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MULTIPLICATIVE INTEGRABILITY OF RIEMANN, LEBESGUE, DENJOY, PERRON AND KURZWEIL INTEGRALS

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Abstract

In this paper we explore multiplicative integrability of these eponymous integrals: Riemann, Lebesgue, Denjoy, Perron, and Kurzweil. For an integrable function f, in each of these senses, we investigate and compare the conditions on a function g that makes the product fg integrable. We conclude that all existing integrals do not preclude a more powerful integral yet to be developed.

1 List of Symbols

ACG_*	generalized absolutely continuous in the restricted sense
$D^+F(x)$	upper right derivate of F at x
$\overline{D}F(x)$	upper derivate of F at x
$(D)\int_a^b f$	Denjoy integral of f
$(K)\int_a^b f$	Kurzweil integral of f
$(L)\int_a^b f$	Lebesgue integral of f

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$(P)\int_a^b f$	Perron integral of f
$\omega(F,[c,d])$	the oscillation of F on $[c,d]$
$V(F,A)$	weak variation of F on A

2 Introduction

The most familiar integral in integration theory is unquestionably the one created by Riemann as it is widely taught in undergraduate level courses. However, it is also widely known that Riemann integral has some defects, the class of Riemann integrable functions is rather small, hence many mathematicians have been trying to introduce other integrals as improving replacements. So we heard of Lebesgue, Denjoy, Perron, and Kurzweil. This paper is to give an exposition of exemplary algebraic property for those integrals to show their advantages and disadvantages: the multiplicative integrability of the integrals developed after Riemann's.

3 Preliminaries

First we recall some definitions of integrable functions of the aforementioned integrals and also related theorems. Let us agree to skip the familiar Riemann and Lebesgue integrals and begin with some notions from [4]:

Definition 1. Let $\omega(F,[c,d]) = \sup\{|F(y)-F(x) : c \le x < y \le d\}$ denote the oscillation of the function F on the interval $[c,d]$.

Definition 2. Let $F : [a,b] \to R$ and let $A \subseteq [a,b]$.

(a) The weak variation of F on A of F on A is defined by

$$V(F,A) = \sup\{\sum_{i=1}^{n} |F(d_i) - F(c_i)|\};$$

where the supremum is taken over all finite collections $\{[c_i,d_i] : 1 \le i \le n\}$ of non-overlapping intervals that have endpoints in A.

(b) The function F is absolutely continuous on A (F is AC on A) if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} |F(d_i) - F(c_i)| < \epsilon$ whenever $\{[c_i,d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals that have endpoints in A and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$. The function F is absolutely continuous in the restricted sense on A (F is AC_* on A) if F is bounded on an interval that contains A and for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \omega(F,[c_i,d_i]) < \epsilon$ whenever $\{[c_i,d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals that have endpoints in A and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$.

(c) The function F is generalized absolutely continuous on A (F is ACG on A) if $F|_A$ is continuous on A and A can be written as a countable union of sets on each of which F is AC. The function F is generalized absolutely continuous in the restricted sense on A (F is ACG_* on A) if $F|_A$ is continuous on A and A can be written as a countable union of sets on each of which F is AC_*.

Proposition 1. *A function $f : [a,b] \to R$ is Lebesgue integrable on $[a,b]$ if and only if there exists an AC function $F : [a,b] \to R$ such that $F' = f$ almost everywhere on $[a,b]$.*

Definition 3. A function $f : [a,b] \to R$ is Denjoy integrable on $[a,b]$ if there exists and ACG_* function $f : [a,b] \to R$ such that $F' = f$ almost everywhere on $[a,b]$. The function f is Denjoy integrable on a measurable set $A \subseteq [a,b]$ if $f\chi_A$ is Denjoy integrable on $[a,b]$.

