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SOME PROPERTIES ON THE LIE DERIVATIVE OF LINEAR CONNECTIONS ON $\mathbb{R}^n$

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Abstract

The aim of this work is to study some properties of the Lie derivative of the linear connection ∇, the conjugate derivative d_∇ with the linear connection and using them for searching the curvature, the torsion of a space $\mathbb{R}^n$ along the linear flat connection ∇.

1 Introduction

The concept of Lie derivative appeared in the early 30s and was related to the works of Slebodzinski, Dantzig, Schouten and Van Campen ([8]). The problem consisted of generalization of the operations which has an invariant sense only when it is applied to a scalar field, to the case of tensor field and the connection object. The Lie differentiation theory plays an important role in studying automorphisms of differential geometric structures. In a more developed form, this theory is presented by K. Yano ([20]).

In some recent decades, the Lie derivative of forms and its application was investigated by many authors (see [7], [6], [5], [17],[18], [19],[20] and the references given therein). In 2010, Sultanov used the Lie derivative of the linear connection to study the curvature tensor and the sorsion tensor on linear algebras (see [19], pp. 362-412). In 2012, basing on the Lie derivative of real-valued

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forms on the Riemannian $n-$dimensional manifold, N. H. Quang, K. P. Chi and B. C. Van constructed the Lie derivative of the currents on Riemann manifolds and given some applications on Lie groups (see [12]). The primary goal of our work is the extension of the operations of Lie derivative to objects defined on the vector-valued differential forms of a manifold. The main goal of the present work is to investigate some properties on the Lie derivative and the conjugate derivative d_∇ on $\mathbb{R}^n$. For an application, we give some results for searching the curvature, the torsion of a space $\mathbb{R}^n$ along the linear flat connection ∇.

2 Notation and Preliminaries

We denote the vector space of all smooth vector fields on $\mathbb{R}^n$ by $\mathfrak{B}(\mathbb{R}^n)$ and $\mathfrak{F}(\mathbb{R}^n)$ is the vector space of smooth functions on $\mathbb{R}^n$. Let ∇ be a linear connection on $\mathbb{R}^n$ and D be an usual directional derivative that give rise to a linear connection on $\mathbb{R}^n$. More precisely, if $X = X^i\partial_i$ and $Y = Y^j\partial_j$, then we define

$$\nabla_{X^i\partial_i}\left(Y^j\partial_j\right) = X^i\partial_i\left(Y^j\right)\partial_j$$

The torsion tensor R of $\mathbb{R}^n$ is defied by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \forall X, Y \in \mathfrak{B}(\mathbb{R}^n).$$

If $T = 0$, then the linear connection $\bigtriangledown$ of $\mathbb{R}^n$ is said to be *flat*. Then D is the linear flat connection on $\mathbb{R}^n$.

The curvature tensor R of $\mathbb{R}^n$ is defied by

$$R(X,Y,Z) = \bigtriangledown_X \bigtriangledown_Y Z - \bigtriangledown_Y \bigtriangledown_X Z - \bigtriangledown_{[X,Y]}Z,$$

for any $X, Y, Z \in \mathfrak{B}(\mathbb{R}^n)$.

In particular, if $\nabla = D$, then we obtain $T = 0$ and $R = 0$. A vector field X is called parallel if $\nabla_Y X = 0, \forall Y \in \mathfrak{B}(\mathbb{R}^n)$.

Now, let F by any normed vector space and F has finite dimension m. A smooth differential $k-$form on $\mathbb{R}^n$ with values in F, for short, $k-$form on $\mathbb{R}^n$, is any smooth function, $\omega : \mathbb{R}^n \to \bigwedge^k(\mathbb{R}^n, F)$. The vector space of all $k-$forms on $\mathbb{R}^n$ is denoted $\Omega^k(\mathbb{R}^n, F)$. The vector space, $\Omega^*(\mathbb{R}^n, F) = \bigoplus_{k\geq 0} \Omega^k(\mathbb{R}^n, F)$, is the set of differential forms on $\mathbb{R}^n$ with values in F (see [3], pp. 307).

