East-West J. of Mathematics: Vol. 17, No 2 (2015) pp. 151-165

UPPER SEMI-CONTINUITY FOR SOLUTION MAPPINGS OF WEAK QUASI-EQUILIBRIUM PROBLEMS

Truong Thi Thuy Duong

Banking Academy, Ha Noi, Vietnam
email: thuyduongktv@yahoo.com.vn

Abstract

In this paper, we consider the weak quasi-equilibrium problems and show some sufficient conditions on the existence of their solutions and the upper semicontinuity of solution mappings.

1 Introduction

In recent years, there are many authors studying the generalized quasi - equilibrium problem. They often care about the existence of solutions and the stability of the solution mapping of problems. Namely, in [3], [9], [12] considered the existence of solutions of version of generalized quasi-equilibrium problems. In [2], [11], [13], [15] have obtained the lower and upper semicontinuity of the solution mappings in the some versions of parametric Ky Fan inequality.

Throughout this paper, unless otherwise specify, X, Y, T, Z are supposed to be locally convex Hausdorff topological vector spaces. Assume that $D \subset X, K \subset Z$ are nonempty subsets. Given multivalued mappings $A : D \times K \to 2^D, B : D \times K \to 2^K, F : T \times D \times K \times K \to 2^Y$ and $C : T \times D \times K \to 2^Y$ is a cone multivalued mapping with convex nonempty cone values. For any $t \in T$, we are interested in the following problems:

(P_1) Find $(\bar{y}, \bar{x}) \in D \times K$ such that

$$\bar{x} \in A(\bar{y}, \bar{x}); \bar{y} \in B(\bar{y}, \bar{x});$$
$$F(t, \bar{y}, \bar{x}, z) \not\subseteq \text{int}C(t, \bar{y}, \bar{x}), \text{ for all } z \in A(\bar{y}, \bar{x}).$$

Key words: Quasi-equilibrium problems, upper and lower C- continuous multivalued mappings, upper semicontinuity.
(2010) Mathematics Subject Classification: 49J27, 49J53, 91B50, 90C48.

(P_2) Find $(\bar{y},\bar{x}) \in D \times K$ such that

$$\bar{x} \in A(\bar{y},\bar{x}); \bar{y} \in B(\bar{y},\bar{x});$$
$$F(t,\bar{y},\bar{x},z) \cap \mathrm{int}C(t,\bar{y},\bar{x}) = \emptyset, \text{ for all } z \in A(\bar{y},\bar{x}).$$

(P_3) Find $(\bar{y},\bar{x}) \in D \times K$ such that

$$\bar{x} \in A(\bar{y},\bar{x}); \bar{y} \in B(\bar{y},\bar{x});$$
$$\exists z \in A(\bar{y},\bar{x}) \text{ such that } F(t,\bar{y},\bar{x},z) \not\subseteq \mathrm{int}C(t,\bar{y},\bar{x}).$$

(P_4) Find $(\bar{y},\bar{x}) \in D \times K$ such that

$$\bar{x} \in A(\bar{y},\bar{x}); \bar{y} \in B(\bar{y},\bar{x});$$
$$\exists z \in A(\bar{y},\bar{x}) \text{ such that } F(t,\bar{y},\bar{x},z) \cap \mathrm{int}C(t,\bar{y},\bar{x}) = \emptyset.$$

The aim of this paper is considering the existence of solution and the upper semicontinuity of solution mappings of problems $(P_i), i = 1,2,3,4$.

2 Preliminaries

Let us recall that the domain and the graph of a multivalued mapping $G : D \longrightarrow 2^Y$ are defined by

$$\mathrm{dom}G = \{x \in D |\ G(x) \neq \emptyset\},$$
$$\mathrm{Gr}(G) = \{(x,y) \in D \times Y |\ y \in G(x)\},$$

respectively. The mapping G is said to be closed (resp. open) if the graph $\mathrm{Gr}(G)$ is a closed (resp. open) subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $clG(D)$ of its range $G(D)$ is a compact set in Y. It is said to be upper (lower) semicontinuous (briefly, usc (respectively, lsc)) at $\bar{x} \in D$ if for each open set V containing $G(\bar{x})$ (respectively, $G(\bar{x}) \cap V \neq \emptyset$), there exists an open set U of $\bar{x}$ that $G(x) \subseteq V$ (respectively, $G(x) \cap V \neq \emptyset$) for each $x \in U$ and G is said to be usc (lsc) on D if it is usc (respectively, lsc) at every point $x \in D$. We say that the mapping G has open lower sections if the set $G^{-1}(y) = \{x \in D \mid y \in G(x)\}$ is open.

The definition following is extracted in [13].

Definition 2.1. Let $G : D \longrightarrow 2^Y$ be multivalued mapping and $C : D \longrightarrow 2^Y$ be a cone multivalued mapping. G is called C- lower semicontinuous (shortly, C-lsc) at $\bar{x} \in D$ if there exists a compact set $W(\bar{x}) \subset C(\bar{x})$ such that, for any open set $\mathcal{N}$ with $G(\bar{x}) \cap \mathcal{N} \neq \emptyset$, we can find a neighborhood $U(\bar{x})$ of $\bar{x}$ such that $G(x) \cap (\mathcal{N} - W(\bar{x})) \neq \emptyset$, for all $x \in U(\bar{x})$. This set $W(\bar{x})$ is called the set associated to the C- lower semicontinuous of G at $\bar{x}$. G is called C-upper semicontinuous (shortly, C-usc) at $\bar{x} \in D$ if there exists a compact set

$W(\bar{x}) \subset C(\bar{x})$ such that, for any open set $\mathcal{N}$ with $G(\bar{x}) \subset \mathcal{N}$, we can find a neighborhood $U(\bar{x})$ of $\bar{x}$ such that $G(x) \cap (W(\bar{x}) + \mathcal{N}) \neq \emptyset$, for all $x \in U(\bar{x})$. This set $W(\bar{x})$ is called the set associated to the C- upper semicontinuous of G at $\bar{x}$.

