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SEMISIMPLE ELEMENTS IN $Hyp_G(2)$

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Abstract

In semigroup theory, it is of interest to consider various types of semigroup elements, including regular, quasi-regular, completely quasi-regular, and semisimple elements. It is known that in general, regular implies completely quasi-regular which in turn implies semisimple. In this paper, we consider a particular semigroup, consisting of a type of mappings called generalized hypersubstitutions. The main result of this paper is that for this particular semigroup, any semisimple element is also regular, making these conditions equivalent.

1 Introduction

The concept of a regular element was first introduced by John von Neumann in 1936 under the name of "regular rings", during his study of von Neumann algebras. It is an important role in many branch of mathematics, especially in semigroup theory. A semigroup element a is semisimple if it can be factored into a product $a = xayaz$ for some elements x, y and z in the semigroup. The purpose of this paper is to characterize semisimple elements of a particular semigroup which is called the monoid of all generalized hypersubstitutions of type $\tau = (2)$. The concept of generalized hypersubstitutions was introduced by S. Leeratanavalee and K. Denecke [1]. It is the convenient method to describe the considered tree transformations. In this particular semigroup, each of mappings can be coded by a term, to which is associated a length. The semigroup operation is composition of mappings, and for the most part the term from a composition of mappings is longer than the terms in the composition. Using this basic length observation, W. Puninagool and S. Leeratanavalee de-

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termined the set of all regular elements and the set of all idempotent elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ (see [2]).

We recall first the definition of a regular element and the briefly concept of a generalized hypersubstitution. Let S be a semigroup. An element $a \in S$ is said to be *regular* if there exists $b \in S$ such that $a = aba$ and S is called a *regular semigroup* if every element of S is regular.

Let $\tau = (n_i)_{i\in I}$ be a type indexed by a set I, f_i be an operation symbol of arity n_i for $n_i \in \mathbb{N}$. Let $X_n := \{x_1, x_2, \ldots, x_n\}$ be an n-element alphabet and $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. An *n-ary term of type τ*, for simply an n-ary term, is defined inductively as follows:

(i) The variables $x_1, x_2, \ldots, x_n$ are n-ary terms.

(ii) If $t_1, t_2, \ldots, t_{n_i}$ are n-ary terms then $f_i(t_1, t_2, \ldots, t_{n_i})$ is an n-ary term.

Let $W_\tau(X_n)$ be the smallest set which contains $x_1, x_2, \ldots, x_n$ and is closed under finite application of (ii). Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ and called the set of all terms of type τ.

A mapping σ from the set of n_i-ary operation symbols to the set $W_\tau(X)$ which does not necessarily preserve the arity, is called a *generalized hypersubstitution of type τ*. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we define at first the concept of a *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then $S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Any generalized hypersubstitution σ induces a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ which is defined in the following inductive way :

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_i, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i supposed that $\hat{\sigma}[t_j], 1 \leq j \leq n_j$ are already defined.

We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then we have the following proposition.

Theorem 1.1. ([1]) *For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have* *(i)* $S^n(\sigma[t], \sigma[t_1], \ldots, \sigma[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. Then $\underline{Hyp_G(\tau)} = (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid where σ_{id} is the identity element.

2 Semisimple Elements

In this section, we characterize semisimple elements of $Hyp_G(2)$. Firstly, we recall some definitions which will be used throughout this paper.

Definition 2.1. Let S be a semigroup. An element a of a semigroup S is called

(i) *left quasi-regular* if there exist $x, y \in S$ such that $xaya = a$;

(ii) *right quasi-regular* if there exist $x, y \in S$ such that $axay = a$;

(iii) *completely quasi-regular* if a is both left and right quasi-regular;

(iv) *semisimple* if there exist $x, y, z \in S$ such that $xayaz = a$.

Remark 2.2. In general, for any semigroup S and $a \in S$, we have the following relationship: a is regular $\Rightarrow$ a is completely quasi-regular $\Rightarrow$ a is left quasi-regular or right quasi-regular $\Rightarrow$ a is semisimple.

Next, we fix a type $\tau = (2)$ with the binary operation f and for $t \in W_{(2)}(X)$ we denote by σ_t means the generalized hypersubstitution of type (2) which maps f to the term t. For $\sigma_t \in Hyp_G(2)$, we denote

$R_1 := \{\sigma_t \mid t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')\}$,
$R_2 := \{\sigma_t \mid t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')\}$,
$R_3 := \{\sigma_t \mid t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')\}$,
$R_4 := \{\sigma_t \mid t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')\}$,
$R_5 := \{\sigma_t \mid t \in \{x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}\}$ and
$R_6 := \{\sigma_t \mid var(t) \cap \{x_1, x_2\} = \emptyset\}$.

