ON AN ALTERNATIVE FUNCTIONAL EQUATION RELATED TO THE JENSEN'S FUNCTIONAL EQUATION

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Abstract

Given an integer $\lambda \neq 1$, we study the alternative Jensen's functional equation

$$f(xy^{-1}) - 2f(x) + f(xy) = 0$$
 or $f(xy^{-1}) - 2f(x) + \lambda f(xy) = 0$,

where f is a mapping from a group (G, \cdot) to a uniquely divisible abelian group (H, +). We prove that for $\lambda \neq -3$, the above functional equation is equivalent to the classical Jensen's functional equation. Furthermore, if G is a 2-divisible group, then we can strengthen the results by the showing that the equivalence is valid for all integers $\lambda \neq 1$.

1 Introduction

The alternative functional equations related to the (classical) Cauchy equation problem

$$f(x+y) = f(x) + f(y)$$
 (1.1)

have been widely studied. For example, in 1974, Kannappan and Kuczma [3] investigated the alternative Cauchy functional equation of the form

$$(f(x+y) - af(x) - bf(y)) (f(x+y) - f(x) - f(y)) = 0, (1.2)$$

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where f is a function form an abelian group to a commutative integral domain with identity and of characteristic zero. Afterwards, Ger[2] extended the result in [3] to the more general alternative functional equation of the form

$$(f(x+y) - af(x) - bf(y)) (f(x+y) - cf(x) - df(y)) = 0.$$

In 1978, Kuczma [4] established the equivalence of (1.2) and the classical Cauchy functional equation (1.1) in the case when a=b=-1 and the domain is a semigroup. Later on, Forti[1] established the general solution of the alternative Cauchy functional equation of the form

$$(cf(x+y) - af(x) - bf(y) - d) (f(x+y) - f(x) - f(y)) = 0.$$

Inspired by the work on the alternative Cauchy functional equation, Nakmahachalasint[5] has proved the analogous results on the alternative Jensen's functional equation of the form

$$f(x) \pm 2f(xy) + f(xy^2) = 0$$

on a semigroup. His work represents a significant extension of the work of NG [6] and Parnami and Vasudeva [7] on the classical Jensen's functional equation

$$f(x) - 2f(xy) + f(xy^2) = 0$$

on a group.

In this paper, given an integer $\lambda \neq 1$, we investigate the alternative Jensen's functional equation of the form

$$f(xy^{-1}) - 2f(x) + f(xy) = 0$$
 or $f(xy^{-1}) - 2f(x) + \lambda f(xy) = 0$, (AJ)

where f is a mapping from a group (G, \cdot) to a uniquely divisible abelian group (H, +). Note that when $\lambda = 1$, (AJ) is just the classical Jensen's functional equation. We will prove that for $\lambda \neq -3$, (AJ) is equivalent to the classical Jensen's functional equation in the sense that

$$f(xy^{-1}) - 2f(x) + f(xy) = 0, (J)$$

for all $x, y \in G$. Furthermore, if the domain G is a 2-divisible group, then we will show that (AJ) is equivalent to the classical Jensen's functional equation (J) for all $\lambda \neq 1$.

2 Auxiliary Lemmas

Let (G, \cdot) be a group and (H, +) be a uniquely divisible abelian group. Given an integer λ and a function $f: G \to H$. For every pair of $x, y \in G$, we define

$$F_y^{(\lambda)}(x):=f(xy^{-1})-2f(x)+\lambda f(xy).$$

Furthermore, for $\lambda \neq 1$, we let $\mathcal{P}f_y^{(\lambda)}(x)$ be the statement

$$\mathcal{P} f_y^{(\lambda)}(x) := \Big(F_y^{(1)}(x) = 0 \quad \text{or} \quad F_y^{(\lambda)}(x) = 0\Big).$$

Denote the set of all solutions of (AJ) by

$$\mathcal{A}_{(G,H)}^{(\lambda)} := \{ f : G \to H \mid \mathcal{P}f_y^{(\lambda)}(x) \text{ for all } x, y \in G \}.$$

Finally, the set of solution of the Jensen's functional equation will be denoted by

$$\mathcal{J}_{(G,H)} := \{ f : G \to H \mid F_y^{(1)}(x) = 0 \text{ for all } x, y \in G \}.$$

It should be note that $\mathcal{J}_{(G,H)} \subseteq \mathcal{A}_{(G,H)}^{(\lambda)}$

The following Lemma will be crucial and we make use of it extensively in the proofs. The reader should keep in mind of this fact.

Lemma 1. Let $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ and let $x, y \in G$. $F_y^{(1)}(x) = 0$ and $F_y^{(\lambda)}(x) = 0$ if and only if f(xy) = 0.