Definition 4. Let $F : [a,b] \to R$. Define the upper right and lower right derivates of F at $x \in [a,b)$ by

$$D^+F(x) = \limsup_{\delta \to 0^+} \left\{ \frac{F(y) - F(x)}{y - x} : x < y < x + \delta \right\}$$

$$D_+F(x) = \liminf_{\delta \to 0^+} \left\{ \frac{F(y) - F(x)}{y - x} : x < y < x + \delta \right\}$$

and define the upper left and lower left derivates of F at $x \in (a,b]$ by

$$D^-F(x) = \limsup_{\delta \to 0^+} \left\{ \frac{F(y) - F(x)}{y - x} : x - \delta < y < x \right\}$$

$$D_-F(x) = \liminf_{\delta \to 0^+} \left\{ \frac{F(y) - F(x)}{y - x} : x - \delta < y < x \right\}$$

Definition 5. Define the upper and lower derivates of F at $x \in [a,b]$ by

$$\overline{D}F(x) = \limsup_{\delta \to 0^+} \left\{ \frac{F(y) - F(x)}{y - x} : 0 < |y - x| < \delta \right\} = \max\{D^+F(x), D^-F(x)\};$$

$$\underline{D}F(x) = \liminf_{\delta \to 0^+} \left\{ \frac{F(y) - F(x)}{y - x} : 0 < |y - x| < \delta \right\} = \min\{D_+F(x), D_-F(x)\};$$

Definition 6. Let $f : [a,b] \to R \cup \{\pm\infty\}$.

(a) A function $M : [a,b] \to R$ is a major function of f on $[a,b]$ if $\underline{D}M(x) > -\infty$ and $\underline{D}M(x) \geq f(x)$ for all $x \in [a,b]$.

(b) A function $m : [a,b] \to R$ is a minor function of f on $[a,b]$ if $\overline{D}m(x) < +\infty$ and $\overline{D}m(x) \le f(x)$ for all $x \in [a,b]$.

Definition 7. A function $f : [a,b] \to R \cup \{\pm\infty\}$. is Perron integrable on $[a,b]$ if f has at least one major and one minor function on $[a,b]$ and the numbers

$$\inf\{M_a^b : M \text{ is a major function of } f \text{ on } [a,b]\}$$

$$\sup\{m_a^b : m \text{ is a minor function of } f \text{ on } [a,b]\}$$

are equal. This common value is the Perron integral of f on $[a,b]$ and will be denoted by $(P)\int_a^b f$. The function f is Perron integrable on a measurable set $A \subseteq [a,b]$ if $f\chi_A$ is Perron integrable on $[a,b]$.

Definition 8. Let $\delta(\cdot)$ be a positive function defined on the interval $[a,b]$. A tagged interval $(x,[c,d])$ consists of an interval $[c,d] \subseteq [a,b]$ and a point $x \in [c,d]$. The tagged interval $(x,[c,d])$ is subordinate to δ if

$$[c,d] \subseteq (x-\delta(x), x+\delta(x)).$$

The letter $\mathcal{P}$ will be used to denote finite collections of non-overlapping tagged intervals. Let $\mathcal{P} = \{(x_i,[c_i,d_i]) : 1 \le i \le n\}$ be such a collection in $[a,b]$. We adopt the following terminology:

(a) The points $\{x_i\}$ are the tags of $\mathcal{P}$ and the intervals $\{[c_i,d_i]\}$ are the intervals of $\mathcal{P}$.

(b) If $(x_i,[c_i,d_i])$ is subordinate to δ for each i, then $\mathcal{P}$ is subordinate to δ.

(c) Let $A \subseteq [a,b]$. If $\mathcal{P}$ is subordinate to δ and each $x_i \in A$, then $\mathcal{P}$ is $A-$subordinate to δ.

(d) If $\mathcal{P}$ is subordinate to δ and $[a,b] = \cup_{i=1}^n [c_i,d_i]$, then $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate ot δ.