In 2011, W. Puninagool and S. Leeratanavalee showed that $\bigcup_{i=1}^{6} R_i$ is the set of all regular elements in $Hyp_G(2)$ and $(\bigcup_{i=3}^{6} R_i) \setminus \{\sigma_{f(x_2,x_1)}\} = E(Hyp_G(2))$ where $E(Hyp_G(2))$ is the set of all idempotent elements in $Hyp_G(2)$.

In 2010, W. Puninagool and S. Leeratanavalee [3] generalized the concept of complexity of terms, compositions and hypersubstitutions to complexity of terms, superpositions and generalized hypersubstitutions and proved the following lemma.

Lemma 2.3. *Let* $s, t_1, \ldots, t_m \in W_\tau(X)$. *Then*
$$op(S^m(s, t_1, \ldots, t_m)) = \sum_{j=1}^{m} vb_j(s) \cdot op(t_j) + op(s),$$
where $op(t)$ *is the total number of all operation symbols occurring in the term* t, *and* $vb_i(t)$ *is the* x_i*-variable count of the term* t.

For any $t \in W_{(2)}(X)$ and $x \in X$, we define semigroup words $Lp(t), Rp(t)$ over the alphabet $\{f\}$ inductively as follows:

(i) If $t = f(x, t_2), t_2 \in W_{(2)}(X)$, then $Lp(t) = f$.

(ii) If $t = f(t_1, x), t_1 \in W_{(2)}(X)$, then $Rp(t) = f$.

(iii) If $t = f(t_1, t_2), t_1 \notin X$, then $Lp(t) = f(Lp(t_1))$.

(iv) If $t = f(t_1, t_2), t_2 \notin X$, then $Rp(t) = f(Rp(t_2))$.

We denote the $f-$count of $Lp(t)$ and $Rp(t)$ by $length(Lp(t))$ and $length(Rp(t))$, respectively .

For $t \in W_{(2)}(X)$, we introduce the following notations:
$leftmost(t) :=$ the first variable (from the left) occurring in t,
$rightmost(t) :=$ the last variable occurring in t,
$var(t) :=$ the set of all variables occurring in t,
$W^G_{(2)}(\{x_1\}) := \{s \in W_{(2)}(X) \mid x_1 \in var(s), x_2 \notin var(s)\}$,
$W^G_{(2)}(\{x_2\}) := \{s \in W_{(2)}(X) \mid x_2 \in var(s), x_1 \notin var(s)\}$,
$W(\{x_1\}) := W^G_{(2)}(\{x_1\}) \backslash \{x_1\}$,
$W(\{x_2\}) := W^G_{(2)}(\{x_2\}) \backslash \{x_2\}$,
$W^G := \{t \in W_{(2)}(X) \mid t \notin X, x_1, x_2 \notin var(t)\}$,
$W^G_{(2)}(\{x_1, x_2\}) := \{t \in W_{(2)}(X) \mid x_1, x_2 \in var(t)\}$,
$E^G(\{x_1, x_2\}) := \{\sigma_t \in Hyp_G(2) \mid t \in W^G_{(2)}(\{x_1, x_2\})\}$.

Then we have the following lemmas which are useful for characterize semisimple elements of the monoid $Hyp_G(2)$.

Lemma 2.4. *Let* $u \in W_{(2)}(X), \sigma_t \in Hyp_G(2)$ *and* $x \in \{x_1, x_2\}$. *If* $x \notin var(u)$, *then* $x \notin var(\hat{\sigma}_t[u])$ *(*x *is not a variable occurring in the term* $(\sigma_t \circ_G \sigma_u)(f)$*).*

Lemma 2.5. *Let* $\sigma_{f(c,d)} \in Hyp_G(2) \backslash \{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$ *and* $u \in W_{(2)}(X) \backslash X$. *If* $\sigma_{f(c,d)} \in E^G(\{x_1, x_2\})$, *then the term* w *corresponding to the term* $\sigma_{f(c,d)} \circ_G \sigma_u(f)$ *is longer than* u.

Lemma 2.6. *If* $f(c, d) \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G (x_1 \notin var(f(c, d))$ *or* $x_2 \notin var(f(c, d)))$, *then for any* $u, v \in W_{(2)}(X)$ *the term* w *corresponding to* $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)}(f)$ *is in* $W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$.