Proof. Assume that $F_y^{(1)}(x) = 0$ and $F_y^{(\lambda)}(x) = 0$. Therefore, $F_y^{(\lambda)}(x) - F_y^{(1)}(x) = 0$, i.e., $(\lambda - 1)f(xy) = 0$. Since $\lambda \neq 1$, we get f(xy) = 0. Conversely, assume that f(xy) = 0. Since $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$, we have $F_y^{(1)}(x) = 0$ or $F_y^{(\lambda)}(x) = 0$. As f(xy) = 0, therefore $F_y^{(1)}(x) = 0$ and $F_y^{(\lambda)}(x) = 0$.

We will prove some fundamental lemmas concerning the property of $\mathcal{P}f_y^{(\lambda)}(x)$.

Lemma 2. Let $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ and let $x, y \in G$. If $F_y^{(1)}(x) \neq 0$, then $f(xy^{-1}) = f(xy)$.

Proof. Assume that $F_y^{(1)}(x) \neq 0$. The alternatives in $\mathcal{P}f_y^{(\lambda)}(x)$ give $F_y^{(\lambda)}(x) = 0$. Similarly, from $F_{y^{-1}}^{(1)}(x) = F_y^{(1)}(x) \neq 0$ and $\mathcal{P}f_{y^{-1}}^{(\lambda)}(x)$, we get $F_{y^{-1}}^{(\lambda)}(x) = 0$. Observe that $F_y^{(\lambda)}(x) - F_{y^{-1}}^{(\lambda)}(x) = 0$, which simplifies to

$$(1 - \lambda)(f(xy^{-1}) - f(xy)) = 0.$$

Since $\lambda \neq 1$, we must have $f(xy^{-1}) = f(xy)$ as desired.

Lemma 3. Let $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ and let $x, y \in G$. If $F_y^{(1)}(x) \neq 0$ and $F_y^{(1)}(xy) \neq 0$, then $\lambda = -3$.

Proof. Assume that $F_y^{(1)}(x) \neq 0$ and $F_y^{(1)}(xy) \neq 0$. Hence $f(xy^2) \neq 0$ by Lemma 1. By Lemma 2, we get

$$f(xy^{-1}) = f(xy)$$
 and $f(x) = f(xy^2)$. (2.1)

From $F_y^{(1)}(x) \neq 0$ and $F_y^{(1)}(xy) \neq 0$, the alternatives in $\mathcal{P}f_y^{(\lambda)}(x)$ and $\mathcal{P}f_y^{(\lambda)}(xy)$ give $F_y^{(\lambda)}(x) = 0$ and $F_y^{(\lambda)}(xy) = 0$, respectively. Substituting f(x) from (2.1) into $F_y^{(\lambda)}(xy) = 0$, we have

$$(\lambda + 1)f(xy^2) - 2f(xy) = 0. (2.2)$$

Substituting $f(xy^{-1})$ and f(x) from (2.1) into $F_y^{(\lambda)}(x) = 0$, we obtain that

$$(\lambda + 1)f(xy) - 2f(xy^2) = 0. (2.3)$$

By (2.2) and (2.3), we get

$$(\lambda^2 + 2\lambda - 3)f(xy^2) = 0.$$

Since $\lambda \neq 1$ and $f(xy^2) \neq 0$, we must have $\lambda = -3$.

Lemma 4. Let $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ and let $x, y \in G$. If $F_y^{(1)}(x) \neq 0$, then $F_y^{(1)}(xy) \neq 0$.

Proof. Suppose $F_y^{(1)}(x) \neq 0$ but $F_y^{(1)}(xy) = 0$. Since H is a uniquely divisible abelian group, we let f(xy) = 2a. Assume that $F_y^{(1)}(x) \neq 0$. By Lemma 2, we get

$$f(xy^{-1}) = 2a. (2.4)$$

From $F_y^{(1)}(x) \neq 0$, the alternatives in $\mathcal{P}f_y^{(\lambda)}(x)$ gives $F_y^{(\lambda)}(x) = 0$. We also have $2a \neq 0$ by Lemma 1, that is, $a \neq 0$. Substituting (2.4) into $F_y^{(\lambda)}(x) = 0$, we obtain that

$$f(x) = (\lambda + 1)a. \tag{2.5}$$

Substituting (2.5) into $F_y^{(1)}(xy) = 0$, we have

$$f(xy^2) = (3 - \lambda)a. \tag{2.6}$$

Consider the following two possible cases in $\mathcal{P}f_y^{(\lambda)}(xy^{-1})$.

