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ON RIGHT STRONGLY PRIME TERNARY RINGS

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Abstract

A ternary ring R is right strongly prime if every nonzero ideal of R contains a finite subset G such that the right annihilator of G with respect to a finite subset of R is zero. Examples are ternary integral domain and simple ternary rings with a unital element 'e' or an identity element. All the strongly prime ternary rings are prime. In this paper we study right strongly prime ternary rings and obtain some characterizations of it. Lastly we characterize strongly prime radical of a ternary ring.

1 Introduction

A ring R is prime if for given $r, t \in R\backslash\{0\}$, there exists $s \in R$ such that $rst \neq 0$. If for each nonzero element r of R, we can restrict the choice of 's' to a finite set(independent of t but depending on r), then we have a ring that is stronger than prime. Considering these Handelman and Lawrence[6] introduced the following notion. Given a nonzero element r of R, a right insulator for r, is defined to be a finite subset F of R, such that $ann(\{rs : s \in F\}) = \{0\}$. Now R is called right strongly prime if each nonzero element of R has a right insulator, i.e for each for each $r \in R\backslash\{0\}$ there is a finite subset F of R such that $xFy = \{0\}$ implies that $y = 0$. Although prime is a symmetric notion, Handelman and Lawrence[6] showed that strongly prime is not. Although primitive group rings

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were the original motivation for introducing strongly prime rings, it became apparent that strongly prime rings are interesting by themselves. In 2007, T.K. Dutta and M.L.Das[4] introduced and studied (right)strongly prime semiring.

In the year 1971, W.G.Lister[7] introduced the notion of ternary ring. In this paper we introduce the notion of right strongly prime ternary ring. We obtain some elementary properties of right strongly prime ternary rings. We also obtain some characterizations of right strongly prime ternary rings. Lastly we introduce the notion of super sp-system in a ternary ring and characterize strongly prime radical by the above notion.

Some earlier work on ternary rings of the authors may be found in [9], [10], [11].

2 Preliminaries

Definition 2.1. *[10] A nonempty set R together with a binary operation, called addition and a ternary multiplication denoted by juxtaposition, is called a ternary ring if R is an additive commutative group satisfying the following properties:*
(i) $(abc)de = a(bcd)e = ab(cde)$,
(ii) $(a+b)cd = acd + bcd$,
(iii) $a(b+c)d = abd + acd$,
(iv) $ab(c+d) = abc + abd$ *for all* $a, b, c, d, e \in R$.

Definition 2.2. *Let R be a ternary ring. The additive identity '0' of R satisfies the property $0xy = x0y = xy0 = 0$ for all $x, y \in R$. This element '0' is called the zero element or simply the zero of the ternary ring R.*

Definition 2.3. *A ternary ring R admits an identity provided that there exist elements $\{(e_i, f_i) \in R \times R(i = 1, 2, ..., n)\}$ such that $\sum_{i=1}^{n} e_i f_i x = \sum_{i=1}^{n} e_i x f_i = \sum_{i=1}^{n} x e_i f_i = x$ for all $x \in R$. In this case the ternary ring R is said to be a ternary ring with identity $\{(e_i, f_i) : i \in 1, 2, ..., n\}$. In particular, if there exists an element $e \in R$ such that $eex = exe = xee = x$ for all $x \in R$ then e is called a unital element of the ternary ring R.*

It is obvious that $xye = (exe)ye = ex(eye) = exy$ and $xye = x(eye)e = xe(yee) = xey$ for all $x, y \in R$. Hence the following result follows.

Proposition 2.4. *If e is a unital element of a ternary ring R, then $exy = xey = xye$, for all $x, y \in R$.*

Definition 2.5. *An additive subgroup T of a ternary ring R is called a ternary subring of R if $t_1 t_2 t_3 \in T$ for all $t_1, t_2, t_3 \in T$.*

Definition 2.6. *A ternary ring R is called commutative if $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$, where σ is a permutation of $\{1, 2, 3\}$ for all $x_1, x_2, x_3 \in R$.*

We now define left(right, lateral) ideal of a ternary ring.

