# HOMODERIVATIONS AND COMMUTATIVITY OF \*-PRIME RINGS

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#### Abstract

In this paper, we prove the commutativity of \*-prime rings admitting homoderivations which commute with \* and satisfy certain conditions on \*-ideals.

## 1 Introduction

Throughout this paper R represents a ring with center Z(R). For any  $x,y\in R$ , the commutator xy-yx will be denoted by [x,y], while the the anticommutator xy+yx will be denoted by  $x\circ y$ . An additive mapping  $*:R\to R$  is called an involution on R if  $(xy)^*=y^*x^*$  and  $(x^*)^*=x$  for all  $x,y\in R$ . A ring R equipped with an involution \* is called a ring with an involution \* or a \*-ring. The set of symmetric and skew elements of R will be denoted by  $S_*(R)=\{x\in R|x^*=\pm x\}$  (see [3]). An ideal I of R is a \*-ideal if  $I^*=I$ . A ring R with an involution \* is \*-prime if  $xRy=0=xRy^*$  implies that x=0 or y=0 (or equivalently  $xRy=0=x^*Ry$  implies that x=0 or y=0). Clearly, every prime ring having an involution \* is \*-prime but the converse is not true in general. However, if R is a \*-prime ring such that  $x\in R$  and xRx=0, then  $xRxRx^*=0$ . By \*-primeness of R, it follows that x=0 or  $xRx^*=0$ . If  $xRx^*=0$ , then xRx=0 is a semiprime ring.

An additive mapping  $h: R \to R$  is called a homoderivation on R if h(xy) = h(x)h(y) + h(x)y + xh(y) for all  $x, y \in R$ . An example of such mapping is to let h(x) = f(x) - x for all  $x \in R$  where f is an endomorphism on R.

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For  $S \subseteq R$ , a mapping  $f: R \to R$  is said to be centralizing on S if  $[x, f(x)] \in Z(R)$  for all  $x \in S$ ; and f is called zero-power valued on S if  $f(S) \subseteq S$  and if for each  $x \in S$ , there exists a positive integer n(x) > 1 such that  $f^{n(x)}(x) = 0$ .

Ashraf and Siddeeque [1] and Oukhtite and Salhi [4] proved the commutativity of \*-prime rings under suitable differential conditions. In this paper, we prove commutativity theorems analogous to some of the results presented in [1, 4] using the concept of homoderivations. In particular, under some restrictions, we prove the commutativity of \*-prime rings satisfying any of the following conditions on \*-ideals:

i 
$$[h(x), x] \in Z(R)$$
,  
ii  $[ah(x), x] = 0$  where  $0 \neq a \in S_*(R)$ ,  
iii  $h([x, y]) = 0$ ,  
iv  $h(x \circ y) = 0$ ,  
v  $h([x, y]) = [x, y]$ , or  
vi  $h(x \circ y) = x \circ y$ .

# 2 Preliminary Results

We start with the following lemma which is essential for proving our results.

**Lemma 2.1 ([4], Lemma 1).** Let R be a \*-prime ring and let I be a nonzero \*-ideal of R. If  $x, y \in R$  are such that  $xIy = 0 = xIy^*$ , then x = 0 or y = 0 (or equivalently  $xIy = 0 = x^*Iy$ , then x = 0 or y = 0).

Now we prove the following lemmas which will frequently be used in developing the proofs of our main results.

**Lemma 2.2.** Let R be a \*-prime ring, I a nonzero \*-ideal of R, and h a nonzero homoderivation on R which commutes with \*. If [x,R]Ih(x)=0 for all  $x \in I$ , then R is commutative.

**Proof.** By hypothesis, we have

$$[x, R]Ih(x) = 0 \qquad \text{for all } x \in I. \tag{1}$$

For any  $x \in I$ , we have  $t = x - x^* \in I$ . It follows by (1) that [t, r]Ih(t) = 0 for

all  $r \in R$ . Since  $t^* = (x - x^*)^* = x^* - x = -t$ , we find that

$$\begin{split} ([t,r])^*Ih(t) &= (tr - rt)^*Ih(t) \\ &= (r^*t^* - t^*r^*)Ih(t) \\ &= (-r^*t + tr^*)Ih(t) \\ &= ([t,r^*])Ih(t) \\ &= 0. \end{split}$$

Thus,  $[t, r]Ih(t) = 0 = ([t, r])^*Ih(t)$  for all  $r \in R$ . According to Lemma 2.1, we have

$$[t, r] = 0$$
 or  $h(t) = 0$ . (2)

Therefore, for each  $x \in I$ , we have either

$$[x,r] = [x^*,r]$$
 or  $h(x) = h(x^*)$ . (3)

Suppose that  $h(x) = h(x^*)$ . Since h commutes with \*,  $h(x) = (h(x))^*$ . Therefore,  $0 = [x, r]Ih(x) = [x, r]I(h(x))^*$  for all  $r \in R$ . Thus, h(x) = 0 or [x, r] = 0, by Lemma 2.1.