1. Assume that $F_y^{(1)}(xy^{-1}) \neq 0$. By Lemma 2, we get $f(xy^{-2}) = f(x)$. By (2.5), we have

$$f(xy^{-2}) = (\lambda + 1)a. (2.7)$$

Substituting (2.5), (2.6) and (2.7) into $\mathcal{P}f_{u^2}^{(\lambda)}(x)$, we obtain that

$$(2-2\lambda)a = 0$$
 or $(-1+2\lambda - \lambda^2)a = 0$.

Since $\lambda \neq 1$, we get a = 0, a contradiction.

2. Assume that $F_y^{(1)}(xy^{-1}) = 0$. Substituting (2.4) and (2.5) into $F_y^{(1)}(xy^{-1}) = 0$, we have

$$f(xy^{-2}) = (3 - \lambda)a. (2.8)$$

Substituting (2.5), (2.6) and (2.8), in $\mathcal{P}f_{y^2}^{(\lambda)}(x)$, we get

$$(4-4\lambda)a = 0$$
 or $(1-\lambda^2)a = 0$.

Since $\lambda \neq 1$ and $a \neq 0$, $\lambda = -1$. Thus (2.6) simplifies to

$$f(xy^2) = 4a. (2.9)$$

Substituting (2.9) into $\mathcal{P}f_y^{(-1)}(xy^2)$, we obtain that

$$f(xy^3) = \pm 6a. (2.10)$$

Substituting (2.4), (2.10) into $\mathcal{P}f_{y^2}^{(-1)}(xy)$ and simplifying, we get a=0, a contradiction.

Therefore, we must have $F_y^{(1)}(xy) \neq 0$ as desired.

Lemma 5. Let $f \in \mathcal{A}_{(G,H)}^{(\lambda)}$ and let $x, y \in G$. If $F_y^{(1)}(x) \neq 0$, then $F_y^{(1)}(xy^n) \neq 0$ for all $n \in \mathbb{Z}$.

Proof. By applying Lemma 4 repeatedly, we get

$$F_y^{(1)}(xy^n) \neq 0 \ \text{ for all } \ n \geq 1.$$

Similarly, by substituting y by y^{-1} in the previous arguments, we have

$$F_y^{(1)}(xy^n) \neq 0$$
 for all $n \leq -1$.

Consequently, we conclude that $F_y^{(1)}(xy^n) \neq 0$ for all $n \in \mathbb{Z}$.

3 Main Results

In this section, we prove our main results of this paper.

Theorem 6. If $\mathcal{A}^{(\lambda)}_{(G,H)} \setminus \mathcal{J}_{(G,H)} \neq \phi$, then $\lambda = -3$. Moreover, if $f \in \mathcal{A}^{(\lambda)}_{(G,H)} \setminus \mathcal{J}_{(G,H)}$ and $x, y \in G$, then $f(xy^n) = (-1)^n a$ for all $n \in \mathbb{Z}$ and for some $a \in H$. **Proof.** Assume that $\mathcal{A}_{(G,H)}^{(\lambda)} \setminus \mathcal{J}_{(G,H)} \neq \emptyset$. Let $f \in \mathcal{A}_{(G,H)}^{(\lambda)} \setminus \mathcal{J}_{(G,H)}$ and $x, y \in G$. Thus $F_y^{(1)}(x) \neq 0$. Lemma 5 gives $F_y^{(1)}(xy^n) \neq 0$ for all $n \in \mathbb{Z}$. By Lemma 2, we get

$$f(xy^{n+1}) = f(xy^{n-1}) \text{ for all } n \in \mathbb{Z}.$$
(3.1)

Since $F_y^{(1)}(x) \neq 0$ and $F_y^{(1)}(xy) \neq 0$, Lemma 3 gives $\lambda = -3$. From $F_y^{(1)}(xy^n) \neq 0$ for all $n \in \mathbb{Z}$, the alternatives $\mathcal{P}f_y^{(-3)}(xy^n)$ give $F_y^{(-3)}(xy^n) = 0$, i.e.,

$$f(xy^{n+1}) - 2f(xy^n) - 3f(xy^{n-1}) = 0. (3.2)$$

By (3.1) and (3.2), we have

$$f(xy^n) = -f(xy^{n-1}) \text{ for all } n \in \mathbb{Z}.$$
 (3.3)

Thus f(xy) = -f(x) when n = 1, and $f(xy^{-1}) = -f(x)$ when n = 0. For any $n \ge 2$, by (3.3), we obtain that

$$f(xy^n) = (-1)f(xy^{n-1}) = (-1)^2 f(xy^{n-2}) = \dots = (-1)^n f(x)$$

and

$$f(xy^{-n}) = (-1)f(xy^{-n+1}) = (-1)^2 f(xy^{-n+2}) = \dots = (-1)^n f(x).$$

Therefore,
$$f(xy^n) = (-1)^n f(x)$$
 for all $n \in \mathbb{Z}$ as desired.