Definition 2.7. *An additive subgroup I of a ternary ring R is called a left(right, lateral) ideal of R if r_1r_2i (respectively ir_1r_2, r_1ir_2) $\in I$ for all $r_1, r_2 \in R$ and $i \in I$. If I is a left, a right and a lateral ideal of R then I is called an ideal of R. If I is a left and right ideal of R then I is called a two sided ideal of R.*

Definition 2.8. *A ternary ring R is called a simple ternary ring if $R^3 \neq (0)$ and if it contains no nonzero proper ideal i.e (0) and R are the only ideals of R.*

Definition 2.9. *A ternary ring R is said to be zero divisor free(ZDF) if for $a, b, c \in R$, $abc = 0$ implies $a = 0$ or $b = 0$ or $c = 0$.*

Definition 2.10. *A commutative ternary ring with identity is called a ternary integral domain if it is zero divisor free(ZDF).*

Definition 2.11. *An element 'r' of a ternary ring R is said to be invertible in R if there exists an element r' in R(called inverse of r) such that $rr'x = r'rx = xrr' = xr'r = x$ for all $x \in R$ and the element 'r' is also called a unit in R.*

Definition 2.12. *A non-trivial ternary ring R with identity is said to be a division ternary ring if for every element $a(\neq 0) \in R$ there exists an element $b \in R$ such that $abx = xab = xba = bax = x$ for all $x \in R$.*

In the following proposition, we describe the principal left(right, lateral) ideal of a ternary ring.

Proposition 2.13. *Let R be a ternary ring and $r \in R$. Then the principal*

(i) left ideal generated by 'r' is given by $< r >_l = RRr + nr$.

(ii) right ideal generated by 'r' is given by $< r >_r = rRR + nr$.

(iii) two-sided ideal generated by 'r' is given by $< r >_t = RRr + rRR + RRrRR + nr$.

(iv) lateral ideal generated by 'r' is given by $< r >_m = RrR + RRrRR + nr$.

(v) ideal generated by 'r' is given by $< r > = RRr + rRR + RrR + RRrRR + nr$, where $n \in \mathbb{Z}$(set of all integers).

If R contains identity or a unital element then $< r > = RRrRR$.

Proposition 2.14. *Let R be a non-trivial division ternary ring. Then R does not contain any left, right and lateral ideals.*

Proposition 2.15. *A division ternary ring does not contain any divisors of zero.*

Definition 2.16. *A proper ideal P of a ternary ring R is called a prime ideal of R if for any three ideals A, B, C of R, $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.*

Definition 2.17. *A ternary ring R is called a prime ternary ring if the zero ideal (0) is a prime ideal of R.*

Throughout the paper by R^* we mean $R^* = R\backslash(0)$.

3 Right strongly prime ternary rings

Definition 3.1. *Let R be a ternary ring and $r \in R^*$. The right insulator for 'r' is a finite subset F of R such that annihilator of r with respect to F i.e $ann(r, F) = \{0\}$ or equivalently for each $r \in R^*$ there exists a finite subset F of R(called right insulator) such that $rFt = \{0\} \Rightarrow t = 0$, for all $t \in R$.*

In the year 1964, N. Nobusawa[8] introduced the notion of $\Gamma-$ring. Later W.E. Barnes[1] weakened the defining condition for Nobusawa's Γ-ring. The notion of Barnes Γ-ring reduces to the notion of ternary ring introduced by W.G. Lister[7] whenever $\Gamma = R$. In the year 1988 G.L.Booth[2] introduced the notion of right strongly prime Γ-ring as follows-

Definition 3.2. *A $\Gamma-$ ring M is called right strongly prime if for every $x \neq 0 \in M$, there exist finite subsets F of M and ϕ and Λ of Γ such that $x\phi F\Lambda y = 0$ implies that $y = 0$ for all $y \in M$.*

We shall show that our definition of right strongly prime ternary ring and the notion of right strongly prime Γ-ring introduced by G.L Booth[2] whenever $\Gamma = R$ are equivalent.

Proposition 3.3. *A ternary ring R is right strongly prime if and only if for every $x \in R^*$, there exist finite subsets F_1, F_2, F_3 of R such that $xF_1F_2F_3y = \{0\}$ implies $y = 0$ for all $y \in R$.*

Proof. Suppose R is a right strongly prime ternary ring. Let $x \in R^*$. Then there exists a finite subset F of R such that

$$xFy = \{0\} \Rightarrow y = 0 \tag{1}$$

for all $y \in R$. Let $F_1 = F$, $F_2 = \{x\}$, $F_3 = F$. Suppose $xF_1F_2F_3y = \{0\}$. Then $xFxFy = \{0\}$. By (1) $xFy = \{0\}$. This again implies that $y = 0$ for all $y \in R$.