Definition 2.2. a. Let $F : D \times K \times K \rightarrow 2^Y$ be multivalued mapping, $C : D \times K \rightarrow 2^Y$ be cone multivalued mapping. We say that F is called diagonally upper (lower) C-quasiconvex-like in the third variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x \in co\{x_1, ..., x_n\}$, $x = \sum_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0, \sum_{j=1}^{n} \alpha_j = 1$, there is an index $j \in \{1, ..., n\}$ it holds

$$F(y, x, x_j) \subseteq F(y, x, x) + C(y, x)$$

$$(\text{respectively, } F(y, x, x) \subseteq \ F(y, x, x_j) - C(y, x)).$$

b. Let $G : D \rightarrow 2^Y$ be multivalued mapping. We say that G is upper (lower) C-quasiconvex-like on D if for any $x_1, x_2 \in D, t \in [0, 1]$, either

$$\begin{aligned} &G(x_1) \subseteq G(tx_1 + (1-t)x_2) + C \\ \text{or,}\quad &G(x_2) \subseteq G(tx_1 + (1-t)x_2) + C \end{aligned}$$

$$\begin{aligned} &(\text{respectively, either } G(tx_1 + (1-t)x_2) \subseteq G(x_1) - C \\ \text{or,}\quad &G(tx_1 + (1-t)x_2) \subseteq G(x_2) - C \end{aligned}$$

holds.

In the proof of the main results in Section 3, we need the following theorems.

Theorem 2.3. ([7]) *Let X, Y be Hausdorff topological spaces, $F : X \rightarrow 2^Y$ be a multivalued map.*

(i) If F is an usc with closed values, then F is closed.

(ii) If Y is a compact space and F is closed, then F is usc.

Theorem 2.4. ([10]). *Let X be a locally convex Hausdorff topological vector space, D be a nonempty convex compact subset of X and $F : D \rightarrow 2^D$ be a usc multivalued mapping with nonempty convex closed values. Then there exists $\bar{x} \in D$ such that $\bar{x} \in F(\bar{x})$.*

Next, we recall that a multivalued mapping $H : D \rightarrow 2^X$ is said to be KKM (see, for example, in [6]), if for any finite set $\{t_1, ..., t_n\} \subset D$, it implies that $co\{t_1, ..., t_n\} \subseteq \bigcup_{j=1}^{n} H(t_j)$.

Theorem 2.5. ([4]) *Let D be a nonempty convex subset of a Hausdorff topological vector space X and let $f : D \rightarrow 2^X$ be KKM-map. For each $x \in D$, if $f(x)$ is closed and for at least one $x' \in D, f(x')$ is compact, then $\bigcap_{x \in D} f(x) \neq \emptyset$.*

Lemma 2.6. ([7]) *Let E_1, E_2 and Z be real Hausdorff topological vector spaces, X and Y be nonempty subsets of E_1 and E_2, respectively. If $F : X \times Y \to 2^Z$ is a closed multivalued mapping and $S : X \to 2^Y$ is an usc multivalued mapping with compact values, then $T : X \to 2^Z$ defined by $T(x) = \cup_{y \in S(x)} F(x, y) = F(x, S(x))$ is a closed multivalued mapping.*

Proposition 2.7. ([14]) *Let $x_0 \in X$ and let $F : X \to 2^Y$ and $F(x_0)$ be compact set. Then, F is C-lsc (resp. C-usc) at x_0 if only if there exists a compact set $S(x_0) \subset C(x_0)$ such that for any neighborhood V of the origin in Y, there exists a neighborhood $U(x_0)$ of x_0 such that*

$$F(x_0) \subset F(x) + V + S(x_0), \forall x \in U(x_0)$$

$$(resp. F(x) \subset F(x_0) + V + S(x_0), \forall x \in U(x_0)).$$

3 The upper semicontinuous of solution mappings

Let D, K be convex compact sets. Given multivalued mappings A, B, F and C with nonempty values as in Introduction. Denote

$$H = \{(y, x) \in D \times K : x \in A(y, x), y \in B(y, x)\}.$$

We defined the multivalued mappings $S_i : T \to 2^{D \times K}, i = 1, 2, 3, 4$ by

$$S_1(t) = \{(y, x) \in H \mid F(t, y, x, z) \not\subseteq \text{int}C(t, y, x), \text{ for all } z \in A(y, x)\}.$$

$$S_2(t) = \{(y, x) \in H \mid F(t, y, x, z) \cap \text{int}C(t, y, x) = \emptyset, \text{ for all } z \in A(y, x)\}.$$

$$S_3(t) = \{(y, x) \in H \mid \exists z \in A(y, x), F(t, y, x, z) \not\subseteq \text{int}C(t, y, x)\}.$$

$$S_4(t) = \{(y, x) \in H \mid \exists z \in A(y, x), F(t, y, x, z) \cap \text{int}C(t, y, x) = \emptyset\}.$$

We will establish the sufficient conditions for the upper semicontinuity of $(S_i), i = 1, 2, 3, 4$. In the Theorems following, we assume that the sets $S_i(t), i = 1, 2, 3, 4$ are nonempty for all $t \in T, \text{int}C(t, y, x) \neq \emptyset$ for all $y \in D, x \in K, t \in T$.

Theorem 3.1. *Let A be continuous multivalued mapping with convex closed values, B be upper semicontinuous with convex closed values. For each $t_0 \in T$, assume the following conditions hold:*
(i) int$C$ is open on $\{t_0\} \times D \times K$;
(ii) F is a closed map on $\{t_0\} \times D \times K \times K$.
Then S_1 is usc at t_0.

Proof. Since $D \times K$ is a compact set, to proof the upper semicontinuity of S_1 we will show that S_1 is closed.

Assume that $x_\beta \to x_0, y_\beta \to y_0, t_\beta \to t_0, (y_\beta, x_\beta) \in S_1(t_\beta)$. Then $(y_\beta, x_\beta) \in H$ and $F(t_\beta, y_\beta, x_\beta, z) \nsubseteq \text{int}C(t_\beta, y_\beta, x_\beta),$ for all $z \in A(y_\beta, x_\beta)$.

For the upper semicontinuity of A, B with nonempty closed values in compact set imply the closed property of A and B. This means $y_0 \in B(y_0, x_0), x_0 \in A(y_0, x_0)$ and so $(y_0, x_0) \in H$.

The lower semicontinuity of A and $x_\beta \to x_0, y_\beta \to y_0$ follow that for any $z_0 \in A(y_0, x_0)$, there exists $z_\beta \in A(y_\beta, x_\beta)$ such that $z_\beta \to z_0$. Thus, $F(t_\beta, y_\beta, x_\beta, z_\beta) \nsubseteq \text{int}C(t_\beta, y_\beta, x_\beta)$. So $F(t_\beta, y_\beta, x_\beta, z_\beta) \cap (Y \setminus \text{int}C(t_\beta, y_\beta, x_\beta)) \neq \emptyset$. The openness of int$C$ implies that $Y \setminus \text{int}C$ is a closed map. Since F and $Y \setminus \text{int}C$ are closed, we have $F(t_0, y_0, x_0, z_0) \cap (Y \setminus \text{int}C(t_0, y_0, x_0)) \neq \emptyset$. This reduces $F(t_0, y_0, x_0, z_0) \nsubseteq \text{int}C(t_0, y_0, x_0)$ for any $z_0 \in A(y_0, x_0)$. Thus, $(y_0, x_0) \in S_1(t_0)$. This shows that S_1 is closed at t_0. The proof is complete.