Observe that $\Omega^0(\mathbb{R}^n, F) = C^\infty(\mathbb{R}^n, F)$, the vector space of smooth functions on $\mathbb{R}^n$ with values in F and $\Omega^1(\mathbb{R}^n, F) = C^\infty(\mathbb{R}^n, Hom(\mathbb{R}^n, F))$, the set of smooth functions from $\mathbb{R}^n$ to the set of linear maps from $\mathbb{R}^n$ to F. Also, $\Omega^k(\mathbb{R}^n, F) = (0)$ for $k > n$. Pick any basis, $(f_1, ..., f_m)$, of F. Then, as every differential $k-$form, $\omega \in \Omega^k(\mathbb{R}^n, F)$, can be written uniquely as (see [3], pp.

311)

$$\omega = \sum_{i=1}^{m} \omega_i . f_i, \tag{2.1}$$

where $\omega_1, ..., \omega_m$ are smooth real-valued differential forms in $\Omega^k(\mathbb{R}^n, \mathbb{R})$ and we view f_i as the constant map with value f_i from $\mathbb{R}^n$ to F. Then, as

$$d\omega_p (X_p) = \sum_{i=1}^{m} (\omega_i')_p (X_p) . f_i, \tag{2.2}$$

for all $X_p \in T_p\mathbb{R}^n$, we see that (see [3], pp. 311)

$$d\omega = \sum_{i=1}^{m} d\omega_i . f_i. \tag{2.3}$$

Actually, because $d\omega$ is defined independently of bases, the f_i do not need to be linearly independent; any choices of vectors and forms such that

$$\omega = \sum_{i=1}^{p} \omega_i . f_i, \tag{2.4}$$

will do.

Let G, H be normal vector space. Given a bilinear map, $\phi : F \times G \to H$, a simple calculation shows that for all $\omega \in \Omega^k(\mathbb{R}^n, F)$ and for all $\eta \in \Omega^k(\mathbb{R}^n, G)$, we have (see [3], pp. 311)

$$\omega \wedge_\phi \eta = \sum_{i=1}^{m} \sum_{j=1}^{m'} \omega_i \wedge \eta_j . \phi (f_i, g_j), \tag{2.5}$$

with $\omega = \sum_{i=1}^{m} \omega_i . f_i$ and $\eta = \sum_{j=1}^{m'} \omega_j . f_j$, where $(f_1, ..., f_m)$ is a basis of F and $(g_1, ..., g_{m'})$ is a basis of G.

If F, G, H are finite dimensions and $\phi : F \times G \to H$ is a bilinear map, then for all $\omega \in \Omega^k(\mathbb{R}^n, F)$ and for all $\eta \in \Omega^l(\mathbb{R}^n, G)$, we have (see [3], pp. 312)

$$d (\omega \wedge_\phi \eta) = d\omega \wedge_\phi \eta + (-1)^k \omega \wedge_\phi d\eta. \tag{2.6}$$

Similar to Equation (2.3), we have the inner product of the smooth vector-valued differential forms

$$X \lrcorner \omega = \sum_{i=1}^{m} (X \lrcorner \omega_i) . f_i, \tag{2.7}$$

where $\omega = \sum\limits_{i=1}^{m} \omega_i.f_i$, and $X\lrcorner\omega_1, ..., X\lrcorner\omega_m$ are the inner products of the smooth real-valued differential forms in $\Omega^k(\mathbb{R}^n, \mathbb{R})$ and $X\lrcorner\omega$ is given by:

$$(X\lrcorner\omega)(X_1, ..., X_{k-1}) = \omega(X, X_1, ..., X_{k-1}), \tag{2.8}$$

for all $X_1, X_2, ..., X_{k-1} \in \mathfrak{B}(\mathbb{R}^n)$.

If F, G, H are finite dimensions and $\phi : F \times G \to H$ is a bilinear map, then for all $\omega \in \Omega^k(\mathbb{R}^n, F)$ and all $\psi \in \Omega^l(\mathbb{R}^n, G)$, we easily get the following properties of the inner product of a smooth vector-valued differential form

$$X\lrcorner(\omega \wedge_\phi \psi) = (X\lrcorner\omega) \wedge_\phi \psi + (-1)^k \omega \wedge_\phi (X\lrcorner\psi)\,. \tag{2.9}$$