Definition 9. A function $f : [a,b] \to R$ is Kurzweil integrable on $[a,b]$ if there exists a real number L with the following property: for each $\epsilon > 0$, there exists a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - L| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to δ. The number L will be denoted by $(K)\int_a^b f$. The function f is Kurzweil integrable on a measurable set $A \subseteq [a,b]$ if $f\chi_A$ is Kurzweil integrable on $[a,b]$.

We will end this section with a very useful lemma for Kurzweil integral theory:

Lemma 1. *(Saks-Henstock Lemma). Let $f : [a,b] \to R$ be Kurzweil integrable on $[a,b]$ and $F(x) = \int_a^x f$ for each $x \in [a,b]$. For $\epsilon > 0$, assume that $\delta : [a,b] \to R$ is a positive function such that $|f(\mathcal{P}) - F(\mathcal{P})| < \epsilon$ for a tagged partition $\mathcal{P}$ of*

$[a,b]$ that is subordinate to δ. Then for a tagged partition $\mathcal{P}_0 = \{(x_i,[c_i,d_i]) : i = 1,2,...,n\}$ subordinate to δ, we have

$$|f(\mathcal{P}_0) - F(\mathcal{P}_0)| \leq \epsilon \quad \textit{and} \quad \sum_{i=1}^{n} |f(x_i)(d_i - c_i) - (F(d_i) - F(c_i))| \leq 2\epsilon.$$

Interested reader can look up for the proof in [4] or [1].

4 Multiplicative Integrability of Integrals

In this section we will explore the multiplicative integrability over an interval of integrable functions in the sense of Riemann, Lebesgue, Denjoy, Perron, and Kurzweil.

4.1 Riemann Integral

From elementary theorem in mathematical analysis we know that the product of two Riemann integrable functions is also Riemann integrable. We will re-visit this theorem by precisely re-state it here:

Theorem 1. *Let $f : [a,b] \to R$ and $g : [a,b] \to R$ be Riemann integrable functions over $[a,b]$. Then the product fg is Riemann integrable over $[a,b]$.*

4.2 Lebesgue Integral

The product of two Lebesgue integrable functions may not be Lebesgue integrable, as we can see from a counterexample:

Consider the function f defined by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & \text{if } 0 < x \leq 1; \\ 0, & \text{if } x = 0. \end{cases}$$

We see that f is Lebesgue integrable on the interval $[0,1]$ but f^2 is not Lebesgue integrable on $[0,1]$. Nonetheless, with the added condition that at least one of the two Lebesgue integrable functions is bounded, their product will now be Lebesgue integrable as shown in the following theorem:

Theorem 2. *Let f and g be Lebesgue integrable functions. If g is bounded then the product fg is Lebesgue integrable.*

The proof is an easy consequence from Theorems in Chapter 5 of [3].

4.3 Denjoy Integral

Now for Denjoy integral, the boundedness condition of one of the two Denjoy integrable functions is far from sufficient. We can see from the following example that the product of a Denjoy integrable function and a bounded Denjoy integrable function may not be Denjoy integrable:

Consider the function $f, g : [0,1] \to R$ defined by, for each $n \in N$,

$$f(x) = \begin{cases} 2^n \frac{(-1)^n}{\sqrt{n}}, & \text{if } x \in (2^{-n}, 2^{-n+1}); \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) := sgn(f(x)) = \begin{cases} 1, & \text{if } f(x) > 0; \\ 0, & \text{if } f(x) = 0; \\ -1, & \text{if } f(x) < 0. \end{cases}$$

The reader can easily prove that f is a Denjoy integrable function (though not Lebesgue integrable!) and g is a bounded integrable function; but $fg = |f|$ is not Denjoy integrable.