Theorem 3.2. *Let A be continuous multivalued mapping with convex closed values, B be upper semicontinuous with convex closed values. For each $t_0 \in T$, assume the following conditions hold:*

(i) intC is open on $\{t_0\} \times D \times K$;

(ii) F is $(-C)$-lsc with compact values on $\{t_0\} \times D \times K \times K$.

Then S_2 is usc at t_0.

Proof. Since $D \times K$ is a compact set, to proof the upper semicontinuity of S_2 we will show that S_2 is closed.

Setting

$$\mathcal{S}_2(t) = \{(y, x) \in D \times K \mid F(t, y, x, z) \cap \text{int}C(t, y, x) = \emptyset, \text{ for all } z \in A(y, x)\}.$$

$$\mathcal{S}'_2(t) = D \times K \setminus \mathcal{S}_2(t).$$

Taking arbitrary $(y_0, x_0) \in \mathcal{S}'_2(t_0)$, there exists $z_0 \in A(y_0, x_0)$ such that $F(t_0, y_0, x_0, z_0) \cap \text{int}C(t_0, y_0, x_0) \neq \emptyset$. Then there exists a point f_0 such that $f_0 \in F(t_0, y_0, x_0, z_0), f_0 \in \text{int}C(t_0, y_0, x_0)$.

Let $W_0(y_0, x_0)$ be the compact set associated to the definition of the $(-C)$-lower semicontinuity property of F, by $C(t_0, y_0, x_0)$ is convex cone and $W_0(y_0, x_0) \subseteq -C(t_0, y_0, x_0)$, therefore $f_0 - W_0(y_0, x_0) \subseteq \text{int}C(t_0, y_0, x_0)$. Since int$C$ is open, there exist a balanced neighborhood V of the origin in Y and neighborhoods $U_1(y_0)$ of $y_0, U_1(x_0)$ of $x_0, U_1(t_0)$ of t_0 such that

$$f_0 - W_0(y_0, x_0) + V \subset \text{int}C(t, y, x) \text{ for all } (t, y, x) \in U_1(t_0) \times U_1(y_0) \times U_1(x_0). \tag{1}$$

Since F is $(-C)$-lsc with compact values, there exist neighborhoods $U_2(y_0) \subset U_1(y_0), U_2(x_0) \subset U_1(x_0), U(t_0) \subset U_1(t_0)$ and $U(z_0)$ such that

$$F(t_0, y_0, x_0, z_0) \subset F(t, y, x, z) + W_0(y_0, x_0) + V \tag{2}$$

for all $(t, y, x, z) \in U(t_0) \times U_2(y_0) \times U_2(x_0) \times U(z_0)$.

Observe now that $U(z_0)$ is open and $A(y_0,x_0)\cap U(z_0)\neq\emptyset$, on the other hand A is lsc, then there are neighborhoods $U(y_0)\subset U_2(y_0), U(x_0)\subset U_2(x_0)$ such that

$$A(y,x)\cap U(z_0)\neq\emptyset \text{ for all } (y,x)\in U(y_0)\times U(x_0).$$

Therefore, there exists z such that $z\in A(y,x)$ and $z\in U(z_0)$.

Since $(t,y,x,z)\in U(t_0)\times U(y_0)\times U(x_0)\times U(z_0)$, and since (1),(2) hold, there is a point $f\in F(t,y,x,z)$ satisfies $f_0\in f+W_0(y_0,x_0)+V$. This implies $f\in f_0-W_0(y_0,x_0)+V$. So $f\in \text{int}C(t,y,x)$. This implies that

$$F(t,y,x,z)\cap \text{int}C(t,y,x)\neq\emptyset$$

for all $(t,y,x,z)\in U=U(t_0)\times U(y_0)\times U(x_0)\times U(z_0)$. Hence, $U\subset \text{Gr}\mathcal{S}'_2$, and then $\mathcal{S}'_2$ is open multivalued mapping. This shows that $\mathcal{S}_2$ is closed multivalued mapping.

We see that $S_2(t)=\mathcal{S}_2(t)\cap H$. The upper semicontinuity with closed values in compact set of A and B imply that H is a closed set. For $\mathcal{S}_2$ is closed maps and H is a closed set, it is easy denotation S_2 is a closed map. The proof is complete.

Similar proof of above Theorems, we will show the upper semicontinuity of S_3, S_4 with weaker hypothesis of A.

Theorem 3.3. *Let A be usc with convex compact values, B be usc with convex closed values. For each $t_0\in T$, assume the following conditions hold:*

(i) int$C$ is open on $\{t_0\}\times D\times K$;

(ii) F is a closed map on $\{t_0\}\times D\times K\times K$.

Then S_3 is usc at t_0.

Proof. Since $D\times K$ is a compact set, to proof the upper semicontinuity of S_3 we will show that S_3 is closed.

Assume that $x_\beta\to x_0, y_\beta\to y_0, t_\beta\to t_0, (y_\beta,x_\beta)\in S_3(t_\beta)$. Then $(y_\beta,x_\beta)\in H$ and there is $z\in A(y_\beta,x_\beta)$ such that $F(t_\beta,y_\beta,x_\beta,z)\not\subseteq \text{int}C(t_\beta,y_\beta,x_\beta)$. So $F(t_\beta,y_\beta,x_\beta,z)\cap(Y\setminus \text{int}C(t_\beta,y_\beta,x_\beta))\neq\emptyset$ with $z\in A(y_\beta,x_\beta)$. Therefore

$$F(t_\beta,y_\beta,x_\beta,A(y_\beta,x_\beta))\cap(Y\setminus \text{int}C(t_\beta,y_\beta,x_\beta))\neq\emptyset. \qquad (3)$$

For A is usc with nonempty compact values and B is usc with closed values in compact set imply the closed property of A and B. This means $y_0\in B(y_0,x_0), x_0\in A(y_0,x_0)$ and so $(y_0,x_0)\in H$.

Since A is usc with compact values and F is closed, the multivalued mapping G defined by $G(t,y,x)=\cup_{z\in A(y,x)}F(t,y,x,z)=F(t,y,x,A(y,x))$, for all $(t,y,x)\in T\times D\times K$ is closed.