We have known that the Cartans formula (see [6], pp.35) for the Lie derivative of real-valued differential forms on manifold states $L_X\omega = d(X\lrcorner\omega) + X\lrcorner(d\omega), \forall\omega \in \Omega^k(\mathbb{R}^n)$, that is, $L_X = doX\lrcorner + X\lrcorner od$. The following Formula (2.10) gives the Cartans formula for the Lie derivative of vector-valued differential forms on $\mathbb{R}^n$. For all $\omega \in \Omega^k(\mathbb{R}^n, F), \omega = \sum\limits_{i=1}^{m} \omega_i.f_i$, and $X \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned} d\,(X\lrcorner\omega) + X\lrcorner d\omega &= \sum_{i=1}^{m} d\,(X\lrcorner\omega_i).f_i + \sum_{i=1}^{m} (X\lrcorner d\omega_i).f_i \\ &= \sum_{i=1}^{m} \left[d\,(X\lrcorner\omega_i) + (X\lrcorner d\omega_i)\right].f_i \\ &= \sum_{i=1}^{m} (L_X\omega_i)\,.f_i \\ &= L_X\omega \end{aligned} \tag{2.10}$$

For all $\omega \in \Omega^k(\mathbb{R}^n, F)$, and $X \in \mathfrak{B}(\mathbb{R}^n)$, we have the formula for the Lie derivative of the vector-valued differential $k-$forms ω on $\mathbb{R}^n$

$$(L_X\omega)(X_1, ..., X_k) = L_X(\omega(X_1, ..., X_k)) - \sum_{i=1}^{k} \omega(X_1, ..., L_X X_i, ..., X_k), \tag{2.11}$$

that is the analogous result in ([19], pp. 378).

3 The Lie derivative and the conjugate derivative on $\mathbb{R}^n$

In this section, we consider in case of the normed vector space $F = \mathfrak{B}(\mathbb{R}^n)$. Then a smooth diffferential k-form ω on $\mathbb{R}^n$ with values in $\mathfrak{B}(\mathbb{R}^n)$ is k-form linear, antisymmetric

$$\omega : \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{B}(\mathbb{R}^n) \times \cdots \times \mathfrak{B}(\mathbb{R}^n) \to \mathfrak{B}(\mathbb{R}^n).$$

The main goal of the present work is to investigate some properties on the Lie derivative and the conjugate derivative $d_\bigtriangledown$ on $\mathbb{R}^n$.

Definition 3.1. Given $X \in \mathfrak{B}(\mathbb{R}^n)$ and $\bigtriangledown$ is a linear connection on $\mathbb{R}^n$. The map

$$L_X \bigtriangledown : \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{B}(\mathbb{R}^n) \to \mathfrak{B}(\mathbb{R}^n)$$

satisfying the condition

$$(L_X \bigtriangledown)(Y, Z) = L_X(\bigtriangledown_Y Z) - \bigtriangledown_{L_X Y} Z - \bigtriangledown_Y (L_X Z), \tag{3.1}$$

for all $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$ is called the Lie derivative of the linear connection $\bigtriangledown$ along a vector field X.

Remark 3.2. *For all $X_1, X_2 \in \mathfrak{B}(\mathbb{R}^n)$, we have*

$$L_{X_1+X_2} \bigtriangledown = L_{X_1} \bigtriangledown + L_{X_2} \bigtriangledown \tag{3.2}$$

Proof. For all $X_1, X_2 \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}
(L_{X_1+X_2} \bigtriangledown)(Y, Z) &= L_{X_1+X_2}(\bigtriangledown_Y Z) - \bigtriangledown_{L_{X_1+X_2} Y} Z - \bigtriangledown_Y (L_{X_1+X_2} Z) \\
&= [X_1 + X_2, \bigtriangledown_Y Z] - \bigtriangledown_{[X_1+X_2, Y]} Z - \bigtriangledown_Y ([X_1 + X_2, Z]) \\
&= [X_1, \bigtriangledown_Y Z] + [X_2, \bigtriangledown_Y Z] - \bigtriangledown_{[X_1, Y]} Z - \bigtriangledown_{[X_2, Y]} Z - \\
&\qquad - \bigtriangledown_Y ([X_1, Z]) - \bigtriangledown_Y ([X_2, Z]) \\
&= \big([X_1, \bigtriangledown_Y Z] - \bigtriangledown_{[X_1, Y]} Z - \bigtriangledown_Y ([X_1, Z])\big) + \\
&\qquad + \big([X_2, \bigtriangledown_Y Z] - \bigtriangledown_{[X_2, Y]} Z - \bigtriangledown_Y ([X_2, Z])\big) \\
&= (L_{X_1} \bigtriangledown)(Y, Z) + (L_{X_2} \bigtriangledown)(Y, Z) \\
&= (L_{X_1} \bigtriangledown + L_{X_2} \bigtriangledown)(Y, Z).
\end{aligned}$$

