m-1]}(y,x)$ *holds true:*

$$ {}_HE_n^{[\alpha,m-1]}(y,x) = \sum_{k=0}^{[\frac{n}{2}]} E_{n-2k}^{[\alpha,m-1]}(y) \frac{x^k}{(n-2k)!k!} \tag{3.15} $$

Proof. We replace x by y and y by x in equation (2.1) to get

$$ \sum_{n=0}^{\infty} {}_HE_n^{[\alpha,m-1]}(y,x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{[\alpha,m-1]}(y) \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{x^k t^{2k}}{k!} \tag{3.16} $$

Now replacing n by n-2k and comparing the coefficients of t, we get the result (3.15).

4. GENERAL SYMMETRY IDENTITIES

In this section, we give general symmetry identities for the generalized Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x,y)$ by applying the generating function (1.13) and (2.1). Throughout this section α will be taken as an arbitrary real or complex parameter.

Theorem 4.1 *For all integers $a>0, b>0, n\geq 0$ and $m\geq 1$, the following identity holds true:*

$$\sum_{k=0}^{n} {}_HE_{n-k}^{[\alpha,m-1]}(bx,b^2y)\,{}_HE_{k}^{[\alpha,m-1]}(ax,a^2y)\frac{a^k b^{n-k}}{(n-k)!k!}$$

$$=\sum_{k=0}^{n} {}_HE_{n-k}^{[\alpha,m-1]}(ax,a^2y)\,{}_HE_{k}^{[\alpha,m-1]}(bx,b^2y)\frac{b^k a^{n-k}}{(n-k)!k!} \tag{4.1}$$

Proof. Start with

$$g(t)=\left(\frac{2^{2m}}{(e^{at}+\sum\limits_{h=0}^{m-1}\frac{a^ht^h}{h!})(e^{bt}+\sum\limits_{h=0}^{m-1}\frac{b^ht^h}{h!})}\right)^{\alpha} e^{abxt+a^2b^2yt^2} \tag{4.2}$$

Then the expression for g(t) is symmetric in a and b and we can expand g(t) into series in two ways to obtain

$$g(t)=\sum_{n=0}^{\infty} {}_HE_{n}^{[\alpha,m-1]}(bx,b^2y)\frac{(at)^n}{n!}\sum_{k=0}^{\infty} {}_HE_{k}^{[\alpha,m-1]}(ax,a^2y)\frac{(bt)^k}{k!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_HE_{n-k}^{[\alpha,m-1]}(bx,b^2y)\frac{(a)^{n-k}}{(n-k)!}\,{}_HE_{k}^{[\alpha,m-1]}(ax,a^2y)\frac{(b)^k}{k!}\frac{(t)^n}{n!}$$

On the similar lines we can show that

$$g(t)=\sum_{n=0}^{\infty} {}_HE_{n}^{[\alpha,m-1]}(ax,a^2y)\frac{(b)^{n-k}}{(n-k)!}\,{}_HE_{k}^{[\alpha,m-1]}(bx,b^2y)\frac{(a)^k}{k!}\frac{(t)^n}{n!}$$

By comparing the coefficients of t on the right hand sides of the last two equations we arrive the desired result.

Remark 1. For $\alpha=1$, the above result reduces to a similar known result of Pathan [13] for Bernoulli polynomials. Further by taking m=1 in Theorem

4.1, we immediately deduce the following result.

Corollary 4.1 *For all integers $a > 0$, $b > 0$ and $n \geq 0$, the following identity holds true:*

$$\sum_{k=0}^{n} {}_HE_{n-k}^{(\alpha)}(bx, b^2y)_HE_k^{(\alpha)}\frac{a^k b^{n-k}}{(n-k)!k!} = \sum_{k=0}^{n} {}_HE_{n-k}^{(\alpha)}(ax, a^2y)_HE_k^{(\alpha)}\frac{b^k a^{n-k}}{(n-k)!k!} \tag{4.3}$$

Remark 2. By setting $b = 1$ in Theorem (4.1), we immediately get the following corollary.

Corollary 4.2 *For all integers $a > 0$, $n \geq 0$ and $m \geq 1$, the following identity holds true:*

$$\sum_{k=0}^{n} {}_HE_{n-k}^{[\alpha,m-1]}(x,y)_HE_k^{[\alpha,m-1]}\frac{a^k}{(n-k)!k!} = \sum_{k=0}^{n} {}_HE_{n-k}^{[\alpha,m-1]}(ax,a^2y)_HE_k^{[\alpha,m-1]}\frac{a^{n-k}}{(n-k)!k!} \tag{4.4}$$

Theorem 4.2 *For each pair of integers a and b and all integers and $n \geq 1$, the following identity holds true:*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-1)^i(-1)^j{}_HE_{n-k}^{(\alpha)}\left(bx+\frac{b}{a}i+j, b^2z\right)E_k^{\alpha}(ay)\frac{(a)^{n-k}}{(n-k)!}\frac{b^k}{k!}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-1)^i(-1)^j{}_HE_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j, a^2z\right)E_k^{\alpha}(ay)\frac{(b)^{n-k}}{(n-k)!}\frac{a^k}{k!} \tag{4.5}$$