Theorem 3.1 shows that the alternative Jensen's functional equation (AJ) is equivalent to the Jensen's functional equation (J) when $\lambda \neq -3$. However, when $\lambda = -3$, (AJ) is not necessarily equivalent to (J) as illustrated by the following example.

Example 7. Given $a \in H \setminus \{0\}$. Let $f : \mathbb{Z} \to H$ be a function such that

$$f(n) = (-1)^n a$$
 for all $n \in \mathbb{Z}$.

First, we will show that $f \in \mathcal{A}_{(\mathbb{Z},H)}^{(-3)}$. Given $n,m \in \mathbb{Z}$. If m is odd, then we see that n-m and n+m have the same parity whereas n and n+m have the opposite. Therefore, f(n-m)-2f(n)-3f(n+m)=0. Otherwise, if m is even, then n-m,n,n+m all have the same parity, i.e., f(n-m)-2f(n)+f(n+m)=0. Next, we will prove that $f \notin \mathcal{J}_{(\mathbb{Z},H)}$. Note that f(0)-2f(1)+f(2)=4a. From $a\neq 0$ and H is uniquely divisible, we get $4a\neq 0$. Thus $f\in \mathcal{A}_{(\mathbb{Z},H)}^{(-3)}\setminus\mathcal{J}_{(\mathbb{Z},H)}$.

In the case that G is a 2—divisible group, our main result is stronger in sense that (AJ) is actually equivalent to the classical Jensen's functional equation (J) as the following theorem.

Theorem 8. Let (G, \cdot) be a 2-divisible group. Then $\mathcal{A}_{(G,H)}^{(\lambda)} = \mathcal{J}_{(G,H)}$ for all $\lambda \neq 1$.

Proof. It is only left to prove that $\mathcal{A}_{(G,H)}^{(\lambda)} \setminus \mathcal{J}_{(G,H)} = \phi$. Assume contradictorily that $\mathcal{A}_{(G,H)}^{(\lambda)} \setminus \mathcal{J}_{(G,H)} \neq \phi$. Let $f \in \mathcal{A}_{(G,H)}^{(\lambda)} \setminus \mathcal{J}_{(G,H)}$ and $x,y \in G$. Since G is a 2-divisible group, there exists a $z \in G$ such that $y = z^2$. By setting y = z in Theorem 6, we obtain that $\lambda = -3$ and $f(xz^n) = (-1)^n a$ for all $n \in \mathbb{Z}$ and for some $a \in H$. We can calculate

$$f(xz^{-2}) - 2f(x) + f(xz^{2}) = (-1)^{-2}a - 2a + (-1)^{2}a = 0.$$

Since $y = z^2$, we conclude that $f(xy^{-1}) - 2f(x) + f(xy) = 0$ and therefore $f \in \mathcal{J}_{(G,H)}$. Hence, it is a contradiction to the fact that $f \notin \mathcal{J}_{(G,H)}$.

4 Concrete Examples

In this section, we give the general solution when the domain is a cyclic group. The following theorems show how Theorem 6 can be applied to certain cases.

Theorem 9. Let (G, \cdot) be an infinite cyclic group with $G = \langle g \rangle$.

1. If
$$\lambda \neq -3$$
, then $\mathcal{A}_{(G,H)}^{(\lambda)} = \mathcal{J}_{(G,H)}$.

2. If $\lambda = -3$, then $\mathcal{A}^{(-3)}_{(G,H)} \setminus \mathcal{J}_{(G,H)}$ is non-empty and $\mathcal{A}^{(-3)}_{(G,H)} \setminus \mathcal{J}_{(G,H)} = \{f : G \to H \mid f(g^n) = (-1)^n a \text{ for all } n \in \mathbb{Z} \text{ and for some } a \in H\}.$

Proof. If $\lambda \neq -3$, then Theorem 3.1 gives $\mathcal{A}_{(G,H)}^{(\lambda)} \setminus \mathcal{J}_{(G,H)} = \phi$, i.e., $\mathcal{A}_{(G,H)}^{(\lambda)} = \mathcal{J}_{(G,H)}$ and therefore (1). Next, we assume that $\lambda = -3$. First, we will show that $\mathcal{A}_{(G,H)}^{(-3)} \setminus \mathcal{J}_{(G,H)}$ is non-empty. Let $f: G \to H$ be

$$f(g^n) = (-1)^n a$$
 for all $n \in \mathbb{Z}$ and for some $a \in H \setminus \{0\}$.