Lemma 2.7. *Let* $s, u, v \notin X$ *and* $\sigma_s \circ_G \sigma_u = \sigma_v$. *Then the following statements hold.*

(i) If $leftmost(s) = x_1$, then $length(Lp(v)) = length(Lp(s))length(Lp(u))$. If $leftmost(s) = x_2$, then $length(Lp(v)) = length(Lp(s))length(Rp(u))$.

(ii) If $rightmost(s) = x_1$, then $length(Rp(v)) = length(Rp(s))length(Lp(u))$. If $rightmost(s) = x_2$, then $length(Rp(v)) = length(Rp(s))length(Rp(u))$.

Lemma 2.8. *Let $s \in W_{(2)}(X) \setminus X, x_1, x_2 \in var(s), t \in W_{(2)}(X)$ and $x_i \in X$ where $i \in \mathbb{N}$. If $x_i \in var(t)$, then $x_i \in var(\hat{\sigma}_s[t])$.*

For more detail see [4].

Theorem 2.9. *Let $s \in W_{(2)}(X) \setminus X, x_1, x_2 \in var(s)$. If σ_s is semisimple, then σ_s is regular.*

Proof. Let σ_s be semisimple and $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$ and $x_1, x_2 \in var(s)$. There exist $t_1, t_2, t_3 \in W_{(2)}(X) \setminus X$ such that $\sigma_{t_1} \circ_G \sigma_s \circ_G \sigma_{t_2} \circ_G \sigma_s \circ_G \sigma_{t_3} = \sigma_{f(s_1,s_2)}$. Suppose that σ_s is not regular, we have $op(s) > 1$. Since $x_1, x_2 \in var(s)$ and $\sigma_{t_1} \circ_G \sigma_s \circ_G \sigma_{t_2} \circ_G \sigma_s \circ_G \sigma_{t_3} = \sigma_{f(s_1,s_2)}$, thus by Lemma 2.6, we get $x_1, x_2 \in var(t_1)$. We set $\sigma_s \circ_G \sigma_{t_2} \circ_G \sigma_s \circ_G \sigma_{t_3} = \sigma_{f(t'_1,t'_2)}$ and then $x_1, x_2 \in var(f(t'_1, t'_2))$. Thus $x_1, x_2 \in var(t_2)$. By Lemma 2.8, we get $op(f(t'_1, t'_2)) > op(s)$. We claim that $op(\hat{\sigma}_{t_1}[f(t'_1, t'_2)]) > op(s)$. Then by Lemma 2.3, we have

$$\begin{aligned}
op(\hat{\sigma}_{t_1}[f(t'_1, t'_2)]) &= op(S^2(t_1, \hat{\sigma}_{t_1}[t'_1], \hat{\sigma}_{t_1}[t'_2])) \\
&= vb_1(t_1) \cdot op(\hat{\sigma}_{t_1}[t'_1]) + vb_2(t_1) \cdot op(\hat{\sigma}_{t_1}[t'_2]) + op(t_1) \\
&\geq op(\hat{\sigma}_{t_1}[t'_1]) + op(\hat{\sigma}_{t_1}[t'_2]) + 1 \\
&\geq op(t'_1) + op(t'_2) + 1 \\
&= op(f(t'_1, t'_2)) \\
&> op(s)
\end{aligned}$$

which is a contradiction. Therefore σ_s is regular. □

Theorem 2.10. *Let $s \in W_{(2)}(X) \setminus X$ such that $x_1 \in var(s)$ and $\sigma_{f(x_m,s)}$ is semisimple, where $m \in \mathbb{N}$ with $m > 2$. Then $\sigma_{f(x_m,s)}$ is regular.*

Proof. Let $\sigma_{f(x_m,s)}$ be semisimple, there exist $t_1, t_2, t_3 \in W_{(2)}(X) \setminus X$ such that $\sigma_{t_1} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3} = \sigma_{f(x_m,s)}$. Suppose that $\sigma_{f(x_m,s)}$ is not regular, we have $op(f(x_m, s)) > 1$ and $length(Rp(f(x_m, s))) \geq 2$. We set $\sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3} = \sigma_{f(x_m,t'_2)}$. Then $t_1 \neq f(x_2, x_1)$. Suppose that $t_1 = f(x_1, x_2)$, we have $\sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3} = \sigma_{f(x_m,s)}$. If $rightmost(s) = x_m$, then $leftmost(t_2) \neq x_m$. If $leftmost(t_2) = x_1$, then $x_1, x_2 \notin var(\sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)})$ which is a contradiction. If $leftmost(t_2) = x_2$, then $vb(\sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3}(f)) > vb(f(x_m, s))$ which is a contradiction. Since $rightmost(s) \neq x_2$, we get $rightmost(s) = x_1$ and then $leftmost(t_2) \neq x_m$. If $leftmost(t_2) = x_1$, then $x_1, x_2 \notin var(\sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G$