The openness of int$C$ implies that $Y \setminus \text{int}C$ is a closed map. Combination (3) with the closedness of G and $Y \setminus \text{int}C$, we have $F(t_0, y_0, x_0, A(y_0, x_0)) \cap (Y \setminus \text{int}C(t_0, y_0, x_0)) \neq \emptyset$. This reduces

$$F(t_0, y_0, x_0, A(y_0, x_0)) \not\subseteq \text{ int}C(t_0, y_0, x_0).$$

Thus, there is $z \in A(y_0, x_0)$ such that $F(t_0, y_0, x_0, z) \not\subseteq \text{int}C(t_0, y_0, x_0)$. This implies $(y_0, x_0) \in S_3(t_0)$. This shows that S_3 is closed at t_0. The proof is complete.

Theorem 3.4. *Let A be usc with convex compact values, B be usc with convex closed values. For each $t_0 \in T$, assume the following conditions hold:*
(i) int$C$ is open on $\{t_0\} \times D \times K$;
(ii) F is $(-C)$-lsc with compact values on $\{t_0\} \times D \times K \times K$.
Then S_4 is usc at t_0.

Proof. Since $D \times K$ is a compact set, to proof the upper semicontinuity of S_4 we will show that S_4 is closed.

Setting

$$\mathcal{S}_4(t) = \{(y, x) \in D \times K \mid \exists z \in A(y, x), F(t, y, x, z) \cap \text{int}C(t, y, x) = \emptyset\}.$$

$$\mathcal{S'}_4(t) = D \times K \setminus \mathcal{S}_4(t).$$

Taking arbitrary $(y_0, x_0) \in \mathcal{S'}_4(t_0)$, then

$$F(t_0, y_0, x_0, z_0) \cap \text{int}C(t_0, y_0, x_0) \neq \emptyset \text{ for all } z_0 \in A(y_0, x_0).$$

This shows that there exists a point f_0 such that $f_0 \in F(t_0, y_0, x_0, A(y_0, x_0)), f_0 \in \text{int}C(t_0, y_0, x_0)$.

Let $W_0(y_0, x_0)$ be the compact set associated to the definition of the $(-C)$-lower semicontinuity property of F, by $C(t_0, y_0, x_0)$ is convex cone and $W_0(y_0, x_0) \subseteq -C(t_0, y_0, x_0)$, therefore $f_0 - W_0(y_0, x_0) \subset \text{int}C(t_0, y_0, x_0)$. Since int$C$ is open, there exist a balanced neighborhood V of the origin in Y and neighborhoods $U_1(y_0)$ of $y_0, U_1(x_0)$ of $x_0, U_1(t_0)$ of t_0 such that

$$f_0 - W_0(y_0, x_0) + V \subset \text{int}C(t, y, x), \ \forall (t, y, x) \in U_1(t_0) \times U_1(y_0) \times U_1(x_0). \quad (4)$$

Since F is $(-C)$-lsc with compact values, there exist neighborhoods $U_2(y_0) \subset U_1(y_0), U_2(x_0) \subset U_1(x_0), U(t_0) \subset U_1(t_0)$ and $U_A(y_0, x_0)$ such that

$$F(t_0, y_0, x_0, A(y_0, x_0)) \subset F(t, y, x, z) + W_0(y_0, x_0) + V \quad (5)$$

for all $(t, y, x, z) \in U(t_0) \times U_2(y_0) \times U_2(x_0) \times U_A(y_0, x_0)$. For $A(y_0, x_0)$ is a compact set, we can cover it by a finite number n of neighborhood $U(z_i), z_i \in A(y_0, x_0)$. Therefore,

$$U = (\cup_{i=1}^n U(z_i)) \cap U_A(y_0, x_0) \text{ is a neighborhood of } A(y_0, x_0).$$

On the other hand, A is usc, then there are neighborhoods $U(y_0) \subset U_2(y_0)$, $U(x_0) \subset U_2(x_0)$ such that

$$A(y,x) \subset U \text{ for all } (y,x) \in U(y_0) \times U(x_0).$$

Therefore, for all $z \in A(y,x)$ and $z \in U$, and (5) becomes

$$F(t_0, y_0, x_0, A(y_0, x_0)) \subset F(t, y, x, A(y,x)) + W_0(y_0, x_0) + V \tag{6}$$

for all $(t, y, x, A(y,x)) \in U(t_0) \times U(y_0) \times U(x_0) \times U$.

Since $(t, y, x, A(y,x)) \in U(t_0) \times U(y_0) \times U(x_0) \times U$, and since (4),(6) hold, there is a point $f \in F(t, y, x, A(y,x))$ and $f_0 \in f + W_0(y_0, x_0) + V$, this implies $f \in f_0 - W_0(y_0, x_0) + V$. So $f \in \text{int}C(t, y, x)$. This implies that $F(t, y, x, A(y,x)) \cap \text{ int}C(t, y, x) \neq \emptyset$ for all $(t, y, x, A(y,x)) \in \mathcal{U} = U(t_0) \times U(y_0) \times U(x_0) \times U$. This means $F(t, y, x, z) \cap \text{int}C(t, y, x) \neq \emptyset$ for all $z \in A(y,x), (t, y, x) \in U(t_0) \times U(y_0) \times U(x_0)$. Hence, $U(t_0) \times U(y_0) \times U(x_0) \subset \text{Gr}\mathcal{S}'_4$, and then $\mathcal{S}'_4$ is open multivalued mapping. This shows that $\mathcal{S}_4$ is a closed multivalued mapping.

We see that $S_4(t) = \mathcal{S}_4(t) \cap H$. The upper semicontinuity with closed values in compact set of A and B imply that H is a closed set. For $\mathcal{S}_4$ is closed maps and H is a closed set, it is easy denotation S_4 is a closed map. The proof is complete.

4 Existence of solutions

In this section, we prove the existence solution of $(P_i), i = 1, 2, 3, 4$. Let the sets D, K be convex compact, the multivalued mappings A, B, F, C be defined as the same as in Section 3.

Theorem 4.1. *Let A be a multivalued mapping with convex closed values, B be upper semicontinuous with convex closed values. For each $t \in T$, assume the following conditions hold:*

(i) A has open lower sections, H is a closed set;

(ii) int$C$ is a open map;

(iii) F is a closed map, F is a diagonally lower $(-C)$-quasiconvex-like in the fourth variable and $F(t, y, x, x) \nsubseteq$ int$C(t, y, x)$.

