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ON AN ALTERNATIVE FUNCTIONAL EQUATION RELATED TO THE JENSEN'S FUNCTIONAL EQUATION

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Abstract

Given an integer $\lambda \neq 1$, we study the alternative Jensen's functional equation

$$f(xy^{-1}) - 2f(x) + f(xy) = 0 \quad \text{or} \quad f(xy^{-1}) - 2f(x) + \lambda f(xy) = 0,$$

where f is a mapping from a group $(G, \cdot)$ to a uniquely divisible abelian group $(H, +)$. We prove that for $\lambda \neq -3$, the above functional equation is equivalent to the classical Jensen's functional equation. Furthermore, if G is a 2-divisible group, then we can strengthen the results by the showing that the equivalence is valid for all integers $\lambda \neq 1$.

1 Introduction

The alternative functional equations related to the (classical) Cauchy equation problem

$$f(x + y) = f(x) + f(y) \tag{1.1}$$

have been widely studied. For example, in 1974, Kannappan and Kuczma [3] investigated the alternative Cauchy functional equation of the form

$$(f(x + y) - af(x) - bf(y))\,(f(x + y) - f(x) - f(y)) = 0, \tag{1.2}$$

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where f is a function form an abelian group to a commutative integral domain with identity and of characteristic zero. Afterwards, Ger[2] extended the result in [3] to the more general alternative functional equation of the form

$$(f(x+y)-af(x)-bf(y))\,(f(x+y)-cf(x)-df(y))=0.$$

In 1978, Kuczma [4] established the equivalence of (1.2) and the classical Cauchy functional equation (1.1) in the case when $a=b=-1$ and the domain is a semigroup. Later on, Forti[1] established the general solution of the alternative Cauchy functional equation of the form

$$(cf(x+y)-af(x)-bf(y)-d)\,(f(x+y)-f(x)-f(y))=0.$$

Inspired by the work on the alternative Cauchy functional equation, Nakmahachalasint[5] has proved the analogous results on the alternative Jensen's functional equation of the form

$$f(x)\pm 2f(xy)+f(xy^2)=0$$

on a semigroup. His work represents a significant extension of the work of NG [6] and Parnami and Vasudeva [7] on the classical Jensen's functional equation

$$f(x)-2f(xy)+f(xy^2)=0$$

on a group.

In this paper, given an integer $\lambda\neq 1$, we investigate the alternative Jensen's functional equation of the form

$$f(xy^{-1})-2f(x)+f(xy)=0 \quad \text{or} \quad f(xy^{-1})-2f(x)+\lambda f(xy)=0, \qquad \text{(AJ)}$$

where f is a mapping from a group $(G,\cdot)$ to a uniquely divisible abelian group $(H,+)$. Note that when $\lambda=1$, (AJ) is just the classical Jensen's functional equation. We will prove that for $\lambda\neq -3$, (AJ) is equivalent to the classical Jensen's functional equation in the sense that

$$f(xy^{-1})-2f(x)+f(xy)=0, \qquad \text{(J)}$$

for all $x,y\in G$. Furthermore, if the domain G is a 2-divisible group, then we will show that (AJ) is equivalent to the classical Jensen's functional equation (J) for all $\lambda\neq 1$.

2 Auxiliary Lemmas

Let $(G,\cdot)$ be a group and $(H,+)$ be a uniquely divisible abelian group. Given an integer λ and a function $f:G\to H$. For every pair of $x,y\in G$, we define

$$F_y^{(\lambda)}(x):=f(xy^{-1})-2f(x)+\lambda f(xy).$$

Furthermore, for $\lambda \neq 1$, we let $\mathcal{P}f_y^{(\lambda)}(x)$ be the statement

$$\mathcal{P}f_y^{(\lambda)}(x) := \Big(F_y^{(1)}(x) = 0 \quad \text{or} \quad F_y^{(\lambda)}(x) = 0\Big).$$

Denote the set of all solutions of (AJ) by

$$\mathcal{A}_{(G,H)}^{(\lambda)} := \{f : G \to H \mid \mathcal{P}f_y^{(\lambda)}(x) \ \text{ for all } x, y \in G\}.$$

Finally, the set of solution of the Jensen's functional equation will be denoted by

$$\mathcal{J}_{(G,H)} := \{f : G \to H \mid F_y^{(1)}(x) = 0 \ \text{ for all } x, y \in G\}.$$

It should be note that $\mathcal{J}_{(G,H)} \subseteq \mathcal{A}_{(G,H)}^{(\lambda)}$

The following Lemma will be crucial and we make use of it extensively in the proofs. The reader should keep in mind of this fact.