□

Proposition 3.3. *Suppose that ∇ be a linear flat connection on $\mathbb{R}^n$. Then the map $L_X \bigtriangledown : \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{B}(\mathbb{R}^n) \to \mathfrak{B}(\mathbb{R}^n)$ is a bilinear.*

Proof. For all $Y_1, Y_2, Z \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}
(L_X \bigtriangledown)(Y_1 + Y_2, Z) &= L_X(\bigtriangledown_{Y_1+Y_2} Z) - \bigtriangledown_{L_X Y_1+Y_2} Z - \bigtriangledown_{Y_1+Y_2} (L_X Z) \\
&= [X, \bigtriangledown_{Y_1+Y_2} Z] - \bigtriangledown_{[X, Y_1+Y_2]} Z - \bigtriangledown_{Y_1+Y_2} ([X, Z]) \\
&= [X, \bigtriangledown_{Y_1} Z] + [X, \bigtriangledown_{Y_2} Z] - \bigtriangledown_{[X, Y_1]} Z - \bigtriangledown_{[X, Y_2]} Z \\
&\qquad - \bigtriangledown_{Y_1} ([X, Z]) - \bigtriangledown_{Y_2} ([X, Z]) \\
&= (L_X \bigtriangledown)(Y_1, Z) + (L_X \bigtriangledown)(Y_2, Z).
\end{aligned}$$

For all $Y, Z \in \mathfrak{B}(\mathbb{R}^n), \varphi \in \mathfrak{F}(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, we have

$$\begin{aligned}(L_X\triangledown)(\varphi Y, Z) &= L_X(\triangledown_{\varphi Y}Z) - \triangledown_{L_X\varphi Y}Z - \triangledown_{\varphi Y}(L_X Z)\\ &= [X, \triangledown_{\varphi Y}Z] - \triangledown_{[X,\varphi Y]}Z - \triangledown_{\varphi Y}([X,Z])\\ &= [X, \varphi.\triangledown_Y Z] - \triangledown_{X[\varphi].Y+\varphi[X,Y]}Z - \varphi.\triangledown_Y([X,Z])\\ &= X[\varphi].\triangledown_Y Z + \varphi.[X,\triangledown_Y Z] - X[\varphi].\triangledown_Y Z-\\ &\qquad - \varphi.\triangledown_{[X,Y]}Z - \varphi.\triangledown_Y([X,Z])\\ &= \varphi.\big([X,\triangledown_Y Z] - \triangledown_{[X,Y]}Z - \triangledown_Y([X,Z])\big)\\ &= \varphi.(L_X\triangledown)(Y,Z).\end{aligned}$$

For all $Y, Z_1, Z_2 \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}(L_X\triangledown)(Y, Z_1+Z_2) &= L_X(\triangledown_Y(Z_1+Z_2)) - \triangledown_{L_XY}(Z_1+Z_2) - \triangledown_Y(L_X(Z_1+Z_2))\\ &= [X, \triangledown_Y(Z_1+Z_2)] - \triangledown_{[X,Y]}(Z_1+Z_2) - \triangledown_Y([X, Z_1+Z_2])\\ &= [X,\triangledown_Y Z_1] + [X,\triangledown_Y Z_2] - \triangledown_{[X,Y]}Z_1 - \triangledown_{[X,Y]}Z_2\\ &\qquad - \triangledown_Y([X,Z_1]) - \triangledown_Y([X,Z_2])\\ &= (L_X\triangledown)(Y,Z_1) + (L_X\triangledown)(Y,Z_2).\end{aligned}$$