Now suppose that  $[x, r] = [x^*, r]$ . Observe that

$$([x,r])^*Ih(x) = (xr - rx)^*Ih(x)$$

$$= (r^*x^* - x^*r^*)Ih(x)$$

$$= [r^*, x^*]Ih(x)$$

$$= [r^*, x]Ih(x)$$

$$= -[x, r^*]Ih(x)$$

Therefore,  $[x, r]Ih(x) = 0 = ([x, r])^*Ih(x)$  for all  $r \in R$  and so h(x) = 0 or [x, r] = 0, by Lemma 2.1.

Hence, both conditions in (3) imply that for each  $x \in I$ , either

$$h(x) = 0$$
 or  $x \in Z(R)$ .

Notice that the sets of  $x \in I$  for which these two conditions hold are additive subgroups of I whose union is I; but since a group cannot be the union of two of its proper subgroups, we have either

$$h(I) = 0$$
 or  $I \subseteq Z(R)$ . (4)

If h(I) = 0, then h(x) = 0 for all  $x \in I$ . Therefore, for all  $r \in R$ , 0 = h(xr) = h(x)h(r) + h(x)r + xh(r) = xh(r). Hence, Ih(r) = 0 for all  $r \in R$ . This implies that

$$IRh(r) = 0 = I^*Rh(r)$$
 for all  $r \in R$ .

By \*-primeness of R, h=0 which is a contradiction. From (4), it follows that  $I \subseteq Z(R)$ . Let  $r, s \in R$  and  $x \in I$ . Then, rsx = rxs = srx and so [r, s]x = 0. Thus, [r, s]I = 0 and

$$[r, s]RI = 0 = [r, s]RI^*$$
 for all  $r, s \in R$ .

By \*-primeness of R, [r, s] = 0 for all  $r, s \in R$ . Hence, R is commutative.  $\square$ 

**Lemma 2.3.** Let R be a \*-prime ring, I a nonzero \*-ideal of R, and h a nonzero homoderivation on R which commutes with \*. If h is zero-power valued on I and [h(x), x] = 0 for all  $x \in I$ , then R is commutative.

**Proof.** By hypothesis, we have

$$[h(x), x] = 0 \qquad \text{for all } x \in I. \tag{5}$$

Linearizing (5), we obtain

$$[h(x), y] + [h(y), x] = 0$$
 for all  $x, y \in I$ . (6)

Replacing y by yx, we get [h(x), yx] + [h(yx), x] = 0. Expanding this and using (5), we get

$$[h(x), y]x + [h(y), x]h(x) + [h(y), x]x + [y, x]h(x) = 0$$
 for all  $x, y \in I$ .

Applying (6), we get

$$[h(y) + y, x]h(x) = 0$$
 for all  $x, y \in I$ .

Since h is zero-power valued on I, we can replace y by  $y - h(y) + h^2(y) + \cdots + (-1)^{n(y)-1}h^{n(y)-1}(y)$  to get

$$[x, y]h(x) = 0$$
 for all  $x, y \in I$ .

Replacing y by ry for arbitrary  $r \in R$ , we obtain 0 = [x, ry]h(x) = [x, r]yh(x) for all  $x, y \in I$ . Hence, [x, R]Ih(x) = 0 for all  $x \in I$ . By Lemma 2.2, R is commutative.

**Lemma 2.4.** Let R be a \*-prime ring and let I be a nonzero \*-ideal of R. If  $x \in R$  and x centralizes I, then  $x \in Z(R)$ .

**Proof.** Let  $x \in R$ . Suppose that [x, u] = 0 for all  $u \in I$ . Then, for arbitrary  $r \in R$ , we have 0 = [x, ru] = [x, r]u for all  $u \in I$ . That is, [x, R]I = 0. Thus,

$$[x, R]RI = 0 = [x, R]RI^*.$$

Since R is \*-prime, we conclude that [x, R] = 0 and hence  $x \in Z(R)$ .

## 3 The Main Results

The study of centralizing mappings and commutativity of certain rings began in the 1950's. Posner [5] established the commutativity of prime rings admitting nonzero centralizing derivations. El Sofy [2] proved an analogous result concerning homodervations. More recently, Oukhtite and Salhi [4] proved the commutativity of \*-prime rings applying Posner's conditions on \*-ideals. Motivated by this work, we explore the commutativity of \*-prime rings admitting centralizing homoderivations and we prove the following theorem.