Neither adding continuity condition will help achieve our purpose in this case. For example, let the function f be as above and $g : [0,1] \to R$ defined by, for each $n \in N$,

$$g(x) = \begin{cases} \frac{2sgn[\frac{(-1)^n}{\sqrt{n}}]}{\sqrt{n}} & \text{if } x = 2^{-n-1} + 2^{-n}; \\ 0, & \text{if } x = 0, 2^{-n}, 2^{-n+1}; \\ x, & \text{otherwise,} \end{cases}$$

Now g is continuous on $[0,1]$ but the product fg is still not Denjoy integrable on $[0,1]$, since

$$\begin{aligned} \int_0^1 fg &= \sum_{n=1}^{\infty} \int_{(2^{-n}, 2^{-n+1})} fg \\ &= \sum_{n=1}^{\infty} 2^n \frac{(-1)^n}{\sqrt{n}} \int_{(2^{-n}, 2^{-n+1})} g \\ &= \sum_{n=1}^{\infty} 2^n \frac{(-1)^n}{\sqrt{n}} \cdot \frac{2sgn[\frac{(-1)^n}{\sqrt{n}}]}{2^n \sqrt{n}} \\ &= \sum_{n=1}^{\infty} \frac{|\frac{(-1)^n}{\sqrt{n}}|}{\sqrt{n}} = +\infty, \end{aligned}$$

Multiplicative integrability for Denjoy can be carried out by Denjoy integration by parts by adding absolute continuity condition for one function. The following theorem prove the integrability and provide us the formula for integration simultaneously:

Theorem 3. *Let $f : [a, b] \to R$ be Denjoy integrable on $[a, b]$. For each $x \in [a, b]$, let $F(x) = \int_a^x f$. If $G : [a, b] \to R$ is absolutely continuous on $[a, b]$, then fG is Denjoy integrable on $[a, b]$ and*

$$(D)\int_a^b fG = F(b)G(b) - (L)\int_a^b FG'.$$

Proof. Since F is ACG_* on $[a, b]$, it follows that FG is ACG_* on $[a, b]$. Then $(FG)'$ is Denjoy integrable on $[a, b]$ by definition of the integral. Now FG' is the product of a Lebesgue integrable function and a bounded measurable function, so it is Lebesgue integrable on $[a, b]$. Since $fG = (FG)' - FG'$ a.e. on $[a, b]$, fG is Denjoy integrable on $[a, b]$, and we have that

$$(D)\int_a^b fG = (D)\int_a^b (FG)' - (D)\int_a^b FG' = F(b)G(b) - (L)\int_a^b FG'.$$

□

4.4 Perron Integral

Intensive studies by H. Hake, P.S. Aleksandrov, H. Looman, R. Henstock, and Y. Kubota, see [9, 5, 4, 7, 6], have shown that Denjoy, Perron, and Kurzweil integrals are equivalent. But some concepts in the definitions of each theory certainly give rise to seemingly different conditions for multiplicative integration. As for Perron integration by parts we have:

Theorem 4. *Let $f : [a, b] \to R$ be Perron integrable on $[a, b]$. For each $x \in [a, b]$, let $F(x) = \int_a^x f$. Assume that the upper and lower derivates of the variation of G are finite nearly everywhere on $[a, b]$. If $G : [a, b] \to R$ is absolutely continuous on $[a, b]$, then the product fG is Perron integrable on $[a, b]$ and we have*

$$(P)\int_a^b fG = F(b)G(b) - (L)\int_a^b FG'.$$

Proof. We can assume, without loss of generality, that G is a non-decreasing function on $[a, b]$ and $G(a) = 0$.
Let $\epsilon > 0$. Since f is Perron integrable on $[a, b]$, there exists a continuous major function M and a continuous minor function m of f on $[a, b]$ such that $M(a) = 0 = m(a)$ and $M_a^b - m_a^b < \epsilon$. Since G and M are differentiable a.e. on $[a, b]$,

$$(GM)'(x) = G(x)M'(x) + M'(x)M(x) \geq f(x)G(x) + F(x)M'(x)$$

a.e. on $[a, b]$. We have that G is non-decreasing on $[a, b]$ by our assumption and $M - F$ is also non-decreasing on $[a, b]$ with $M(a) - F(a) = 0$. Let

$$A = \{x \in (a, b) : \overline{D}G(x) < +\infty \text{ and } \underline{D}M(x) > -\infty\}.$$