Proof. Let

$$g(t) = \frac{(2a)^{\alpha}(2b)^{\alpha}(e^{abt}+1)^2 e^{ab(x+y)t+a^2b^2zt^2}}{(e^{abt}+1)^{\alpha+1}(e^{bt}+1)^{\alpha+1}}$$

$$g(t) = \left(\frac{2a}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2zt^2}\left(\frac{e^{abt}+1}{e^{bt}+1}\right)\left(\frac{bt}{e^{bt}+1}\right)^{\alpha} e^{abyt}\left(\frac{e^{abt}+1}{e^{at}+1}\right)$$

$$= \left(\frac{2a}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2zt^2}\sum_{i=0}^{a-1}e^{bti}\left(\frac{bt}{e^{bt}+1}\right)^{\alpha} e^{abyt}\sum_{j=0}^{b-1}e^{atj} \tag{4.6}$$

$$= \left(\frac{2a}{e^{at}+1}\right)^{\alpha} e^{abxt+a^2b^2zt^2}\sum_{i=0}^{a-1}e^{bti}\left(\frac{bt}{e^{bt}+1}\right)^{\alpha} e^{abyt}\sum_{j=0}^{b-1}e^{atj}$$

$$= \left(\frac{2a}{e^{at}+1}\right)^{\alpha} e^{a^2b^2zt^2} \sum_{i=0}^{a-1}\sum_{j=0}^{b-1} e^{(bx+\frac{b}{a}i+j)at} \sum_{k=0}^{\infty} E_k^{\alpha}(ay)\frac{(bt)^k}{k!}$$

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} {}_HE_n^{(\alpha)}\left(bx+\frac{b}{a}i+j, b^2z\right)\frac{(at)^n}{n!}\sum_{k=0}^{\infty} E_k^{\alpha}(ay)\frac{(bt)^k}{(k)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-1)^i(-1)^j {}_HE_n^{(\alpha)}\left(bx+\frac{b}{a}i+j, b^2z\right)\frac{(at)^{n-k}}{(n-k)!}\frac{(bt)^k}{k!} \tag{4.7}$$

On the other hand

$$g(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-1)^i(-1)^j {}_HE_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j, a^2z\right)E_k^{\alpha}(ay)\frac{(bt)^{n-k}}{(n-k)!}\frac{(at)^k}{k!} \tag{4.8}$$

By comparing the coefficients of t on the right hand sides of the last two equations,we arrive at the desired result.

Theorem 4.3 *For each pair of integers a and b and all integers and $n \geq 0$, the following identity holds true:*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-1)^i(-1)^j(a)^{n-k}(b)^k {}_HB_{n-k}^{(\alpha)}\left(bx+\frac{b}{a}i+j, b^2z\right)B_k^{\alpha}(ay+\frac{a}{b}j)$$

$$= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-1)^i(-1)^j(b)^{n-k}(a)^k {}_HE_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j, a^2z\right)E_k^{\alpha}(by+\frac{b}{a}j) \tag{4.9}$$

Proof. The proof is analogous to Theorem (4.2) but we need to write equation (4.6) in the form

$$g(t) = \sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-1)^i(-1)^j {}_HE_n^{(\alpha)}\left(bx+\frac{b}{a}i+j, b^2z\right)\frac{(at)^n}{n!}\sum_{k=0}^{\infty} E_k^{\alpha}(ay+\frac{a}{b}j)\frac{(bt)^k}{k!} \tag{4.10}$$

On the other hand equation (4.6) can be shown equal to

$$g(t) = \sum_{n=0}^{\infty}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-1)^i(-1)^j {}_HE_n^{(\alpha)}\left(ax+\frac{a}{b}i+j, a^2z\right)\frac{(bt)^n}{n!}\sum_{k=0}^{\infty} E_k^{\alpha}(by+\frac{b}{a}j)\frac{(at)^k}{k!} \tag{4.11}$$

Next making change of index and by equating the coefficients of t^n to zero in (4.10) and (4.11), we get the result

Remark 1. By setting $y = 0$ in Theorem (4.3), we immediately get the following corollary.

Corollary 4.3. For all integers $a > 0, b > 0$ and $n \geq 0$ the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^i (-1)^j (a)^{n-k} (b)^k {}_H E_{n-k}^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z \right) E_k^{\alpha} (\frac{a}{b} j)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-1)^i (-1)^j (b)^{n-k} (a)^k {}_H E_{n-k}^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z \right) E_k^{\alpha} (\frac{b}{a} j) \tag{4.12}$$

Theorem 4.4 For all integers $a > 0, b > 0$ and $n \geq 0$ the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} E_{n-k}^{\alpha} b^{n-k} a^k \sum_{i=0}^{a-1} (-1)^i {}_H E_k^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z \right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} E_{n-k}^{\alpha} a^{n-k} b^k \sum_{i=0}^{b-1} (-1)^i {}_H E_k^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z \right) \tag{4.13}$$