$\sigma_{f(x_m,s)})$ which is a contradiction. This implies $leftmost(t_2) = x_2$. So that $length(Rp((\sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3})(f)))$

$$\begin{aligned}
&= length(Rp((\sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3})(f)))\\
&= length(Rp(f(x_m, s)))length(Lp(\sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3})(f))\\
&> length(Rp(f(x_m, s)))
\end{aligned}$$

which is a contradiction. If $op(t_1) > 1$ and $x_1, x_2 \in var(t_1)$, then by Lemma 2.3, we have

$$\begin{aligned}
op(\hat{\sigma}_{t_1}[f(x_m, t'_2)]) &= op(S^2(t_1, \hat{\sigma}_{t_1}[x_m], \hat{\sigma}_{t_1}[t'_2]))\\
&= vb_1(t_1) \cdot op(\hat{\sigma}_{t_1}[x_m]) + vb_2(t_1) \cdot op(\hat{\sigma}_{t_1}[t'_2]) + op(t_1)\\
&> op(\hat{\sigma}_{t_1}[x_m]) + op(\hat{\sigma}_{t_1}[t'_2]) + 1\\
&\geq op(x_m) + op(t'_2) + 1\\
&= op(f(x_m, t'_2))\\
&\geq op(f(x_m, s))
\end{aligned}$$

which is a contradiction. If $x_1 \in var(t_1)$ and $x_2 \notin var(t_1)$, then $x_1, x_2 \notin var(\sigma_{t_1} \circ_G \sigma_{f(x_m,s)}(f))$ which is a contradiction. If $x_1 \notin var(t_1)$ and $x_2 \in var(t_1)$ and $rightmost(s) = x_m$, we get $x_1, x_2 \notin var(\sigma_{t_1} \circ_G \sigma_{f(x_m,s)}(f))$ which is a contradiction. Since $rightmost(s) \neq x_2$, we get $rightmost(s) = x_1$ and then $rightmost(t_1) \neq x_m$. This implies $rightmost(t_1) = x_2$. If $leftmost(t_2) = x_1$ or $leftmost(t_2) = x_m$, then $x_1, x_2 \notin var(\sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)}(f))$ which is a contradiction. So $leftmost(t_2) = x_2$ and $length(Rp((\sigma_{t_1} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3})(f)))$

$$\begin{aligned}
&= length(Rp(t_1))length(Rp((\sigma_{f(x_m,s)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3})(f)))\\
&= length(Rp(t_1))length(Rp(f(x_m, s)))length(Lp(\sigma_{t_2} \circ_G \sigma_{f(x_m,s)} \circ_G \sigma_{t_3})(f))\\
&> length(Rp(f(x_m, s)))
\end{aligned}$$

which is a contradiction. Therefore $\sigma_{f(x_m,s)}$ is regular. □

Theorem 2.11. *Let $s \in W_{(2)}(X) \setminus X$ such that $x_1 \in var(s)$ and $\sigma_{f(s,x_m)}$ is semisimple, where $m \in \mathbb{N}$ with $m > 2$. Then $\sigma_{f(s,x_m)}$ is regular.*

Proof. The proof is similar to the proof of Theorem 2.10. □

Theorem 2.12. *Let $s \in W_{(2)}(X) \setminus X$ such that $x_2 \in var(s)$ and $\sigma_{f(s,x_m)}$ is semisimple, where $m \in \mathbb{N}$ with $m > 2$. Then $\sigma_{f(s,x_m)}$ is regular.*

Proof. The proof is similar to the proof of Theorem 2.10. □

Theorem 2.13. *Let $s \in W_{(2)}(X) \setminus X$ such that $x_2 \in var(s)$ and $\sigma_{f(x_m,s)}$ is semisimple, where $m \in \mathbb{N}$ with $m > 2$. Then $\sigma_{f(x_m,s)}$ is regular.*