Let $r \in A$. Then there are positive numbers K and δ such that $(r - \delta, r + \delta) \subseteq (a, b)$. If $x \in (r - \delta, r + \delta)$, then

$$0 \le \frac{G(x) - G(r)}{x - r} \le K \text{ and } \frac{M(x) - M(r)}{x - r} \ge -K.$$

It follows that

$$\begin{aligned} \frac{G(x)M(x) - G(r)M(r)}{x - r} &= G(x)\frac{M(x) - M(r)}{x - r} + M(r)\frac{G(x) - G(r)}{x - r} \\ &\ge -KG(x) - K|M(r)| \end{aligned}$$

for each $x \in (r - \delta, r + \delta)$.

Now $\underline{D}(GM)(r) > -\infty$ since M and G are bounded on $[a, b]$. Also, $\underline{D}(GM) > -\infty$ nearly everywhere on $[a, b]$ since $[a, b] - A$ is countable. It can be shown similarly that $\overline{D}(Gm) \le fG + FG'$ a.e. on $[a, b]$, and that $\overline{D}(Gm) < +\infty$ nearly everywhere on $[a, b]$. We now have that

$$(GM)_a^b - (Gm)_a^b = G(b)(M(b) - m(b)) < \epsilon G(b)$$

and

$$(Gm)_a^b \le F(b)G(b) \le (GM)_a^b,$$

therefore, $fG + FG'$ is Perron integrable on $[a, b]$, and

$$\int_a^b (fG + FG') = F(b)G(b).$$

Next, FG' is Lebesgue integrable on $[a, b]$, it is also Perron integrable on $[a, b]$. Hence $fG = (fG + FG') - FG'$ is Perron integrable on $[a, b]$ and

$$(P) \int_a^b fG = F(b)G(b) - (L) \int_a^b FG'.$$

□

4.5 Kurzweil Integral

Now, for Kurzweil integration by parts:

Theorem 5. *Let $f : [a, b] \to R$ be Kurzweil integrable on $[a, b]$. For each $x \in [a, b]$, let $F(x) = \int_a^x f$. If $G : [a, b] \to R$ is absolutely continuous on $[a, b]$, then fG is Kurzweil integrable on $[a, b]$ and*

$$(K) \int_a^b fG = F(b)G(b) - (L) \int_a^b FG'.$$

Proof. Let $\epsilon > 0$. Since f is Kurzweil integrable on $[a, b]$, there exists a positive function $\delta_0 : [a, b] \to R$ such that $|f(\mathcal{P}) - \int_a^b f| < \epsilon$ if $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to δ_0.
Now F is uniformly continuous on $[a, b]$, so there exists $\lambda > 0$ such that for all $x, y \in [a, b]$, if $|x - y| < \lambda$ then $|F(x) - F(y)| < \epsilon$. Let $\delta : [a, b] \to R$ be defined by

$$\delta(x) = \begin{cases} min\{\delta_0(x), \frac{\lambda}{2}, b - x, x - a\}, & \text{if } x \in (a, b); \\ min\{\delta_0(x), \frac{\lambda}{2}\}, & \text{if } x = a, b, \end{cases}$$

Let $\mathcal{P} = \{(r_k, [x_{k-1}, x_k]) : k = \{1, 2, ..., n\}\}$ be a tagged partition of $[a, b]$ which is subordinate to δ. Also, we assume that $r_1 = a$ and $r_k = b$ and that each tag appears only once.