Then for each $t \in T$, there exists $(\bar{y}, \bar{x}) \in D \times K$ such that

$$\bar{x} \in A(\bar{y}, \bar{x}); \bar{y} \in B(\bar{y}, \bar{x});$$
$$F(t, \bar{y}, \bar{x}, z) \nsubseteq \textit{int}C(t, \bar{y}, \bar{x}), \textit{ for all } z \in A(\bar{y}, \bar{x}).$$

Proof. We define the multivalued mapping $M : D \times K \to 2^K$ by

$$M(y,x) = \{z \in K \mid F(t, y, x, z) \subseteq \text{int}C(t, y, x)\}$$

Assume on the contrary that for all $(y,x) \in H$ such that $F(t,y,x,z) \subseteq \text{int}C(t,y,x)$, for some $z \in A(y,x)$. This means for all $(y,x) \in H$ then $M(y,x) \cap A(y,x) \neq \emptyset$. Hence, the multivalued mapping $Q : D \times K \to 2^K$ by

$$Q(y,x) = \begin{cases} co(M(y,x)) \cap A(y,x), & \text{if } (y,x) \in H; \\ A(y,x), & \text{if } (y,x) \in (D \times K) \setminus H \end{cases}$$

has nonempty values.

Setting $E = \{(y,x) \in D \times K \mid F(t,y,x,z) \nsubseteq \text{ int}C(t,y,x)\}$. We will show that E is a closed set. Indeed, assume $x_\beta \to x, y_\beta \to y, (y_\beta, x_\beta) \in E$. This means $F(t,y_\beta,x_\beta,z) \nsubseteq \text{int}C(t,y_\beta,x_\beta)$. Then

$$F(t,y_\beta,x_\beta,z) \cap (Y \setminus \text{int}C(t,y_\beta,x_\beta)) \neq \emptyset. \tag{7}$$

The openness of int$C$ implies that $Y \setminus \text{int}C$ is a closed map. Since (7) and $F, Y \setminus \text{int}C$ are closed, we have

$$F(t,y,x,z) \cap (Y \setminus -intC(t,y,x)) \neq \emptyset.$$

Thus, $(y,x) \in E$ so E is a closed set. This follows that $M^{-1}(z)$ is a open set.

$$Q^{-1}(z) = (coM)^{-1}(z) \cap A^{-1}(z) \cup (A^{-1}(z) \cap (D \times K \setminus H)).$$

We reduce that Q has nonempty convex values and open lower sections, $D \times K$ is a compact in Hausdorff topological space, we claim from Theorem 8.1.3 of [5] that there exists a continous single-valued map $\phi : D \times K \to K$ such that $\phi(y,x) \in Q(y,x)$.

We see that, the set-valued map $\psi : D \times K \to 2^{D \times K}$ defined by

$$\psi(y,x) = B(y,x) \times \{\phi(y,x)\}$$

is compact usc with nonempty convex closed in $D \times K$, then it has fixed point. This means, there exists $(\bar{y},\bar{x}) \in D \times K$ such that $\bar{x} \in \phi(\bar{y},\bar{x}), \bar{y} \in B(\bar{y},\bar{x})$.

Obviously, $(\bar{y},\bar{x}) \in H$ and $\bar{x} \in Q(\bar{y},\bar{x})$. By the definition of Q, $\bar{x} \in coM(\bar{y},\bar{x})$ and $\bar{x} \in A(\bar{x},\bar{y})$.

Since $\bar{x} \in coM(\bar{y},\bar{x})$, there exists a finite subset $\{x_1,...,x_n\}$ of $M(\bar{y},\bar{x})$ and $\bar{x} = \sum\limits_{i=1}^{n} \alpha_i x_i, \alpha_i \geq 0, \sum\limits_{i=1}^{n} \alpha_i = 1$. Thus

$$F(t,\bar{y},\bar{x},x_i) \subseteq \text{int}C(t,\bar{y},\bar{x}) \text{ for all } i = 1,2,...,n.$$

For F is diagonally lower $(-C)$-quasiconvex-like in the fourth variable, we conclude that there is an index $j \in \{1,...,n\}$ such that

$$F(t,\bar{y},\bar{x},\bar{x}) \subset F(t,\bar{y},\bar{x},x_j) + C(t,\bar{y},\bar{x}) \subseteq \text{int}C(t,\bar{y},\bar{x}).$$

This contrary to (iii). Then for each $t \in T$, there exists $(\bar{y}, \bar{x}) \in D \times K$ such that

$$\bar{x} \in A(\bar{y}, \bar{x}); \bar{y} \in B(\bar{y}, \bar{x});$$
$$F(t, \bar{y}, \bar{x}, z) \not\subseteq \text{int}C(t, \bar{y}, \bar{x}), \text{ for all } z \in A(\bar{y}, \bar{x}).$$

The proof is complete.

Theorem 4.2. *Let A be a multivalued mapping with convex closed values, B be upper semicontinuous with convex closed values. For each $t \in T$, assume the following conditions hold:*

(i) A has open lower sections, H is a closed set;

(ii) int$C$ is a open map;

(iii) For any $z \in K, F(t, ., ., z)$ is $(-C)$-lsc, F is a diagonally upper $(-C)$-quasiconvex-like in the fourth variable and $F(t, y, x, x) \cap intC(t, y, x) = \emptyset$ for all $(y, x) \in D \times K$.

Then for each $t \in T$, there exists $(\bar{y}, \bar{x}) \in D \times K$ such that

$$\bar{x} \in A(\bar{y}, \bar{x}); \bar{y} \in B(\bar{y}, \bar{x});$$
$$F(t, \bar{y}, \bar{x}, z) \cap intC(t, \bar{y}, \bar{x}) = \emptyset, \text{ for all } z \in A(\bar{y}, \bar{x}).$$

Proof. We define the multivalued mapping $N : D \times K \to 2^K$ by

$$N(y, x) = \{z \in K |\ F(t, y, x, z) \cap \text{ int}C(x) \neq \emptyset\}$$

Assume on the contrarily that for all $(y, x) \in H$ such that $F(t, y, x, z) \cap \text{int}C(t, y, x) \neq \emptyset$, for some $z \in A(y, x)$. This means for all $(y, x) \in H$ then $N(y, x) \cap A(y, x) \neq \emptyset$. Hence, the multivalued mapping $Q' : D \times K \to 2^K$ by

$$Q'(y, x) = \begin{cases} co(N(y, x)) \cap A(y, x), & \text{if } (y, x) \in H; \\ A(y, x), & \text{if } (y, x) \in (D \times K) \setminus H \end{cases}$$

has nonempty values.

$$N^{-1}(z) = \{(y, x) \in D \times K |\ F(t, y, x, z) \cap \text{int}C(t, y, x) \neq \emptyset\}.$$

We will show that $N^{-1}(z)$ is open in $D \times K$.