Conversely suppose that for every $r \in R^*$, there exist finite subsets F_1, F_2, F_3 of R such that $xF_1F_2F_3y = \{0\} \Rightarrow y = 0$ for all $y \in R$. Then $F = F_1F_2F_3$ is a finite subset of R. Now $xFy = \{0\}$ implies $xF_1F_2F_3y = \{0\}$ which again implies that $y = 0$ for all $y \in R$. Consequently R is a right strongly prime ternary ring. □

Corollary 3.4. *A ternary ring R is right strongly prime if and only if for every $r \in R^*$, there exists a finite subset F of R such that $rFFFy = \{0\} \Rightarrow y = 0$ for all $y \in R$.*

Proof. The necessity follows by taking $F = F_1 \cup F_2 \cup F_3$ in proposition 3.3.
Converse is obvious. □

Example 3.5. *Let $R = \{ri : r \in \mathbb{R}, i^2 = -1\}$, where $\mathbb{R}$ is the set of all real numbers. Then R is a ternary ring together with usual binary addition and ternary multiplication. Let $ri(\neq 0) \in R$ and $F = \{ri\}$. Then $(ri)Fy = 0$ implies that $y = 0$ for all $y \in R$. Hence R is a right strongly prime ternary ring.*

Theorem 3.6. *A right strongly prime ternary ring is prime.*

Proof. Suppose that R is a right strongly prime ternary ring. Let A, B, C be three ideals of R such that $ABC = (0)$. Suppose that $A \neq (0)$ and $B \neq (0)$. Since $A \neq (0)$, there exists $a(\neq 0) \in A$. Since R is a right strongly prime ternary ring, there exists a finite subset F of R such that $aFy = (0)$ implies that $y = 0$ ...(1) for all $y \in R$. Now $aF(BRBRC) = (aFB)(RBR)C \subseteq ABC = (0)$. This implies that $BRBRC = (0)$ by (1). Since $B \neq (0)$, there exists $b(\neq 0) \in B$, and a finite subset F' of R such that $bF'y = (0) \Rightarrow y = 0$ for all $y \in R$....(2). Now $bF'BRC \subseteq BRBRC = (0)$. So $BRC = (0)$ by (2). Again $bF'c \subseteq BRC = (0)$ for $c \in C$. This implies that $c = 0$. Since c is an arbitrary element of C, we find that $C = (0)$. Similarly we can prove than if $A \neq (0)$, $C \neq (0)$ then $B = (0)$ and if $B \neq (0)$, $C \neq (0)$, then $A = (0)$. Thus (0) is a prime ideal of R and hence R is a prime ternary ring. □

Theorem 3.7. *Let R be a ternary ring . Then the following are equivalent :*

(i) R is right strongly prime;

(ii) If I is a non-zero ideal of R, there exist finite subsets F' of I and F of R such that $F'Fy = \{0\}$ implies that $y = 0$ for all $y \in R$;

(iii) If $x \in R^*$, there exist $r \in R$ and finite subsets F', F of R such that $xrF'Fy = \{0\}$ implies that $y = 0$ for all $y \in R$.

Proof. $(i) \Rightarrow (ii)$ Let I be a nonzero ideal of R. Let $x(\neq 0) \in I$. Then by (i) there exists a finite subset F of R such that $xFy = 0$ implies that $y = 0$ for all $y \in R$. Then $F_1 = \{x\}$ is a finite subset of I and $F_1Fy = \{0\}$ implies that $y = 0$ for all $y \in R$.

$(ii) \Rightarrow (iii)$ Let $a(\neq 0) \in R$. Then $< a >$ is a nonzero ideal of R. So by (ii) there exist finite subsets F_1 of $< a >$ and F_2 of R such that $F_1F_2y = \{0\}$ implies that $y = 0$ for all $y \in R$. Suppose that $aRR = \{0\}$. Then $< a > RR = \{0\}$. Since $F_1F_2a \subseteq < a > RR = \{0\}$, we have $F_1F_2a = \{0\}$. This implies $a = 0$, a contradiction. Therefore $arx \neq 0$ for some $r, x \in R$. Then $I =< arx >$ is a nonzero ideal of R. Again by (ii) there exists a finite subset G' of I and a finite subset G of R such that $G'Gy = \{0\}$ implies that $y = 0$ for all $y \in R$. Now any