Proof. The proof is similar to the proof of Theorem 2.10. □

Theorem 2.14. *Let $s_1, s_2 \in W_{(2)}(X) \setminus X$. If $x_1 \in var(s_1) \cup var(s_2)$ or $x_2 \in var(s_1) \cup var(s_2)$, then $\sigma_{f(s_1,s_2)}$ is not semisimple.*

Proof. Suppose that $s = f(s_1, s_2)$ is semisimple. Since $s_1 \notin X$, we get $length(Lp(s)) \geq 2$. Since $s_2 \notin X$, we get $length(Rp(s)) \geq 2$. Then there exist $t_1, t_2, t_3 \in W_{(2)}(X) \setminus X$ such that $\sigma_{t_1} \circ_G \sigma_s \circ_G \sigma_{t_2} \circ_G \sigma_s \circ_G \sigma_{t_3} = \sigma_{f(s_1,s_2)}$.
Case 1: $x_1, x_2 \in var(s)$. The proof is similar to the proof of Theorem 2.10.
Case 2: $x_1 \in var(s), x_2 \notin var(s)$. We set $\sigma_s \circ_G \sigma_{t_2} \circ_G \sigma_s \circ_G \sigma_{t_3} = \sigma_{f(s_1',s_2')}$.

Case 2.1: $leftmost(s) = x_m$ where $m \in \mathbb{N}$ with $m > 2$. Suppose that $t_1 = f(x_2, x_1)$, then $rightmost(s) = x_m$. If $leftmost(t_2) = x_m$, then $x_1, x_2 \notin var(\sigma_{f(s_1,s_2)} \circ_G \sigma_{t_2})$ which is a contradiction. If $leftmost(t_2) = x_1$ or $leftmost(t_2) = x_2$, then $x_1, x_2 \notin var(\sigma_{f(s_1,s_2)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(s_1,s_2)})$ which is a contradiction. So $t_1 \neq f(x_2, x_1)$. The remainder of proof is similar to the proof of Theorem 2.10.

Case 2.2: $leftmost(s) = x_1$. Then we get $leftmost(t_1) = x_1$ or $leftmost(t_1) = x_2$.
If $leftmost(t_1) = x_1$, then $leftmost(t_2) \neq x_m$ where $m \in \mathbb{N}$ with $m > 2$. So

$$\begin{aligned}
&length(Lp((\sigma_{t_1} \circ_G \sigma_{f(s_1,s_2)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(s_1,s_2)} \circ_G \sigma_{t_3})(f))) \\
&= length(Lp(t_1))length(Lp((\sigma_{f(s_1,s_2)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(s_1,s_2)} \circ_G \sigma_{t_3})(f))) \\
&= length(Lp(t_1))length(Lp(f(s_1,s_2)))length(Lp(\sigma_{t_2} \circ_G \sigma_{f(s_1,s_2)} \circ_G \sigma_{t_3})(f)) \\
&> length(Lp(f(s_1,s_2)))
\end{aligned}$$

which is a contradiction. If $leftmost(t_1) = x_2$, then $rightmost(s) = x_1$. And thus $leftmost(t_2) \neq x_m$. So $leftmost(t_2) = x_1$ or $leftmost(t_2) = x_2$. Hence

$$\begin{aligned}
&length(Rp((\sigma_{t_1} \circ_G \sigma_{f(s_1,s_2)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(s_1,s_2)} \circ_G \sigma_{t_3})(f))) \\
&= length(Rp(t_1))length(Rp((\sigma_{f(s_1,s_2)} \circ_G \sigma_{t_2} \circ_G \sigma_{f(s_1,s_2)} \circ_G \sigma_{t_3})(f))) \\
&= length(Rp(t_1))length(Rp(f(s_1,s_2)))length(Lp(\sigma_{t_2} \circ_G \sigma_{f(s_1,s_2)} \circ_G \sigma_{t_3})(f)) \\
&> length(Rp(f(s_1,s_2)))
\end{aligned}$$

which is a contradiction.

Case 3: $x_1 \notin var(t), x_2 \in var(t)$. The proof is similar to the proof of Case 2. □

Theorem 2.15. *Let $\sigma_s \in Hyp_G(2)$. Then the following are equivalent:*

(a) σ_s *is regular,*

(b) σ_s *is completely quasi-regular,*

(c) σ_s *is left quasi-regular,*

(d) σ_s *is right quasi-regular,*

(e) σ_s *is semisimple.*

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