For all $Y, Z \in \mathfrak{B}(\mathbb{R}^n), \varphi \in \mathfrak{F}(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, we have

$$\begin{aligned}(L_X\triangledown)(Y, \varphi.Z) &= L_X(\triangledown_Y(\varphi.Z)) - \triangledown_{L_XY}(\varphi.Z) - \triangledown_Y(L_X(\varphi.Z))\\ &= [X, \triangledown_Y(\varphi.Z)] - \triangledown_{[X,Y]}(\varphi.Z) - \triangledown_Y([X,\varphi.Z])\\ &= [X, Y[\varphi].Z + \varphi\triangledown_Y Z] - [X,Y][\varphi].Z - \varphi.\triangledown_{[X,Y]}Z-\\ &\quad - \triangledown_Y(X[\varphi].Z + \varphi.[X,Z])\\ &= [X, Y[\varphi].Z] + X[\varphi].\triangledown_Y Z + \varphi.[X,\triangledown_Y Z] - [X,Y][\varphi].Z-\\ &\quad - \varphi.\triangledown_{[X,Y]}Z - \triangledown_Y(X[\varphi].Z) - Y[\varphi].[X,Z] - \varphi.\triangledown_Y[X,Z]\\ &= \varphi.(L_X\triangledown)(Y,Z) + [X,Y[\varphi].Z] + X[\varphi].\triangledown_Y Z - [X,Y][\varphi].Z-\\ &\qquad - \triangledown_Y(X[\varphi].Z) - Y[\varphi].[X,Z]\\ &= \varphi.(L_X\triangledown)(Y,Z) + X[Y[\varphi]].Z + Y[\varphi].[X,Z] + X[\varphi].\triangledown_Y Z-\\ &\quad - [X,Y][\varphi].Z - Y[X[\varphi]].Z - X[\varphi].\triangledown_Y Z - Y[\varphi].[X,Z]\\ &= \varphi.(L_X\triangledown)(Y,Z).\end{aligned}$$

□

Now, we note the linear flat connection $\triangledown$ and X is a parallel vector field on $\mathbb{R}^n$. Following theorem gives an application of the Lie derivative to determine the curvature of $\mathbb{R}^n$.

Theorem 3.4. *Suppose that ∇ be a linear flat connection and X be a parallel vector field on $\mathbb{R}^n$. Then we have*

$$(L_X\triangledown)(Y,Z) = R(X,Y,Z), \forall Y, Z \in \mathfrak{B}(\mathbb{R}^n). \tag{3.3}$$

Proof. For any $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}(L_X\triangledown)(Y,Z) &= L_X(\triangledown_Y Z) - \triangledown_{L_XY}Z - \triangledown_Y(L_X Z)\\ &= [X, \triangledown_Y Z] - \triangledown_{L_XY}Z - \triangledown_Y([X,Z])\\ &= \triangledown_X \triangledown_Y Z - \nabla_{\triangledown_Y Z}X - \triangledown_{[X,Y]}Z - \triangledown_Y \triangledown_X Z + \triangledown_Y \triangledown_Z X\\ &= \triangledown_X \triangledown_Y Z - \triangledown_Y \triangledown_X Z - \triangledown_{[X,Y]}Z\\ &= R(X,Y,Z).\end{aligned}$$

This proves the theorem. □

Corollary 3.5. *Suppose that X be a parallel vector field on $\mathbb{R}^n$ and $\nabla = D$. Then we have $L_X D = 0$.*

Proof. By using Equation (3.3), we have

$$(L_X D)(Y,Z) = R(X,Y,Z) = 0, \forall Y, Z \in \mathfrak{B}(\mathbb{R}^n).$$

Hence, $L_X D = 0$. □

Now, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism and f_* be the push-forward of f. The mapping

$$f_*\triangledown : \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{B}(\mathbb{R}^n) \to \mathfrak{B}(\mathbb{R}^n)$$

is defined by the following formula:

$$(f_*\triangledown)(f_*X, f_*Y) = f_*(\triangledown_X Y), \forall X, Y \in \mathfrak{B}(\mathbb{R}^n). \tag{3.4}$$

Then $f_*\triangledown$ is a linear connection on $\mathbb{R}^n$.