element of G' is of the form $\{narx+\sum_{i=1}^{m} arx\alpha_i\beta_i+\sum_{j=1}^{l} s_j s'_j arx+\sum_{t=1}^{k} e(d_t arxe_t)e+\sum_{p=1}^{s} c_p g_p arx f_p f'_p$, where $m, n, l, k, s \in \mathbb{Z}, \alpha_i, \beta_i, s_i, s'_j, d_t, e_t, c_p, g_p, f_p, f'_p \in R\}$.
Let $F = \{x, x\alpha_i\beta_i, xe_t e, xf_p f'_p : i = 1, 2, .., m; t = 1, 2, ..., k; p = 1, 2.........s; m, k, s \in \mathbb{Z}\}$, and let $arFGy = \{0\}$. Then $G'Gy = \{0\}$. By (ii) $y = 0$.
$(iii) \Rightarrow (i)$ Let $a(\neq 0) \in R$. Then by (iii), there exists $r \in R$ and finite subsets F_2, F_3 of R such that $arF_2F_3y = \{0\}$ implies that $y = 0$ for all $y \in R$. Let $F = rF_2F_3$, which is a finite subset of R. Then $aFy = \{0\}$ implies that $y = 0$ for all $y \in R$. Thus (i) holds. □

Definition 3.8. *Let A be a non-empty subset of a ternary ring R. The right annihilator of A with respect to a nonempty subset B of R, denoted by $r(A, B)$, is defined by $r(A, B) = \{x \in R : ABx = \{0\}\}$.*

Proposition 3.9. *The right annihilator $r(A, B)$ of A with respect to B in a ternary ring R is a right ideal of R.*

Proof. The proof is a routine matter of verification and so we omit it. □

Proposition 3.10. *The right annihilator of a nonempty subset A with respect to a right ideal B of a ternary ring R with a unital element 'e' is an ideal of R.*

Proof. By the proposition 3.9, $r(A, B)$ is a right ideal of R. Now it remains to show that $r(A, B)$ is a left and a lateral ideal of R. Let $r \in r(A, B)$. Then $ABr = \{0\}$. Now since B is a right ideal of R, we find that $AB(xyr) = A(Bxy)r \subseteq A(BRR)r \subseteq ABr = \{0\}$ for all $x, y \in R$ i.e $xyr \in r(A, B)$ i.e $RRr(A, B) \subseteq r(A, B)$. This implies that $r(A, B)$ is a left ideal of R. Again $AB(xry) = AB(exe)(ere)y = A(Bex)(eerey) \subseteq A(BRR)(eerey) \subseteq AB(rey) = (ABr)ey = \{0\}ey = \{0\}$ for all $x, y \in R$, i.e $xry \in r(A, B)$ i.e $Rr(A, B)R \subseteq r(A, B)$. Then $r(A, B)$ is a lateral ideal of R. Hence $r(A, B)$ is an ideal of R. □

Definition 3.11. *A ternary ring R is said to be a bounded right strongly prime ternary ring of bound n if each nonzero element of R has an insulator containing not more than n-elements and at least one element has no insulator with fewer that n-elements.*

Theorem 3.12. *A ternary ring R with identity or a unital element is right strongly prime if and only if every nonzero ideal of R contains a finitely generated left ideal whose right annihilator with respect to a finite subset F of R is zero.*

Proof. Suppose R is a right strongly prime ternary ring and I is a nonzero ideal of R. Let $r(\neq 0) \in I$. Since R is a right strongly prime ternary ring, there exists a finite subset F of R such that $rFFFt = \{0\} \Rightarrow t = 0$. Now $rFF \subseteq I$ and rFF is finite. Let L be the left ideal of R generated by rFF, i.e $L = RRrFF$. So $L \subseteq I$. Let $LFt = 0$. Then $rFFFt \subseteq RRrFFFt = LFt = 0$, since R contains identity or a unital element. Hence $t = 0$. Thus I contains the finitely generated left ideal L whose right annihilator with respect to a finite subset F is zero.

Conversely, suppose the condition holds. Let $r \in R^*$. Now $\langle r \rangle$ is a nonzero ideal of R. By the given condition there exists a finite subset F' of $\langle r \rangle$ such that right annihilator of the left ideal L generated by F' with respect to a finite subset F of R is zero, i.e. $LFy = \{0\} \Rightarrow y = 0$ for all $y \in R$. If possible let $rRR = \{0\}$. Then $\langle r \rangle RR = \{0\}$. Since $F'Fr \subseteq \langle r \rangle = RRrRR$, (since R contains identity or a unital element)$\subseteq \langle r \rangle RR = \{0\}$. Thus we have $F'Fr = \{0\} \Rightarrow LFr = \{0\}$ as $L = RRF'$; but $LFr = \{0\} \Rightarrow r = 0$, a contradiction. Thus $rRR \neq 0$. So there exist $r_1, x \in R$ such that $rr_1x \neq 0$. Let $I = \langle rr_1x \rangle$, $I \neq (0)$. Then by hypothesis there exists a finite subset G' of I such that if $H = RRG'$ then $HH'y = 0 \Rightarrow y = 0$ for all $y \in R$ where H' is a finite subset of R. By theorem 3.7, we have R is a right strongly prime ternary ring. □