Since F is uniformly continuous and G is absolutely continuous, we have

$$\begin{aligned} \left|\sum_{k=1}^{n-1} F(x_k)(G(r_{k+1}) - G(r_k) - (L)\int_a^b FG'\right| &= \left|\sum_{k=1}^{n-1} F(x_k)\int_{r_k}^{r_{k+1}} G' - (L)\int_a^b FG'\right| \\ &= \left|\sum_{k=1}^{n-1}\int_{r_k}^{r_{k+1}} (F(x_k) - F(x))G'(x)dx'\right| \\ &\leq \sum_{k=1}^{n-1}\int_{r_k}^{r_{k+1}} |F(x_k) - F(x)|\left|G'(x)\right| dx \\ &\leq \sum_{k=1}^{n-1}\epsilon\int_{r_k}^{r_{k+1}} \left|G'(x)\right| dx \\ &\leq \epsilon\int_a^b |G'|. \end{aligned}$$

Then by Saks-Henstock Lemma,

$$\begin{aligned} &\left|\sum_{k=1}^{n} f(r_k)G(r_k)(x_k - x_{k-1}) - \left(F(b)G(b) - (L)\int_a^b FG'\right)\right| \\ &= |\sum_{k=1}^{n-1}\left(\sum_{k=1}^{n} f(r_i)(x_i - x_{i-1})\left(G(r_k) - G(r_{k+1})\right)\right) \\ &\quad + \sum_{k=1}^{n} f(r_i)(x_i - x_{i-1})G(r_k) - \left(F(b)G(b) - (L)\int_a^b FG'\right)| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{n-1} |G(r_k) - G(r_{k+1})| \left| \sum_{k=1}^{n} \left(f(r_i)(x_i - x_{i-1}) - F(x_k) \right) \right| \\
&\quad + \left| \sum_{k=1}^{n-1} F(x_k) \left(G(r_{k+1}) - G(r_k) \right) - (L) \int_a^b FG' \right) | \\
&\quad + |G(b)| \left| \sum_{k=1}^{n} f(r_i)(x_i - x_{i-1}) - F(b) \right| \\
&< \epsilon V(G, [a, b]) + \epsilon \int_a^b |G'| + \epsilon |G(b)|.
\end{aligned}
$$

□

5 Conclusion

Integration has evolved for more than 2 millennia by painstaking efforts of numerous mathematicians from the great Archimedes, Newton, Leibniz, Cauchy, Riemann, to the modern Lebesgue, Denjoy, Perron, and Kurzweil. In this paper we sample multiplicative integrability problem just to show the still incessant effort of mathematicians to find the most perfect integral by which we mean the most powerful (can integrate every function) and the most uncomplicated (easiest to understand and use). In our opinion, we have not achieved this end. Some theories are really more powerful than Riemann's but are also more complicated. It makes us feel like using sledgehammer to crack a nut. Maybe we are waiting for another Riesz who makes Lebesgue integral easier and retains its integrability power to come along for later-developed integrals. Or, more importantly, we are waiting for another Lebesgue to create a new, more perfect integral itself. Our belief is that integration theory can still be improved as Lebesgue himself once said at a conference at la Société Mathématique in Copenhagen on May 8, 1926 [8], "*... Messieurs, je m'arrête et je vous remercie de votre courtoise attention; mais il faut un mot de conclusion. Ce sera, si vous le voulez bien, qu'une généralisation faite non pour le vain plaisir de généraliser, mais pous résoudre des problèmes antérieurement posés, est toujours une généralisation féconde. Les divers emplois qu'ont déjà reçus les notions que nous venons d'examiner lo prouveraient surabondamment.*" ("*... Gentlemen, I end now and thank you for your courteous attention; but a word of conclusion is necessary. This is, if you will, that a generalization made not for the vain pleasure of generalizing, but rather for the solution of problems previously posed, is always a fruitful generalization. The diverse applications which have already taken the concepts which we have just examined prove this superabundantly.*")

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