Lemma 1. *Let* $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ *and let* $x, y \in G$.
$F_y^{(1)}(x) = 0$ *and* $F_y^{(\lambda)}(x) = 0$ *if and only if* $f(xy) = 0$.

Proof. Assume that $F_y^{(1)}(x) = 0$ and $F_y^{(\lambda)}(x) = 0$. Therefore, $F_y^{(\lambda)}(x) - F_y^{(1)}(x) = 0$, i.e., $(\lambda - 1)f(xy) = 0$. Since $\lambda \neq 1$, we get $f(xy) = 0$. Conversely, assume that $f(xy) = 0$. Since $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$, we have $F_y^{(1)}(x) = 0$ or $F_y^{(\lambda)}(x) = 0$. As $f(xy) = 0$, therefore $F_y^{(1)}(x) = 0$ and $F_y^{(\lambda)}(x) = 0$. □

We will prove some fundamental lemmas concerning the property of $\mathcal{P}f_y^{(\lambda)}(x)$.

Lemma 2. *Let* $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ *and let* $x, y \in G$.
If $F_y^{(1)}(x) \neq 0$, *then* $f(xy^{-1}) = f(xy)$.

Proof. Assume that $F_y^{(1)}(x) \neq 0$. The alternatives in $\mathcal{P}f_y^{(\lambda)}(x)$ give $F_y^{(\lambda)}(x) = 0$. Similarly, from $F_{y^{-1}}^{(1)}(x) = F_y^{(1)}(x) \neq 0$ and $\mathcal{P}f_{y^{-1}}^{(\lambda)}(x)$, we get $F_{y^{-1}}^{(\lambda)}(x) = 0$. Observe that $F_y^{(\lambda)}(x) - F_{y^{-1}}^{(\lambda)}(x) = 0$, which simplifies to

$$(1 - \lambda)\big(f(xy^{-1}) - f(xy)\big) = 0.$$

Since $\lambda \neq 1$, we must have $f(xy^{-1}) = f(xy)$ as desired. □

Lemma 3. *Let* $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ *and let* $x, y \in G$.
If $F_y^{(1)}(x) \neq 0$ *and* $F_y^{(1)}(xy) \neq 0$, *then* $\lambda = -3$.

Proof. Assume that $F_y^{(1)}(x) \neq 0$ and $F_y^{(1)}(xy) \neq 0$. Hence $f(xy^2) \neq 0$ by Lemma 1. By Lemma 2, we get

$$f(xy^{-1}) = f(xy) \quad \text{and} \quad f(x) = f(xy^2). \tag{2.1}$$

From $F_y^{(1)}(x) \neq 0$ and $F_y^{(1)}(xy) \neq 0$, the alternatives in $\mathcal{P}f_y^{(\lambda)}(x)$ and $\mathcal{P}f_y^{(\lambda)}(xy)$ give $F_y^{(\lambda)}(x) = 0$ and $F_y^{(\lambda)}(xy) = 0$, respectively. Substituting $f(x)$ from (2.1) into $F_y^{(\lambda)}(xy) = 0$, we have

$$(\lambda + 1)f(xy^2) - 2f(xy) = 0. \tag{2.2}$$

Substituting $f(xy^{-1})$ and $f(x)$ from (2.1) into $F_y^{(\lambda)}(x) = 0$, we obtain that

$$(\lambda + 1)f(xy) - 2f(xy^2) = 0. \tag{2.3}$$

By (2.2) and (2.3), we get

$$(\lambda^2 + 2\lambda - 3)f(xy^2) = 0.$$

Since $\lambda \neq 1$ and $f(xy^2) \neq 0$, we must have $\lambda = -3$. □

Lemma 4. *Let $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ and let $x, y \in G$. If $F_y^{(1)}(x) \neq 0$, then $F_y^{(1)}(xy) \neq 0$.*

Proof. Suppose $F_y^{(1)}(x) \neq 0$ but $F_y^{(1)}(xy) = 0$. Since H is a uniquely divisible abelian group, we let $f(xy) = 2a$. Assume that $F_y^{(1)}(x) \neq 0$. By Lemma 2, we get

$$f(xy^{-1}) = 2a. \tag{2.4}$$

From $F_y^{(1)}(x) \neq 0$, the alternatives in $\mathcal{P}f_y^{(\lambda)}(x)$ gives $F_y^{(\lambda)}(x) = 0$. We also have $2a \neq 0$ by Lemma 1, that is, $a \neq 0$. Substituting (2.4) into $F_y^{(\lambda)}(x) = 0$, we obtain that

$$f(x) = (\lambda + 1)a. \tag{2.5}$$

Substituting (2.5) into $F_y^{(1)}(xy) = 0$, we have

$$f(xy^2) = (3 - \lambda)a. \tag{2.6}$$

Consider the following two possible cases in $\mathcal{P}f_y^{(\lambda)}(xy^{-1})$.