Theorem 3.13. *A ternary ring R is right strongly prime if and only if every nonzero ideal of R contains a finite subset G such that right annihilator of G with respect to a finite subset of R is zero.*

Proof. Suppose R is a right strongly prime ternary ring and I is a nonzero ideal of R. Let $a(\neq 0) \in I$. Since R is a right strongly prime ternary ring, there exists a finite subset F of R such that $aFFFt = 0 \Rightarrow t = 0$. Let $G = aFF \subseteq I$. Then G is a finite subset of I and right annihilator of G with respect to F is zero i.e $r(G, F) = 0$

Conversely, suppose that every nonzero ideal of R contains a finite subset whose right annihilator with respect to a finite subset F of R is zero. Let $a(\neq 0) \in R$. Then $\langle a \rangle$ is a nonzero ideal of R. Then there exist a finite subsets F' of $\langle a \rangle$ and a finite subset F of R such that $F'Fy = \{0\} \Rightarrow y = 0$ for all $y \in R$. Now by theorem 3.7, we can show that R is a right strongly prime ternary ring. □

Definition 3.14. *A ternary ring R is said to satisfy descending chain condition(DCC) on right ideals of R if for each sequence of right ideals A_1, A_2, A_3... of R with $A_1 \supseteq A_2 \supseteq A_3$.... there exists a positive integer n(depending on the sequence) such that $A_n = A_{n+1} =$ ....*

Proposition 3.15. *If R is a prime ternary ring with DCC on right annihilator ideals of R then R is a right strongly prime ternary ring.*

Proof. Let I be a nonzero ideal of R and $\mathcal{M}$ denote the class of all right annihilators of the form $r(F, F')$, where F and F' are finite subsets of I and R respectively. Then $\mathcal{M}$ contains a minimal element $J = r(F_0, F_0')$, say. Suppose $J \neq 0$. Since R is a prime ternary ring, $IRJ \neq \{0\}$. Then there exist $x \in I$, $r \in R$ and $y \in J$ such that $xry \neq 0$. Let $F'' = F_0 \cup \{x\}$ and $F''' = F_0' \cup \{r\}$. Since $F_0F_0'r \subseteq F''F'''r$, $r(F'', F''') \subset J$. Again $y \in J$ and $xry \neq 0$ implies that $y \notin r(F'', F''')$ which contradicts the minimality of J. Hence $J = \{0\}$. Therefore by proposition 3.7(ii), R is right strongly prime. □

Proposition 3.16. *Every simple ternary ring with a unital element e is right strongly prime.*

Proof. Let R be a simple ternary ring with a unital element e. Since R is simple, R is the only nonzero ideal of R. Now $F' = \{e\}$ is a finite subset of R and $F'F'y = \{0\}$ i.e $eey = 0$ implies $y = 0$ for all $y \in R$. Consequently R is a right strongly prime ternary ring. □

Definition 3.17. *[5] A class ρ of ternary rings is called hereditary if I is an ideal of a ternary ring R and $R \in \rho$ then $I \in \rho$.*

Proposition 3.18. *The class of all right strongly prime ternary rings is hereditary.*

Proof. Let R be a right strongly prime ternary ring and I be an ideal of R. If $I = (0)$ then I is trivially right strongly prime ternary ring. So we assume that $I \neq (0)$ and let a be a non-zero element of I. Since R is a right strongly prime ternary ring, there exists a right insulator F for 'a', such that

$$aFy = 0 \Rightarrow y = 0 \tag{1}$$

Obviously $F_1 = FaF$ is a finite subset of I. Now suppose that $aF_1y = 0$ where $y \in I$. Then $aFaFy = 0$. Now by (1) we have $aFy = 0$, which again implies that $y = 0$. So I is a right strongly prime ternary ring. Hence the class of all right strongly prime ternary rings is hereditary.

□

Now we consider the matrix ternary ring $M_n(R)$, where R is a ternary ring.