Taking arbitrary $(y_0, x_0) \in N^{-1}(z)$, we reduce

$$F(t, y_0, x_0, z) \cap \text{ int}C(t, y_0, x_0) \neq \emptyset.$$

Then there exists a point f_0 such that $f_0 \in F(t, y_0, x_0, z), f_0 \in \text{ int}C(t, y_0, x_0)$.

Let $W_0(y_0, x_0)$ be the compact set associated to the definition of the $(-C)$-lower semicontinuity property of F, by $C(t, y_0, x_0)$ is convex cone and $W_0(y_0, x_0) \subseteq -C(t, y_0, x_0)$, therefore $f_0 - W_0(y_0, x_0) \subset \text{int}C(t, y_0, x_0)$. Since

$\text{int}C$ is open, for any neighborhood V of the origin in Y, there exist neighborhoods $U_1(y_0), U_1(x_0)$ of y_0, x_0 such that

$$f_0 - W_0(y_0, x_0) + V \subset \text{int}C(t, y, x) \text{ for all } (y, x) \in U_1(y_0) \times U_1(x_0).$$

Since F is $(-C)$-lsc, there exist neighborhoods $U_2(y_0) \subset U_1(y_0), U_2(x_0) \subset U_1(x_0)$ such that

$$F(t, y, x, z) \cap (f_0 - W_0(y_0, x_0) + V) \neq \emptyset \text{ for all } (y, x) \in U_2(y_0) \times U_2(x_0).$$

Thus, there exists a point $f \in F(t, y, x, z)$ and $f \in f_0 - W_0(y_0, x_0) + V$. So $f \in \text{int}C(t, y, x)$. This implies that $F(t, y, x, z) \cap \text{int}C(t, y, x) \neq \emptyset$ for all $(y, x) \in U_2(y_0) \times U_2(x_0)$. So $U_2(y_0) \times U_2(x_0) \subset N^{-1}(z)$, and then N^{-1} is a open set.

On the other hand

$$Q'^{-1}(z) = (coN)^{-1}(z) \cap A^{-1}(z) \cup (A^{-1}(z) \cap (D \times K) \setminus H).$$

Combination the openness of $N^{-1}(z), A^{-1}(z)$ and the closedness of H, we conclude the openness of $Q'^{-1}(z)$. Hence, Q' has nonempty convex values and open lower sections, $D \times K$ is a compact in Hausdorff topological space, we claim from Theorem 8.1.3 of [5] that there exists a continous single-valued map $\phi' : D \times K \to K$ such that $\phi'(y, x) \in Q'(y, x)$.

We see that, the set-valued map $\psi' : D \times K \to 2^{D \times K}$ defined by

$$\psi'(y, x) = B(y, x) \times \{\phi'(y, x)\}$$

is compact usc with nonempty convex closed in $D \times K$, then it has fixed point. This means, there exists $(\bar{y}, \bar{x}) \in D \times K$ such that $\bar{x} \in \phi'(\bar{y}, \bar{x}), \bar{y} \in B(\bar{y}, \bar{x})$.

Obviously, $(\bar{y}, \bar{x}) \in H$ and $\bar{x} \in Q'(\bar{y}, \bar{x})$. By the definition of Q', $\bar{x} \in coN(\bar{y}, \bar{x})$ and $\bar{x} \in A(\bar{x}, \bar{y})$.

Since $\bar{x} \in coN(\bar{y}, \bar{x})$, there exists a finite subset $\{x_1, ..., x_n\}$ of $M(\bar{y}, \bar{x})$ and $\bar{x} = \sum\limits_{i=1}^{n} \alpha_i x_i, \alpha_i \geq 0, \sum\limits_{i=1}^{n} \alpha_i = 1$. Thus

$$F(t, \bar{y}, \bar{x}, x_i) \cap \text{int}C(t, \bar{y}, \bar{x}) \neq \emptyset \text{ for all } i = 1, 2, ..., n.$$

For F is diagonally upper $(-C)$-quasiconvex-like in the fourth variable, we conclude that there is an index $j \in \{1, ..., n\}$ such that

$$F(t, \bar{y}, \bar{x}, x_j) \subset F(t, \bar{y}, \bar{x}, \bar{x}) - C(t, \bar{y}, \bar{x})$$

This reduce

$$(F(t, \bar{y}, \bar{x}, \bar{x}) - C(t, \bar{y}, \bar{x})) \cap \text{int}C(t, \bar{y}, \bar{x}) \neq \emptyset.$$

Therefore $F(t, \bar{y}, \bar{x}, \bar{x}) \cap \text{int}C(t, \bar{y}, \bar{x}) \neq \emptyset$. This contrary to (iii). Then for each $t \in T$, there exists $(\bar{y}, \bar{x}) \in D \times K$ such that

$$\begin{aligned}
&\bar{x} \in A(\bar{y}, \bar{x}); \bar{y} \in B(\bar{y}, \bar{x});\\
&F(t, \bar{y}, \bar{x}, z) \cap \text{int}C(t, \bar{y}, \bar{x}) = \emptyset, \text{ for all } \ z \in A(\bar{y}, \bar{x}).
\end{aligned}$$

The proof is complete.

Theorem 4.3. *Let A be usc multivalued mapping with convex compact values, B be usc multivalued mapping with convex closed values. For each $t \in T$, assume the following conditions hold:*

(i) $intC$ is a open map;

(ii) F is a closed map, F is upper C-quasiconvex-like in the third and fourth variable;

(iii) For all $(t, y, x) \in T \times D \times K, z \notin coN(t, y, x, z)$ where $N(t, y, x, z) = \{\xi \in K \mid F(t, y, \xi, z) \subseteq \; intC(t, y, x)\}$.