Proof. We now use

$$g(t) = \frac{(2a)^{\alpha} (2b)^{\alpha} (1 + (-1)^{a+1} e^{abt}) e^{ab(x+y)t + a^2 b^2 z t^2}}{(e^{at} + 1)^{\alpha} (e^{bt} + 1)^{\alpha + 1}}$$

to find that

$$g(t) = \left(\frac{2a}{e^{at} + 1} \right)^{\alpha} e^{abxt + a^2 b^2 z t^2} \left(\frac{1 - (-e^{bt})^a}{e^{bt} + 1} \right) \left(\frac{2b}{e^{bt} + 1} \right)^{\alpha} e^{abyt}$$

$$g(t) = \left(\frac{2a}{e^{at} + 1} \right)^{\alpha} e^{abxt + a^2 b^2 z t^2} \sum_{i=0}^{a-1} (-e^{bt})^i \sum_{n=0}^{\infty} E_n^{\alpha} (ay) \frac{(bt)^n}{n!}$$

$$g(t) = \left(\frac{2a}{e^{at} + 1} \right)^{\alpha} e^{a^2 b^2 z t^2} \sum_{i=0}^{a-1} (-1)^i e^{(bx + \frac{b}{a} i) at} \sum_{n=0}^{\infty} E_n^{\alpha} (ay) \frac{(bt)^n}{n!}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{a-1} (-1)^i {}_H E_k^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z \right) \frac{(at)^k}{k!} \sum_{n=0}^{\infty} E_n^{\alpha} (ay) \frac{(bt)^n}{n!}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{a-1} (-1)^i {}_H E_k^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z \right) \frac{(a)^k}{k!} \sum_{n=0}^{\infty} E_n^{\alpha} (ay) \frac{(b)^n (t)^{n+k}}{n!}$$

Replacing n by n-k in the above equation, we have

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}E_{n-k}^{\alpha}(ay)a^{k}b^{n-k}\sum_{i=0}^{a-1}(-1)^{i}{}_{H}E_{k}^{(\alpha)}\left(bx+\frac{b}{a}i,b^{2}z\right)\frac{(t)^{n}}{n!}$$

We may also expand g(t) as

$$g(t)=\left(\frac{2b}{e^{bt}+1}\right)^{\alpha}e^{baxt+a^{2}b^{2}zt^{2}}\left(\frac{1-(-e^{at})^{b}}{e^{at}+1}\right)\left(\frac{2a}{e^{at}+1}\right)^{\alpha}e^{bayt}$$

$$g(t)=\left(\frac{2b}{e^{bt}+1}\right)^{\alpha}e^{baxt+a^{2}b^{2}zt^{2}}\sum_{i=0}^{b-1}(-e^{bt})^{i}\sum_{n=0}^{\infty}E_{n}^{\alpha}(by)\frac{(at)^{n}}{n!}$$

$$g(t)=\left(\frac{2b}{e^{bt}+1}\right)^{\alpha}e^{a^{2}b^{2}zt^{2}}\sum_{i=0}^{b-1}(-1)^{i}e^{(ax+\frac{a}{b}i)bt}\sum_{n=0}^{\infty}E_{n}^{\alpha}(by)\frac{(at)^{n}}{n!}$$

$$=\sum_{k=0}^{\infty}\sum_{i=0}^{b-1}(-1)^{i}{}_{H}E_{k}^{(\alpha)}\left(ax+\frac{a}{b}i,a^{2}z\right)\frac{(bt)^{k}}{k!}\sum_{n=0}^{\infty}E_{n}^{\alpha}(by)\frac{(at)^{n}}{n!}$$

$$=\sum_{k=0}^{\infty}\sum_{i=0}^{b-1}(-1)^{i}{}_{H}E_{k}^{(\alpha)}\left(ax+\frac{a}{b}i,a^{2}z\right)\frac{(b)^{k}}{k!}\sum_{n=0}^{\infty}E_{n}^{\alpha}(by)\frac{(a)^{n}(t)^{n+k}}{n!}$$

Replacing n by n-k in the above equation, we have

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}E_{n-k}^{\alpha}(by)b^{k}a^{n-k}\sum_{i=0}^{b-1}(-1)^{i}{}_{H}E_{k}^{(\alpha)}\left(ax+\frac{a}{b}i,a^{2}z\right)\frac{(t)^{n}}{n!}$$

Equating the coefficients of t^n in the two expressions for g(t) gives us the desired result.

Theorem 4.5 For all integers $a>0, b>0, m\geq 1$ and $n\geq 0$ the following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k}E_{n-k}^{\alpha,m}(ay)a^{k}b^{n-k}\sum_{i=0}^{a-1}(-1)^{i}{}_{H}E_{k}^{(\alpha,m)}\left(bx+\frac{b}{a}i,b^{2}z\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}^{\alpha,m}(ay)b^{k}a^{n-k}\sum_{i=0}^{b-1}(-1)^{i}{}_{H}E_{k}^{(\alpha,m)}\left(ax+\frac{a}{b}i,a^{2}z\right)\tag{4.14}$$

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