Proposition 3.6. *Suppose that ∇ be a linear connection on $\mathbb{R}^n$, $\tilde{X} = f_*X, \tilde{Y} = f_*Y, \forall X, Y \in \mathfrak{B}(\mathbb{R}^n)$. Then we have*

$$L_{f_*X}\left(f_*\triangledown\right)(\tilde{X},\tilde{Y}) = f_*\left(\left(L_X\triangledown\right)(X,Y)\right) \tag{3.5}$$

Proof. For all $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}L_{f_*X}\left(f_*\triangledown\right)(\tilde{X},\tilde{Y}) &= (L_{f_*X}\left(f_*\triangledown\right))(f_*Y, f_*Z)\\ &= L_{f_*X}((f_*\triangledown)_{f_*Y}\left(f_*Z\right)) - (f_*\triangledown)_{L_{f_*X}f_*Y}\left(f_*Z\right) - (f_*\triangledown)_{f_*Y}\left(L_{f_*X}\left(f_*Z\right)\right)\\ &= [f_*X, f_*\left(\triangledown_Y Z\right)] - (f_*\triangledown)_{f_*[X,Y]}\left(f_*Z\right) - (f_*\triangledown)_{f_*Y}\left(f_*\left[X,Z\right]\right)\\ &= f_*\left[X, \triangledown_Y Z\right] - f_*\big(\triangledown_{[X,Y]}Z\big) - f_*\left(\triangledown_Y\left[X,Z\right]\right)\\ &= f_*\big([X, \triangledown_Y Z] - \triangledown_{[X,Y]}Z - \triangledown_Y\left[X,Z\right]\big)\\ &= f_*\left((L_X\triangledown)(Y,Z)\right).\end{aligned}$$

□

Definition 3.7. Given a linear connection ∇ on M. The conjugate derivative with the connection ∇ on $\mathbb{R}^n$ is defined by the following formula:

$$\begin{aligned}(d_\nabla \omega)(X_0, X_1, ..., X_k) = \sum_{i=0}^{k} (-1)^i \nabla_{X_i}(\omega(X_0, X_1, ..., \widehat{X_i}, ..., X_k)) + \\ + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i; X_j], X_0, X_1, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_k),\end{aligned} \tag{3.6}$$

for any $X_0, X_1, ..., X_k \in \mathfrak{B}(\mathbb{R}^n)$ and the covariant derivative ∇_X of $\omega \in \Omega^k(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$ along a vector field X on $\mathbb{R}^n$ is defined by the formula:

$$(\nabla_X \omega)(X_1, ..., X_k) = \nabla_X(\omega(X_1, ..., X_k)) - \sum_{i=1}^{k} \omega(X_1, ..., \nabla_X X_i, ..., X_k), \tag{3.7}$$

for any $X_1, ..., X_k \in \mathfrak{B}(\mathbb{R}^n)$

From definition 3.7, we have the following Remark.

Remark 3.8. *Suppose that ∇ be a linear flat connection, ω be the constant differential $1-$form and X be a vector field on $\mathbb{R}^n$. The identity mapping $I : \mathfrak{B}(\mathbb{R}^n) \to \mathfrak{B}(\mathbb{R}^n)$ is defined by $I(X) = X$, for all $X \in \mathfrak{B}(\mathbb{R}^n)$. The following identities holds:*

i) $L_X \omega = 0$;
ii) $\nabla_X \omega = 0$;
iii) $d_\nabla \omega = 0$;
iv) $d_\nabla I = 0$.

Proof. i) For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$(L_X \omega)(Y) = [X, \omega(Y)] - \omega[X, Y] = \nabla_X(\omega(Y)) - \nabla_{\omega(Y)} X - \omega[X, Y].$$

On the other hand, since ω is a constant, we have $\omega = (\lambda_1 \omega_1, ..., \lambda_n \omega_n)$, with $\lambda_i \in \mathbb{R}$ and $\omega_j = \sum_{i=1}^{n} \alpha_{ij} dx_i, \alpha_{ij} \in \mathbb{R}$. Hence,

$$\nabla_X(\omega(Y)) - \nabla_{\omega(Y)} X - \omega[X, Y] = 0.$$

Consequently $L_X \omega = 0$.

ii) For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$(\nabla_X \omega)(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y) = 0$$

iii) For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$d_\nabla \omega(X, Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X, Y]) = 0$$

iv) For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned} d_\nabla I(X,Y) &= \nabla_X(I(Y)) - \nabla_Y(I(X)) - I([X,Y]) \\ &= \nabla_X Y - \nabla_Y X - [X,Y] \\ &= T(X,Y) = 0 \end{aligned}$$

□

Proposition 3.9. *Suppose that ∇ be a linear flat connection on $\mathbb{R}^n$ and $\omega \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$. Then we have*

$$(d_\nabla \omega)(X,Y) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X), \forall X, Y \in \mathfrak{B}(\mathbb{R}^n). \tag{3.8}$$