Then for each $t \in T$, there exists $(\bar{y}, \bar{x}) \in D \times K$ such that

$$\bar{x} \in A(\bar{y}, \bar{x}); \bar{y} \in B(\bar{y}, \bar{x});$$
$$\exists z \in A(\bar{y}, \bar{x}), F(t, \bar{y}, \bar{x}, z) \not\subseteq intC(t, \bar{y}, \bar{x}).$$

Proof. We define the multivalued mapping $M : D \times K \to 2^K, W : D \times K \to 2^{D \times K}$ by

$$M(y, x) = \{\xi \in A(y, x) \mid \exists z \in A(y, x), F(t, y, \xi, z) \not\subseteq \; \text{int}C(t, y, x)\}$$

$$W(y, x) = B(y, x) \times M(y, x).$$

The first, we will show that M is closed. Assume that $x_\beta \to x, y_\beta \to y, \xi_\beta \to \xi, \xi_\beta \in M(y_\beta, x_\beta)$. Then $\xi_\beta \in A(y_\beta, x_\beta)$ and there is $z \in A(y_\beta, x_\beta)$ such that

$$F(t, y_\beta, \xi_\beta, z) \not\subseteq \text{int}C(t, y_\beta, x_\beta).$$

So $F(t, y_\beta, \xi_\beta, A(y_\beta, x_\beta)) \cap Y \setminus \text{int}C(t, y_\beta, x_\beta) \neq \emptyset$.

For the usc with nonempty compact values of A and $\xi_\beta \in A(y_\beta, x_\beta)$ implies $\xi \in A(y, x)$.

Since A is usc with compact values and F is closed, the multivalued mapping G defined by $G(t, y, \xi, x) = \cup_{z \in A(y,x)} F(t, y, \xi, z) = F(t, y, \xi, A(y, x))$, for all $(t, y, x) \in T \times D \times K$ is closed.

The openness of int$C$ implies that $Y \setminus \text{int}C$ is a closed map. Since the closedness of G and $Y \setminus \text{int}C$, we have $F(t, y, \xi, A(y, x)) \cap (Y \setminus \; \text{int}C(t, y, x)) \neq \emptyset$. This reduces $F(t, y, \xi, A(y, x)) \not\subseteq \; \text{int}C(t, y, x)$. Thus, there is $z \in A(y, x)$ such that $F(t, y, \xi, z)) \not\subseteq \; \text{int}C(t, y, x)$. This implies $\xi \in M(y, x)$. This shows that M is closed.

In the next step, we prove $M(y, x)$ is a convex set. Taking arbitrary $\xi_1, \xi_2 \in M(y, x)$. Then there are $z_1, z_2 \in A(y, x)$ such that

$$F(t, y, \xi_1, z_1) \not\subseteq \text{int}C(t, y, x), F(t, y, \xi_2, z_2) \not\subseteq \text{int}C(t, y, x). \tag{8}$$

Since $A(y, x)$ is convex, $\lambda z_1 + (1 - \lambda) z_2 \in A(y, x)$ and $\lambda \xi_1 + (1 - \lambda) \xi_2 \in A(y, x)$ for all $\lambda \in [0, 1]$.

For F is upper$C$-quasiconvex-like in the third and fourth variable implies that for all $\lambda \in [0, 1]$

$$F(t, y, \xi_1, z_1) \subset F(t, y, \lambda \xi_1 + (1 - \lambda) \xi_2, \lambda z_1 + (1 - \lambda) z_2) + C(t, y, x), \tag{9}$$

or

$$F(t,y,\xi_2,z_2) \subset F(t,y,\lambda\xi_1+(1-\lambda)\xi_2,\lambda z_1+(1-\lambda)z_2)+C(t,y,x). \quad (10)$$

Combination conclusions (8), (9), (10), we conclude that

$$F(t,y,\lambda\xi_1+(1-\lambda)\xi_2,\lambda z_1+(1-\lambda)z_2)+C(t,y,x) \not\subseteq \mathrm{int}C(t,y,x).$$

Hence,

$$F(t,y,\lambda\xi_1+(1-\lambda)\xi_2,\lambda z_1+(1-\lambda)z_2) \not\subseteq \mathrm{int}C(t,y,x).$$

This shows $\lambda\xi_1+(1-\lambda)\xi_2 \in M(y,x)$. So $M(y,x)$ is a convex set.

Setting $L : A(y,x) \to A(y,x)$ defined by

$$L(z) = \{\xi \in A(y,x) \mid F(t,y,\xi,z) \not\subseteq \ \mathrm{int}C(t,y,x)\}.$$

Suppose, there exists a finite subset $\{z_1,...,z_n\} \in A(y,x)$ such that $coz_i \not\subset \cup_{i=1}^n L(z_i)$. So we can find $z = \sum_{i=1}^n \alpha_i z_i, \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1$ such that $z \not\subset \cup_{i=1}^n L(z_i)$. This means $F(t,y,z,z_i) \subset \mathrm{int}C(t,y,x)$ for all $i = 1,2,...,n$ or equivalent $z_i \in N(t,y,x,z)$. Thus, $z \in coN(t,y,x,z)$, which contradicts with (iii).

Argument similar to proof the closed property of M, we can easy show that $L(z)$ is a closed set in $A(y,x)$ compact. Hence, according the Theorem 2.4, it follows

$$\cap_{z\in A(y,x)} L(z) \neq \emptyset.$$

This means M has nonempty values.

Combining all these facts proves that M is usc with nonempty closed convex values. Since B is usc with closed convex values, W is too. Hence, W has a fixed point. This follows there exists $\bar{x} \in M(\bar{y},\bar{x}), \bar{y} \in B(\bar{y},\bar{x})$. So $\bar{x} \in A(\bar{y},\bar{x})$ and there is $z \in A(\bar{y},\bar{x})$ such that $F(t,\bar{y},\bar{x},z) \not\subseteq \ \mathrm{int}C(t,\bar{y},\bar{x})$.

Theorem 4.4. *Let A be usc multivalued mapping with convex compact values, B be usc multivalued mapping with convex closed values. For each $t \in T$, assume the following conditions hold:*

(i) $intC$ is a open map;

(iii) For any $z \in K, F(t,.,.,z)$ is $(-C)$-lsc with compact values and F is lower C-quasiconvex-like in the third anf fourth variable;

(iii) For all $(t,y,x) \in T \times D \times K, z \notin coQ(t,y,x,z)$ where $Q(t,y,x,z) = \{\xi \in K \mid F(t,y,\xi,z) \cap \ intC(t,y,x) \neq \emptyset\}$.

Then for each $t \in T$, there exists $(\bar{y},\bar{x}) \in D \times K$ such that

$$\bar{x} \in A(\bar{y},\bar{x}); \bar{y} \in B(\bar{y},\bar{x});$$
$$\exists z \in A(\bar{y},\bar{x}), F(t,\bar{y},\bar{x},z) \cap intC(t,\bar{y},\bar{x}) = \emptyset.$$

Proof. We define the multivalued mapping $N : D \times K \to 2^K, W : D \times K \to 2^{D\times K}$ by

$$N(y,x) = \{\xi \in A(y,x) \mid \exists z \in A(y,x), F(t,y,\xi,z) \cap \text{int}C(t,y,x) = \emptyset\}$$

$$W(y,x) = B(y,x) \times N(y,x).$$

Since int$C$ is open and F is $(-C)$-lsc with compact values, we can prove the closedness of N as the same as the closedness of S_4 in Theorem 3.4.