Let R be a ternary ring with a unital element e and $M_n(R)$ be the set of all square matrices of order $n (n \in \mathbb{N})$ with entries from R. Suppose $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n} \in M_n(R)$.
We define binary addition and ternary multiplication in $M_n(R)$ as follows:

$(a_{ij})_{n\times n}+(b_{ij})_{n\times n} = (a_{ij}+b_{ij})_{n\times n}$ and $(a_{ij})_{n\times n}(b_{ij})_{n\times n}(c_{ij})_{n\times n} = (d_{ij})_{n\times n}$, where $d_{ij} = \sum_{k,l=1}^{n} a_{ik}b_{kl}c_{lj}$; $1 \leq i, j \leq n$.

It can be easily verified that together with above defined addition and multiplication $M_n(R)$ is a ternary ring with unital element. We call $M_n(R)$ the matrix ternary ring.

Let $r \in R^*$. Then the notation rE_{ij} will be used to denote the $n \times n$ matrix in which the $(i,j)th$ entry is r and all other entries are zero. Then we can write $A = \sum_{i,j=1}^{n} a_{ij}E_{ij}$ and it can be easily verified that $(xE_{pq})(yE_{rs})(zE_{uv}) =$
$\begin{cases} (xyz)E_{pv} & \text{if } q = r \text{ and } s = u \\ 0 & \text{if } q \neq r \text{ or } s \neq u \text{ for all } x, y, z \in R \end{cases}$

Proposition 3.19. *If R is an $n \times n$ matrix ternary ring over a ternary integral domain, then R is bounded strongly prime of bound n.*

Proof. Let $A = (a_{ij})_{n\times n} \in R^*$. Then at least one $a_{ij} (1 \leq i, j \leq n)$ is nonzero. Suppose $a_{pq} \neq 0$. We now prove that $\{rE_{qi}\}$ is a right insulator for a_{pq}. Let $B = (b_{ij})_{n\times n}$. Now $A(rE_{qi})B = 0 \Rightarrow a_{pq}rb_{ij} = 0$ for $1 \leq j \leq n \Rightarrow b_{ij} = 0 (1 \leq j \leq n)$, since R is a matrix ternary ring over ternary integral domain. This shows that $\{rE_{qi}\}_{i=1}^{n}$ is an insulator for A. Also the element $E_{11} \in R^*$ has an insulator $\{rE_{1j}\}_{j=1}^{n}$ and no insulator of E_{11} contains less than n elements. Hence R is a bounded strongly prime ternary ring of bound n. □

Proposition 3.20. *If R is a right strongly prime ternary ring and 'i' is a nonzero idempotent element in R then iRi is a right strongly prime ternary subring of R.*

Proof. Obviously iRi is a ternary subring of R. Let iri be a non-zero element of iRi. Then $iRi \in R$. Since R is a right strongly prime ternary ring, corresponding to iRi there exists a right insulator $F = \{f_1, f_2, ..., f_k\}$, (say) in R. Let $F_1 = \{iif_sii : f_s \in F \text{ where } 1 \leq s \leq k\}$. Now

$$\begin{aligned} &(iri)F_1(iyi) = 0 \\ &\Rightarrow (iri)(iif_sii)(iyi) = 0, where\ f_s \in F \\ &\Rightarrow ir(iii)f_s(iii)yi = 0 \\ &\Rightarrow (iri)f_s(iyi) = 0 \\ &\Rightarrow (iri)F(iyi) = 0. \end{aligned}$$

This implies that $iyi = 0$. Thus iRi is a right strongly prime ternary subring of R. □

Proposition 3.21. *Right strongly primeness is morita invariant*

Proof. Let R be a right strongly prime ternary ring and 'i' be a nonzero idempotent element in R. Then by above proposition 3.20, iRi is a right strongly prime ternary subring of R.

Now we prove that if R is a right strongly prime ternary ring then $M_n(R)$ is also a right strongly prime ternary ring. Let B be a nonzero matrix in $M_n(R)$ and let its (p,q)th component b_{pq} is nonzero. Let $\{t_k\}$ be a right insulator for b_{pq} in R. Let A be a nonzero matrix with nonzero (i,j)th component a_{ij}. Then $b_{pq}t_ka_{ij} \neq 0$ for some $t_k \in \{t_k\}$. Now $(p,j)th$ component of $B(t_kE_{qj})A$ is $b_{pq}t_ka_{ij}$. So $A \neq 0 \Rightarrow B(t_kE_{qj})A \neq 0$ for some $t_k \in \{t_k\}$. Contrapositively, $B(t_kE_{ij})A = 0$ for all $t_k \in \{t_k\} \Rightarrow A = 0$. Therefore $\{t_kE_{ij}\}_{i,j,k}$ is a right insulator for B. Hence $M_n(R)$ is a right strongly prime ternary ring. Hence the proposition. □