1. Assume that $F_y^{(1)}(xy^{-1}) \neq 0$. By Lemma 2, we get $f(xy^{-2}) = f(x)$. By (2.5), we have

   $$f(xy^{-2}) = (\lambda + 1)a. \tag{2.7}$$

   Substituting (2.5), (2.6) and (2.7) into $\mathcal{P}f_{y^2}^{(\lambda)}(x)$, we obtain that

   $$(2 - 2\lambda)a = 0 \quad \text{or} \quad (-1 + 2\lambda - \lambda^2)a = 0.$$

   Since $\lambda \neq 1$, we get $a = 0$, a contradiction.

2. Assume that $F_y^{(1)}(xy^{-1}) = 0$. Substituting (2.4) and (2.5) into $F_y^{(1)}(xy^{-1}) = 0$, we have

$$f(xy^{-2}) = (3-\lambda)a. \tag{2.8}$$

Substituting (2.5), (2.6) and (2.8), in $\mathcal{P}f_{y^2}^{(\lambda)}(x)$, we get

$$(4-4\lambda)a = 0 \ \text{ or } \ (1-\lambda^2)a = 0.$$

Since $\lambda \neq 1$ and $a \neq 0$, $\lambda = -1$. Thus (2.6) simplifies to

$$f(xy^2) = 4a. \tag{2.9}$$

Substituting (2.9) into $\mathcal{P}f_y^{(-1)}(xy^2)$, we obtain that

$$f(xy^3) = \pm 6a. \tag{2.10}$$

Substituting (2.4), (2.10) into $\mathcal{P}f_{y^2}^{(-1)}(xy)$ and simplifying, we get $a = 0$, a contradiction.

Therefore, we must have $F_y^{(1)}(xy) \neq 0$ as desired. □

Lemma 5. *Let $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ and let $x, y \in G$.*
If $F_y^{(1)}(x) \neq 0$, then $F_y^{(1)}(xy^n) \neq 0$ for all $n \in \mathbb{Z}$.

Proof. By applying Lemma 4 repeatedly, we get

$$F_y^{(1)}(xy^n) \neq 0 \ \text{ for all } \ n \geq 1.$$

Similarly, by substituting y by y^{-1} in the previous arguments, we have

$$F_y^{(1)}(xy^n) \neq 0 \ \text{ for all } \ n \leq -1.$$

Consequently, we conclude that $F_y^{(1)}(xy^n) \neq 0$ for all $n \in \mathbb{Z}$. □

3 Main Results

In this section, we prove our main results of this paper.

Theorem 6. *If $\mathcal{A}_{(G,H)}^{(\lambda)} \backslash \mathcal{J}_{(G,H)} \neq \phi$, then $\lambda = -3$.*
Moreover, if $f \in \mathcal{A}_{(G,H)}^{(\lambda)} \backslash \mathcal{J}_{(G,H)}$ and $x, y \in G$, then $f(xy^n) = (-1)^n a$ for all $n \in \mathbb{Z}$ and for some $a \in H$.

Proof. Assume that $\mathcal{A}^{(\lambda)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} \neq \phi$. Let $f \in \mathcal{A}^{(\lambda)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$ and $x, y \in G$. Thus $F^{(1)}_y(x) \neq 0$. Lemma 5 gives $F^{(1)}_y(xy^n) \neq 0$ for all $n \in \mathbb{Z}$. By Lemma 2, we get

$$f(xy^{n+1}) = f(xy^{n-1}) \quad \text{for all} \quad n \in \mathbb{Z}. \tag{3.1}$$

Since $F^{(1)}_y(x) \neq 0$ and $F^{(1)}_y(xy) \neq 0$, Lemma 3 gives $\lambda = -3$. From $F^{(1)}_y(xy^n) \neq 0$ for all $n \in \mathbb{Z}$, the alternatives $\mathcal{P}f^{(-3)}_y(xy^n)$ give $F^{(-3)}_y(xy^n) = 0$, i.e.,

$$f(xy^{n+1}) - 2f(xy^n) - 3f(xy^{n-1}) = 0. \tag{3.2}$$

By (3.1) and (3.2), we have

$$f(xy^n) = -f(xy^{n-1}) \quad \text{for all} \quad n \in \mathbb{Z}. \tag{3.3}$$