Let $\xi_1, \xi_2 \in N(y,x)$. Then there are $z_1, z_2 \in A(y,x)$ such that

$$F(t,y,\xi_1,z_1) \cap \text{int}C(t,y,x) = \emptyset, F(t,y,\xi_2,z_2) \cap \text{int}C(t,y,x) = \emptyset.$$

These facts show that

$$(F(t,y,\xi_1,z_1) - C(t,y,x)) \cap \text{int}C(t,y,x) = \emptyset,$$

$$(F(t,y,\xi_2,z_2) - C(t,y,x)) \cap \text{int}C(t,y,x) = \emptyset. \tag{11}$$

Since $A(y,x)$ is convex, $\lambda z_1 + (1-\lambda)z_2 \in A(y,x)$ and $\lambda\xi_1 + (1-\lambda)\xi_2 \in A(y,x)$ for all $\lambda \in [0,1]$.

For F is C-lower quasiconvex-like in the third and fourth variable implies that

$$F(t,y,\lambda\xi_1 + (1-\lambda)\xi_2, \lambda z_1 + (1-\lambda)z_2) \subset F(t,y,\xi_1,z_1) - C(t,y,x), \tag{12}$$

or

$$F(t,y,\lambda\xi_1 + (1-\lambda)\xi_2, \lambda z_1 + (1-\lambda)z_2) \subset F(t,y,\xi_2,z_2) - C(t,y,x). \tag{13}$$

Combination conclusions (11), (12), (13), we conclude that

$$F(t,y,\lambda\xi_1 + (1-\lambda)\xi_2, \lambda z_1 + (1-\lambda)z_2) \cap \text{int}C(t,y,x) = \emptyset.$$

This shows $\lambda\xi_1 + (1-\lambda)\xi_2 \in N(y,x)$. So $N(y,x)$ is a convex set.

Setting $L' : A(y,x) \to A(y,x)$ defined by

$$L'(z) = \{\xi \in A(y,x) \mid F(t,y,\xi,z) \cap \text{int}C(t,y,x) = \emptyset\}.$$

We will show that L' is KKM. Suppose, there exists a finite subset $\{z_1, ..., z_n\} \in A(y,x)$ such that $coz_i \not\subset \cup_{i=1}^n L'(z_i)$. So we can find $z = \sum\limits_{i=1}^{n} \alpha_i z_i, \alpha_i \geq 0, \sum\limits_{i=1}^{n} \alpha_i = 1$ such that $z \not\subset \cup_{i=1}^n L'(z_i)$. This means

$$F(t,y,z,z_i) \cap \text{int}C(t,y,x) \neq \emptyset \text{ for all } i = 1,2,...,n$$

or equivalent $z_i \in Q(t,y,x,z)$. Thus, $z \in coQ(t,y,x,z)$, which contradicts with (iii).

Argument similar to proof the closed property of N, we can easy show that $L'(z)$ is a closed set in $A(y,x)$ compact. Hence, according the Theorem 2.4, it follows

$$\cap_{z\in A(y,x)}L'(z)\neq\emptyset.$$

This means M has nonempty values.

Combining all these facts proves that N is usc with nonempty, closed convex values. Since B is usc with closed convex values, W is too. Hence, W has a fixed point. This follows there exists $\bar{x}\in N(\bar{y},\bar{x}),\bar{y}\in B(\bar{y},\bar{x})$. So $\bar{x}\in A(\bar{y},\bar{x}),y\in B(\bar{y},\bar{x})$ and there is $z\in A(\bar{y},\bar{x})$ such that $F(t,\bar{y},\bar{x},z)\cap \text{int}C(t,\bar{y},\bar{x})=\emptyset$.

References

[1] Aubin, J. P.: Mathematiccal Methods of Game and Economic Theory. *North-Holland, Amsterdam*, 1979.

[2] Chen, B., Huang, N-J.: Continuity of the solution maping to parametric generalized vector Ky Fan inequality problem. J. Global Optimization, 56 (2013) 1515 - 1528.

[3] Duong, T. T. T.: Mixed generalized quasi-equilibrium problems. J. Global Optimization, 56(2) (2013), 647-667.

[4] Fan, K.:*A generalization of Tychonoff's fixed point theorem*. Math. Ann., 142 (1961) 305 - 310.

[5] Klein, E., Thompson, A.C.: Theory of Corrsepondences. Wiley, New York (1984).

[6] Luc, D. T.: *An Abstract problem in Variational Analysis*. J Optim. Theory Appl., 138 (2008), 1, 65-76.

[7] Lin, L.J and Tu, Chin-I: The studies of systems of variational inclusions problems and variational disclusions problems with applications. Nonlinear Analysis, 69 (2008) 1981 -1998.

[8] Luc, D. T, Sarabi, E. and Soubeyran, A.: Existence of solutions in variational relation problems without convexity. J. Math. Analysis and Appl., 364 (2010), 544 - 555.

[9] Lin, L.J. and Tan, N. X.: On quasivariational inclusion problems of type I and related problems. J. Global Optimization. 39, No 3 (2007), 393-407.

[10] Park, S.: *Fixed Points and Quasi-Equilibrium Problems*. Nonlinear Operator Theory. Mathematical and Computer Modelling, 32 (2000), 1297-1304.

[11] Peng, Z. Y., Yang, X. M., Peng, J. W.: On the lower semicontinuity of the solution mappings to parametric weak generalized Ky Fan inequality. L. Optim. Theory Appl. 152, (2012) 256-264.

[12] Tan, N. X.: On the existence of solutions of quasi-variational inclusion problems, J. Optim Theory Appl., 123, 2004, 619-638.

[13] Tuan, L. A., Lee, G. M. and Sach, P. H.: Upper semicontinuity result for the solution mapping of a mixed parametric generalized vector quasiequilibrium problem with moving cone, J. Global Optimization, 47, 2010, 639 -660.

[14] Sach, P. H., Lin, J.L. and Tuan, L.A.: Generalized vector quasivariational inclusion problems with moving cones. L. Optim. Theory Appl. 147, (2010), 607 - 620.

[15] Sach, P. H. and Minh, N. B.: Continuity of solution mappings in some parametric non-weak vector Ky Fan inequalities. J. Global Optimization, 57 (2013), 1401 - 1418.