Given $n, m \in \mathbb{Z}$. If m is odd, then we see that n-m and n+m have the same parity whereas n and n+m have the opposite. Therefore,

$$f(g^{n-m}) - 2f(g^n) - 3f(g^{n+m}) = 0.$$

Otherwise, if m is even, then n-m, n, n+m all have the same parity, i.e.,

$$f(q^{n-m}) - 2f(q^n) + f(q^{n+m}) = 0.$$

Note that $f(e) - 2f(g) + f(g^2) = 4a$. From $a \neq 0$ and H is uniquely divisible, we get $4a \neq 0$. Thus $f \in \mathcal{A}^{(-3)}_{(G,H)} \setminus \mathcal{J}_{(G,H)}$. Since $\mathcal{A}^{(-3)}_{(G,H)} \setminus \mathcal{J}_{(G,H)}$ is non-empty, the proof is complete by setting x = e and y = g in Theorem 6.

Theorem 10. Let (G,\cdot) be a cyclic group of finite order $m \geq 2$ with $G = \langle g \rangle$.

- 1. If $\lambda \neq -3$, then $\mathcal{A}_{(G,H)}^{(\lambda)} = \mathcal{J}_{(G,H)}$.
- 2. If $\lambda = -3$, then
 - (a) if m is odd, then $\mathcal{A}_{(G,H)}^{(-3)} = \mathcal{J}_{(G,H)}$, or
 - (b) if m is even, then $\mathcal{A}^{(-3)}_{(G,H)} \setminus \mathcal{J}_{(G,H)}$ is non-empty and $\mathcal{A}^{(-3)}_{(G,H)} \setminus \mathcal{J}_{(G,H)} = \{f: G \to H \mid f(g^n) = (-1)^n a \text{ for all } n \in \mathbb{Z} \text{ and for some } a \in H\}.$

Proof. If $\lambda \neq -3$, then Theorem 3.1 gives $\mathcal{A}_{(G,H)}^{(\lambda)} \setminus \mathcal{J}_{(G,H)} = \phi$, i.e., $\mathcal{A}_{(G,H)}^{(\lambda)} = \mathcal{J}_{(G,H)}$ and therefore (1). Next, we assume that $\lambda = -3$. We will consider two possible cases of m as follows.

(a) Assume that m is odd. We will show that $\mathcal{A}^{(-3)}_{(G,H)} \setminus \mathcal{J}_{(G,H)} = \phi$. Suppose $\mathcal{A}^{(-3)}_{(G,H)} \setminus \mathcal{J}_{(G,H)} \neq \phi$. Let $f \in \mathcal{A}^{(-3)}_{(G,H)} \setminus \mathcal{J}_{(G,H)}$. By setting x = e and y = g in Theorem 6, we obtain that $f(g^n) = (-1)^n a$ for all $n \in \mathbb{Z}$ and for some $a \in H$. Hence f(e) = a. Since $e = g^m$, we get

$$a = f(e) = f(g^m) = (-1)^m a = -a.$$

We conclude that a=0 and therefore $f \in \mathcal{J}_{(G,H)}$, a contradiction. Thus we must have $\mathcal{A}_{(G,H)}^{(-3)} \setminus \mathcal{J}_{(G,H)} = \phi$.

(b) Assume that m is even. Let $f: G \to H$ be

$$f(g^n) = (-1)^n a$$
 for all $n \in \mathbb{Z}$ and for some $a \in H \setminus \{0\}$.

Given $n, m \in \mathbb{Z}$. If m is odd, then we see that n-m and n+m have the same parity whereas n and n+m have the opposite. Therefore,

$$f(g^{n-m}) - 2f(g^n) - 3f(g^{n+m}) = 0.$$

Otherwise, if m is even, then n - m, n, n + m all have the same parity, i.e.,

$$f(g^{n-m}) - 2f(g^n) + f(g^{n+m}) = 0.$$

Note that $f(e) - 2f(g) + f(g^2) = 4a$. From $a \neq 0$ and H is uniquely divisible, we get $4a \neq 0$. Thus $f \in \mathcal{A}_{(G,H)}^{(-3)} \setminus \mathcal{J}_{(G,H)}$, i.e., $\mathcal{A}_{(G,H)}^{(-3)} \setminus \mathcal{J}_{(G,H)} \neq \phi$. Hence the proof is then complete by setting x = e and y = g in Theorem 6.

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