Proposition 3.22. *Let R be a ternary ring with a unital element 'e'. Then R is a right strongly prime ternary ring if and only if $M_n(R)$ is a right strongly prime ternary ring.*

Proof. Let $M_n(R)$ be a right strongly prime ternary ring and $a(\neq 0) \in R$ and $b(\neq 0) \in R$. Then aE_{11} and bE_{11} are nonzero elements in $M_n(R)$. Since $M_n(R)$ is right strongly prime, there exists a finite subset $A = \{(a_{ij})_{n\times n}\}$ of $M_n(R)$ such that $(aE_{11})(a_{ij})_{n\times n}(bE_{11}) \neq (0)_{n\times n}$.
Now $(aE_{11})(a_{ij})_{n\times n}(bE_{11}) = (d_{ij})_{n\times n}$.

Where $d_{ij} = \begin{cases} aa_{11}b & \text{if } i = j = 1 \\ 0 & \text{elsewhere} \end{cases}$

Since $(d_{ij})_{n\times n} \neq (0)_{n\times n}$, we have $aa_{11}b \neq 0$. Now choose $A' = \{a_{1j} : 1 \leq j \leq n\} \subseteq R$. Then $a\alpha b \neq 0$ for some $\alpha \in A'$ and hence R is a right strongly prime ternary ring.

Converse follows from the proposition 3.21. □

Definition 3.23. *An ideal I of a ternary ring R is called a right strongly prime ideal if the quotient ternary ring R/I is right strongly prime.*

Theorem 3.24. *Let Q be an ideal of a ternary ring R. Then Q is a right strongly prime ideal of R if and only if for every ideal I of R not contained in Q, there exist finite subsets F' and F of I and R respectively such that $F'Fy \subseteq Q$ implies that $y \in Q$ for all $y \in R$*

Proof. Let Q be a right strongly prime ideal of the ternary ring R. Then the quotient ternary ring R/Q is right strongly prime. Let I be a nonzero ideal of R not contained in Q. Then $(I + Q)/Q$ is a nonzero ideal of the right strongly prime quotient ring R/Q. Hence there exists finite subsets $F' = \{i_1/Q, i_2/Q, i_3/Q......, i_n/Q\}$ of $(I+Q)/Q$ and $F''/Q = \{r_1/Q, r_2/Q, ..., r_n/Q\}$ of R/Q such that $F'(F''/Q)(y/Q) = 0/Q$ implies that $y/Q = 0/Q$ for all $y/Q \in R/Q$. Let $F = \{i_1, i_2, i_3, ...i_n\}$. and $F'' = \{r_1, r_2, ..., r_n\}$. Then F is a finite subset of I. Let $FF''y \subseteq Q$. Then $(F/Q)(F''/Q)(y/Q) = 0/Q$ implies that $y/Q = 0/Q$ for all $y/Q \in R/Q$; so $y \in Q$ for all $y \in R$. Conversely, let I/Q be a nonzero ideal of R/Q. Then I is an ideal of R not contained in Q. Then by hypothesis there exist finite subsets F' and F of I and R respectively such that $F'Fy \subseteq Q$ implies that $y \in Q$ for all $y \in R$. Since F' is a finite subset of I, F/Q is finite subset of I/Q. Let $(F'/Q)(F/Q)(y/Q) = 0/Q$. Then $F'Fy \subseteq Q$ and hence $y \in Q$ i.e $y/Q = 0/Q$. Thus by the theorem 3.7 R/Q is a right strongly prime ternary ring. Hence Q is a right strongly prime ideal of R. □

Corollary 3.25. *An ideal I of a ternary ring R is a right strongly prime ideal if for all $a \notin I$, there exist finite subsets F of $< a >$ and F' of R such that $FF'b \subseteq I$ implies that $b \in I$.*

Proof. Since $a \notin I$, $< a >$ is not properly contained in I. Then by Theorem 3.24, there exist finite subsets F and F' of $< a >$ and R respectively such that $FF'b \subseteq I$ implies that $b \in I$. □

4 Right strongly prime radical

Definition 4.1. *[3] A nonempty subset A of a ternary ring R is called an m-system if for each $a, b, c \in A$ there exist elements x_1, x_2, x_3, x_4 of R such that $ax_1bx_2c \in A$ or $ax_1x_2bx_3x_4c \in A$ or $ax_1x_2bx_3cx_4 \in A$ or $x_1ax_2bx_3x_4c \in A$.*

Theorem 4.2. *A proper ideal P of a ternary ring R is prime if and only its complement $R\backslash I$ is an m-system.*

Proof. The proof is a routine matter of verification and so we omit it. □

Now we generalize the above results in case of right strongly prime ternary rings.