Thus $f(xy) = -f(x)$ when $n = 1$, and $f(xy^{-1}) = -f(x)$ when $n = 0$. For any $n \geq 2$, by (3.3), we obtain that

$$f(xy^n) = (-1)f(xy^{n-1}) = (-1)^2 f(xy^{n-2}) = \cdots = (-1)^n f(x)$$

and

$$f(xy^{-n}) = (-1)f(xy^{-n+1}) = (-1)^2 f(xy^{-n+2}) = \cdots = (-1)^n f(x).$$

Therefore, $f(xy^n) = (-1)^n f(x)$ for all $n \in \mathbb{Z}$ as desired. □

Theorem 3.1 shows that the alternative Jensen's functional equation (AJ) is equivalent to the Jensen's functional equation (J) when $\lambda \neq -3$. However, when $\lambda = -3$, (AJ) is not necessarily equivalent to (J) as illustrated by the following example.

Example 7. *Given $a \in H \backslash \{0\}$. Let $f : \mathbb{Z} \to H$ be a function such that*

$$f(n) = (-1)^n a \quad \textit{for all } n \in \mathbb{Z}.$$

First, we will show that $f \in \mathcal{A}^{(-3)}_{(\mathbb{Z},H)}$. Given $n, m \in \mathbb{Z}$. If m is odd, then we see that $n - m$ and $n + m$ have the same parity whereas n and $n + m$ have the opposite. Therefore, $f(n - m) - 2f(n) - 3f(n + m) = 0$. Otherwise, if m is even, then $n - m, n, n + m$ all have the same parity, i.e., $f(n - m) - 2f(n) + f(n + m) = 0$. Next, we will prove that $f \notin \mathcal{J}_{(\mathbb{Z},H)}$. Note that $f(0) - 2f(1) + f(2) = 4a$. From $a \neq 0$ and H is uniquely divisible, we get $4a \neq 0$. Thus $f \in \mathcal{A}^{(-3)}_{(\mathbb{Z},H)} \backslash \mathcal{J}_{(\mathbb{Z},H)}$.

In the case that G is a $2-$divisible group, our main result is stronger in sense that (AJ) is actually equivalent to the classical Jensen's functional equation (J) as the following theorem.

Theorem 8. *Let $(G, \cdot)$ be a 2-divisible group. Then $\mathcal{A}^{(\lambda)}_{(G,H)} = \mathcal{J}_{(G,H)}$ for all $\lambda \neq 1$.*

Proof. It is only left to prove that $\mathcal{A}^{(\lambda)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} = \phi$. Assume contradictorily that $\mathcal{A}^{(\lambda)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} \neq \phi$. Let $f \in \mathcal{A}^{(\lambda)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$ and $x, y \in G$. Since G is a 2-divisible group, there exists a $z \in G$ such that $y = z^2$. By setting $y = z$ in Theorem 6, we obtain that $\lambda = -3$ and $f(xz^n) = (-1)^n a$ for all $n \in \mathbb{Z}$ and for some $a \in H$. We can calculate

$$f(xz^{-2}) - 2f(x) + f(xz^2) = (-1)^{-2}a - 2a + (-1)^2 a = 0.$$

Since $y = z^2$, we conclude that $f(xy^{-1}) - 2f(x) + f(xy) = 0$ and therefore $f \in \mathcal{J}_{(G,H)}$. Hence, it is a contradiction to the fact that $f \notin \mathcal{J}_{(G,H)}$. □

4 Concrete Examples

In this section, we give the general solution when the domain is a cyclic group. The following theorems show how Theorem 6 can be applied to certain cases.

Theorem 9. *Let $(G, \cdot)$ be an infinite cyclic group with $G = \langle g \rangle$.*

1. *If $\lambda \neq -3$, then $\mathcal{A}^{(\lambda)}_{(G,H)} = \mathcal{J}_{(G,H)}$.*
2. *If $\lambda = -3$, then $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$ is non-empty and $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} = \{f : G \to H \mid f(g^n) = (-1)^n a$ for all $n \in \mathbb{Z}$ and for some $a \in H\}$.*

Proof. If $\lambda \neq -3$, then Theorem 3.1 gives $\mathcal{A}^{(\lambda)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} = \phi$, i.e., $\mathcal{A}^{(\lambda)}_{(G,H)} = \mathcal{J}_{(G,H)}$ and therefore (1). Next, we assume that $\lambda = -3$. First, we will show that $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$ is non-empty. Let $f : G \to H$ be

$$f(g^n) = (-1)^n a \quad \text{for all} \quad n \in \mathbb{Z} \quad \text{and for some} \quad a \in H \backslash \{0\}.$$