Proof. For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned} (\nabla_X \omega)(Y) &= \nabla_X(\omega(Y)) - \omega(\nabla_X Y); \\ (\nabla_Y \omega)(X) &= \nabla_Y(\omega(X)) - \omega(\nabla_Y X). \end{aligned}$$

Hence,

$$\begin{aligned} (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) &= \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega(\nabla_X Y - \nabla_Y X) \\ &= \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega[X,Y] \\ &= (d_\nabla \omega)(X,Y), \forall X, Y \in \mathfrak{B}(\mathbb{R}^n). \end{aligned}$$

□

The following theorem gives a description the formula between the Lie derivative and the conjugate derivative d_∇ on $\mathbb{R}^n$.

Theorem 3.10. *Let $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$ and $X \in \mathfrak{B}(\mathbb{R}^n)$. Then we have*

$$(L_X(d_\nabla \theta))(Y,Z) = (d_\nabla(L_X \theta))(Y,Z) + (L_X \nabla)(Y, \theta(Z)) - (L_X \nabla)(Z, \theta(Y)), \tag{3.9}$$

for all $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$.

Proof. For any $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$ and $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, we have

$$\begin{aligned} (L_X(d_\nabla \theta))(Y,Z) &= [X, (d_\nabla \theta)(Y,Z)] - (d_\nabla \theta)([X,Y], Z) - (d_\nabla \theta)(Y, [X,Z]) \\ &= [X, \nabla_Y(\theta(Z))] - [X, \nabla_Z(\theta(Y))] - [X, \theta(Y,Z)] - \\ &- \big(\nabla_{[X,Y]}(\theta(Z)) - \nabla_Z(\theta([X,Y])) - \theta([[X,Y],Z])\big) - \\ &- \big(\nabla_Y(\theta([X,Z])) - \nabla_{[X,Z]}(\theta(Y)) - \theta([Y,[X,Z]])\big) \\ &= [X, \nabla_Y(\theta(Z))] - [X, \nabla_Z(\theta(Y))] - [X, \theta(Y,Z)] - \\ &- \nabla_{[X,Y]}(\theta(Z)) + \nabla_Z(\theta([X,Y])) + \theta([[X,Y],Z]) - \\ &- \nabla_Y(\theta([X,Z])) + \nabla_{[X,Z]}(\theta(Y)) + \theta([Y,[X,Z]]). \end{aligned} \tag{3.10}$$

On the other hand, we have

$$\begin{aligned}(d_\nabla (L_X\theta))(Y,Z) &= \nabla_Y ((L_X\theta)(Z)) - \nabla_Z ((L_X\theta)(Y)) - (L_X\theta)([Y,Z]) \\ &= \nabla_Y ([X,\theta(Z)] - \theta([X,Z])) - \nabla_Z ([X,\theta(Y)] - \theta([X,Y])) - \\ &- ([X,\theta([Y,Z])] - \theta([X,([Y,Z])])) \\ &= \nabla_Y ([X,\theta(Z)]) - \nabla_Y (\theta([X,Z])) - \nabla_Z ([X,\theta(Y)]) + \\ &+ \nabla_Z (\theta([X,Y])) - [X,\theta([Y,Z])] + \theta([X,([Y,Z])]) .\end{aligned} \tag{3.11}$$