Definition 4.3. *A nonempty subset G of a ternary ring R is called an sp-system if for any $g \in G$ there is a finite subset $F_1 \subseteq \langle g \rangle$ and a finite subset F_2 of R such $F_1F_2z \cap G \neq \phi$ for all $z \in G$.*

Proposition 4.4. *A proper ideal I of a ternary ring R is right strongly prime if and only if $R\backslash I$ is an sp-system.*

Proof. Let I be a right strongly prime ideal of R. Let $g \in R\backslash I$. Then $g \notin I$. So there exist finite subsets F_1 of $\langle g \rangle$ and F_2 of R such that $F_1F_2b \subseteq I$ implies $b \in I$, by using corollary 3.25. This implies that $F_1F_2z \cap (R\backslash I) \neq \phi$ for all $z \in (R\backslash I)$. Hence $R\backslash I$ is an sp-system.

Conversely, suppose that $(R\backslash I)$ is an sp-system. Let $a \notin I$. Then $a \in R\backslash I$. So there exist a finite subset F_1 of $\langle a \rangle$ and F_2 of R such that $F_1F_2z \cap (R\backslash I) \neq \phi$ for all $z \in R\backslash I$. Let $F_1F_2b \subseteq I$. Then $F_1F_2b \cap (R\backslash I) = \phi$. If possible, let $b \notin I$. Then $b \in R\backslash I$ which implies that $F_1F_2b \cap (R\backslash I) \neq \phi$, a contradiction. Consequently, $b \in I$ and hence I is a right strongly prime ideal of R. □

Definition 4.5. *Right strongly prime radical of a ternary ring R is defined by $SP(S) = \cap\{I : I$ is a right strongly prime ideal of $S\}$.*

Definition 4.6. *A pair of subsets (G, P) where P is an ideal of a ternary ring R and G is a nonempty subset of R is called a super sp-system of R if $G \cap P$ contains no nonzero element of R and for any $g \in G$ there is a finite subset F_1 of $\langle g \rangle$ and finite subset F_2 of R such that $F_1F_2z \cap G \neq \phi$ for all $z \notin P$.*

Proposition 4.7. *An ideal I of a ternary R is a right strongly prime ideal if and only if $(R \backslash I, I)$ is a super sp-system of R.*

Theorem 4.8. *For any ternary ring R, $SP(S) = \{x \in R :$ whenever $x \in G$ and (G, P) is a super sp-system for some ideal P of R then $0 \in G\}$……(*).*

Proof. Let $x \in SP(S)$. Let $x \in G$ where (G, P) is a super sp-system. If possible, let $0 \notin G$. Then $G \cap P = \phi$. By Zorn's lemma choose an ideal Q with $P \subseteq Q$ and Q is maximal with respect to the property that $G \cap Q = \phi$. We now prove that Q is a right strongly prime ideal of R. Let $a \notin Q$. Then there is a $g \in G$ such that $\langle g \rangle \subseteq Q + \langle a \rangle$. Since (G, P) is a super sp-system, there exists a finite subset $F_1 = \{g_1, g_2, ..., g_k\} \subseteq \langle g \rangle$ and a finite subset F_2 of R such that $F_1F_2z \cap G \neq \phi$ for all $z \notin P$. Since $F_1 \subseteq Q + \langle a \rangle$, each g_i is of the form $g_i = q_i + a_i$ for some $q_i \in Q$ and $a_i \in \langle a \rangle$. Let $F_1' = \{a_1, a_2, ..., a_k\}$. Then $F_1' \subseteq \langle a \rangle$. Let $z \in R$ such that $F_1'F_2z \subseteq Q$. Then $F_1F_2z \subseteq Q$. If $z \notin Q$ then $F_1F_2z \cap G \neq \phi$, because $P \subseteq Q$. But this contradicts $G \cap Q = \phi$. Hence $z \in Q$. So Q is a right strongly prime ideal. But $x \notin Q$, since $G \cap Q = \phi$, and $x \in G$, a contradiction. Hence $0 \in G$.

Conversely, suppose x belongs to the R.H.S of (*). If possible, let $x \notin SP(R)$. Then there exists a right strongly prime ideal I of R such that $x \notin I$. Then $(R \backslash I, I)$ is a super sp-system where $x \in R \backslash I$ but $0 \notin R \backslash I$, a contradiction. Hence the converse follows. ☐

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