Given $n, m \in \mathbb{Z}$. If m is odd, then we see that $n - m$ and $n + m$ have the same parity whereas n and $n + m$ have the opposite. Therefore,

$$f(g^{n-m}) - 2f(g^n) - 3f(g^{n+m}) = 0.$$

Otherwise, if m is even, then $n - m, n, n + m$ all have the same parity, i.e.,

$$f(g^{n-m}) - 2f(g^n) + f(g^{n+m}) = 0.$$

Note that $f(e) - 2f(g) + f(g^2) = 4a$. From $a \neq 0$ and H is uniquely divisible, we get $4a \neq 0$. Thus $f \in \mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$. Since $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$ is non-empty, the proof is complete by setting $x = e$ and $y = g$ in Theorem 6. □

Theorem 10. *Let $(G, \cdot)$ be a cyclic group of finite order $m \geq 2$ with $G = \langle g \rangle$.*

1. *If* $\lambda \neq -3$, *then* $\mathcal{A}^{(\lambda)}_{(G,H)} = \mathcal{J}_{(G,H)}$.

2. *If* $\lambda = -3$, *then*

   (a) *if* m *is odd, then* $\mathcal{A}^{(-3)}_{(G,H)} = \mathcal{J}_{(G,H)}$, *or*

   (b) *if* m *is even, then* $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$ *is non-empty and* $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} = \{f : G \to H \mid f(g^n) = (-1)^n a$ *for all* $n \in \mathbb{Z}$ *and for some* $a \in H\}$.

Proof. If $\lambda \neq -3$, then Theorem 3.1 gives $\mathcal{A}^{(\lambda)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} = \phi$, i.e., $\mathcal{A}^{(\lambda)}_{(G,H)} = \mathcal{J}_{(G,H)}$ and therefore (1). Next, we assume that $\lambda = -3$. We will consider two possible cases of m as follows.

(a) Assume that m is odd. We will show that $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} = \phi$. Suppose $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} \neq \phi$. Let $f \in \mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$. By setting $x = e$ and $y = g$ in Theorem 6, we obtain that $f(g^n) = (-1)^n a$ for all $n \in \mathbb{Z}$ and for some $a \in H$. Hence $f(e) = a$. Since $e = g^m$, we get

$$a = f(e) = f(g^m) = (-1)^m a = -a.$$

We conclude that $a = 0$ and therefore $f \in \mathcal{J}_{(G,H)}$, a contradiction. Thus we must have $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} = \phi$.

(b) Assume that m is even. Let $f : G \to H$ be

$$f(g^n) = (-1)^n a \quad \text{for all} \quad n \in \mathbb{Z} \quad \text{and for some} \quad a \in H \backslash \{0\}.$$

Given $n, m \in \mathbb{Z}$. If m is odd, then we see that $n - m$ and $n + m$ have the same parity whereas n and $n + m$ have the opposite. Therefore,

$$f(g^{n-m}) - 2f(g^n) - 3f(g^{n+m}) = 0.$$

Otherwise, if m is even, then $n - m, n, n + m$ all have the same parity, i.e.,

$$f(g^{n-m}) - 2f(g^n) + f(g^{n+m}) = 0.$$

Note that $f(e) - 2f(g) + f(g^2) = 4a$. From $a \neq 0$ and H is uniquely divisible, we get $4a \neq 0$. Thus $f \in \mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$, i.e., $\mathcal{A}^{(-3)}_{(G,H)} \backslash \mathcal{J}_{(G,H)} \neq \phi$. Hence the proof is then complete by setting $x = e$ and $y = g$ in Theorem 6.

□

References

[1] G.L. Forti, La soluzione generale dellequazione funzionale $\{cf(x+y) - af(x) - bf(y) - d\}\{f(x+y) - f(x) - f(y)\} = 0$, *Matematiche (Catania)* **34** (1979) 219-42.

[2] R. Ger, On an alternative functional equation, *Aequationes Mathematicae* **15** (1977) 145–162.

[3] P.L. Kannappan and M. Kuczma, On a functional equation related to the Cauchy equation, *Ann. Polon. Math.* **30** (1974) 49–55.

[4] M. Kuczma, On Some Alternative Functional Equations, *Aequationes Mathematicae* **2** (1978) 182–98.

[5] P. Nakmahachalasint, An alternative Jensens functional equation on semigroups, *ScienceAsia* **38** (2012) 408-13.

[6] C. NG, Jensen's Functional Equation on Groups, *Aequationes Mathematicae* **39** (1990) 85–99.

[7] J.C. Parnami and H.L. Vasudeva, On Jensen's Functional Equation, *Aequationes Mathematicae* **43** (1992) 211–8.