From Equations (3.10, 3.11), we obtain

$$\begin{aligned}\Big(L_X (d_\nabla\theta) - d_\nabla (L_X\theta)\Big)(Y,Z) &= [X,\nabla_Y (\theta(Z))] - [X,\nabla_Z (\theta(Y))] \\ - [X,\theta(Y,Z)] - \nabla_{[X,Y]} (\theta(Z)) &+ \nabla_Z (\theta([X,Y])) + \theta([[X,Y],Z]) - \\ &- \nabla_Y (\theta([X,Z])) + \nabla_{[X,Z]} (\theta(Y)) + \theta([Y,[X,Z]]) \\ &- \nabla_Y ([X,\theta(Z)]) + \nabla_Y (\theta([X,Z])) + \nabla_Z ([X,\theta(Y)]) - \\ &- \nabla_Z (\theta([X,Y])) + [X,\theta([Y,Z])] - \theta([X,([Y,Z])]) \\ &= [X,\nabla_Y (\theta(Z))] - [X,\nabla_Z (\theta(Y))] - \nabla_{[X,Y]} (\theta(Z)) + \\ &+ \theta([[X,Y],Z]) + \nabla_{[X,Z]} (\theta(Y)) + \theta([Y,[X,Z]]) - \\ &- \nabla_Y ([X,\theta(Z)]) + \nabla_Z ([X,\theta(Y)]) - \theta([X,([Y,Z])]) \\ &= [X,\nabla_Y (\theta(Z))] - [X,\nabla_Z (\theta(Y))] - \nabla_{[X,Y]} (\theta(Z)) + \\ &+ \nabla_{[X,Z]} (\theta(Y)) - \nabla_Y ([X,\theta(Z)]) + \nabla_Z ([X,\theta(Y)]) \\ &= (L_X\nabla)(Y,\theta(Z)) - (L_X\nabla)(Z,\theta(Y)) .\end{aligned}$$

Consequently,

$$(L_X (d_\nabla\theta))(Y,Z) = (d_\nabla (L_X\theta))(Y,Z) + (L_X\nabla)(Y,\theta(Z)) - (L_X\nabla)(Z,\theta(Y)) .$$

This proves the theorem. □

Corollary 3.11. *If X is a parallel vector field on $\mathbb{R}^n$ and $\nabla = D$ then we have*

$$L_X (d_D\theta) = d_D (L_X\theta) , \tag{3.12}$$

for all $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$.

Proof. For all $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, applying Theorem 3.10, we obtain

$$(L_X (d_D\theta))(Y,Z) = (d_D (L_X\theta))(Y,Z) + (L_X\nabla)(Y,\theta(Z)) - (L_X\nabla)(Z,\theta(Y))$$

Since X is parallel vector field on $\mathbb{R}^n$, thus applying Theorem 3.5, we obtain

$$(L_X (d_\nabla\theta))(Y,Z) = (d_\nabla (L_X\theta))(Y,Z) , \text{ for any } Y,Z \in \mathfrak{B}(\mathbb{R}^n).$$

Hence, $L_X (d_\nabla\theta) = d_\nabla (L_X\theta)$, for any $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$. □

Proposition 3.12. *Suppose that* $X, Y, Z \in \mathfrak{B}(\mathbb{R}^n), \omega \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$. *Then we have*

$$(L_X (\nabla_Y \omega))(Z) - (\nabla_Y (L_X \omega))(Z) = (L_X \nabla_Y)(\omega(Z)) - \omega[[X, Y], Z]. \quad (3.13)$$

Proof. For all $X, Y, Z \in \mathfrak{B}(\mathbb{R}^n), \omega \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, we have

$$\begin{aligned}(L_X (\nabla_Y \omega))(Z) &= [X, (\nabla_Y \omega)(Z)] - (\nabla_Y \omega)[X, Z] \\ &= [X, \nabla_Y (\omega(Z)) - \omega[Y, Z]] - \nabla_Y (\omega[X, Z]) + \omega[Y, [X, Z]] \\ &= [X, \nabla_Y (\omega(Z))] - [X, \omega[Y, Z]] - \nabla_Y (\omega[X, Z]) + \omega[Y, [X, Z]]\end{aligned} \quad (3.14)$$

$$\begin{aligned}(\nabla_Y (L_X \omega))(Z) &= \nabla_Y ((L_X \omega)(Z)) - (L_X \omega)[Y, Z] \\ &= \nabla_Y ([X, \omega(Z)] - \omega[X, Z]) - [X, \omega[Y, Z]] + \omega[X, [Y, Z]] \\ &= \nabla_Y [X, \omega(Z)] - \nabla_Y (\omega[X, Z]) - [X, \omega[Y, Z]] + \omega[X, [Y, Z]]\end{aligned} \quad (3.15)$$

From Equations (3.14, 3.15), we obtain

$$\begin{aligned}(L_X (\nabla_Y \omega))(Z) - (\nabla_Y (L_X \omega))(Z) &= [X, \nabla_Y (\omega(Z))] - \nabla_Y [X, \omega(Z)] + [Z, [X, Y]] \\ &= (L_X \nabla_Y)(\omega(Z)) - \omega[[X, Y], Z], \forall \omega \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n)).\end{aligned}$$

□

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