**Theorem 3.1.** Let R be a \*-prime ring with characteristic different from two, I a nonzero \*-ideal of R, and h a nonzero homoderivation on R which commutes with \*. If h is centralizing and zero-power valued on I, then R is commutative.

**Proof.** By hypothesis, we have

$$[h(x), x] \in Z(R)$$
 for all  $x \in I$ . (7)

Linearizing (7), we obtain

$$[h(x), y] + [h(y), x] \in Z(R)$$
 for all  $x, y \in I$ .

Replacing y by  $x^2$ , we get  $[h(x), x^2] + [h(x^2), x] \in Z(R)$  which can be extended to

$$x[h(x), x] + [h(x), x]x + h(x)[h(x), x] + [h(x), x]h(x) + [h(x), x]x + x[h(x), x] \in Z(R)$$

for all  $x \in I$ . Applying (7) yields

$$(4x + 2h(x))[h(x), x] \in Z(R)$$
 for all  $x \in I$ .

Since char  $R \neq 2$ ,

$$(2x + h(x))[h(x), x] \in Z(R)$$
 for all  $x \in I$ .

Thus, for arbitrary  $r \in R$ , we have [(2x+h(x))[h(x),x],r]=0. Expanding this and using (7) yields

$$[2x + h(x), r][h(x), x] = 0 \qquad \text{for all } x \in I, r \in R.$$

In particular, [2x + h(x), x][h(x), x] = 0 for all  $x \in I$ . This can be simplified to

$$[h(x), x]^2 = 0 \qquad \text{for all } x \in I. \tag{8}$$

Since every \*-prime ring is semiprime and since the center of semiprime rings contains no nonzero nilpotent elements, we find that [h(x), x] = 0 for all  $x \in I$  and hence by Lemma 2.3, R is commutative.

Also, Oukhtite and Salhi [4] proved that if a \*-prime ring R with characteristic different from two has a nonzero derivation d which commutes with \* and satisfies [ad(x),x]=0 on \*-ideals, then a=0 or R is commutative. Our next result will provide an analogous conclusion using the concept of homoderivations.

**Theorem 3.2.** Let R be a \*-prime ring with characteristic different from two, I a nonzero \*-ideal of R, and h a nonzero homoderivation on R which commutes with \*. If h is zero-power valued on I and  $a \in S_*(R)$  such that [ah(x), x] = 0 for all  $x \in I$ , then a = 0 or R is commutative.

**Proof.** By hypothesis, we have

$$[ah(x), x] = 0 \qquad \text{for all } x \in I. \tag{9}$$

Linearizing (9), we obtain

$$[ah(x), y] + [ah(y), x] = 0$$
 for all  $x, y \in I$ . (10)

Replacing y by yx, we get [ah(x),yx]+[ah(y)h(x),x]+[ah(y)x,x]+[ayh(x),x]=0 which is equivalent to y[ah(x),x]+[ah(x),y]x+ah(y)[h(x),x]+[ah(y),x]h(x)+[ah(y),x]x+ay[h(x),x]+a[y,x]h(x)+[a,x]yh(x)=0. Applying (9) and (10), yields

$$ah(y)[h(x), x] + [ah(y), x]h(x) + ay[h(x), x] + a[y, x]h(x) + [a, x]yh(x) = 0$$
 for all  $x, y \in I$ .

This can be written as

$$a(h(y)+y)[h(x),x]+[a,x](h(y)+y)h(x)+a[h(y)+y,x]h(x)=0$$
 for all  $x,y\in I$ .

Since h is zero-power valued on I,

$$ay[h(x), x] + [a, x]yh(x) + a[y, x]h(x) = 0$$
 for all  $x, y \in I$ . (11)

Replacing y by ay, we get

$$a^{2}y[h(x), x] + [a, x]ayh(x) + a^{2}[y, x]h(x) + a[a, x]yh(x) = 0$$
 for all  $x, y \in I$ . (12)

Applying (11) to (12) yields [a, x]ayh(x) = 0 for all  $x, y \in I$ . That is,

$$[a, x]aIh(x) = 0 for all x \in I. (13)$$

Observe that for  $x \in I \cap S_*(R)$  we have  $x^* = \pm x$ . Thus, since h commutes with \*, we have  $(h(x))^* = h(x^*) = \pm h(x)$ . So, by (13),  $[a,x]aIh(x) = 0 = [a,x]aI(h(x))^*$  and by Lemma 2.1, it follows that [a,x]a = 0 or h(x) = 0. Now consider  $y \in I$ . Since  $(y+y^*) \in I \cap S_*(R)$ , by the above observation we have

$$[a, y + y^*]a = 0$$
 or  $h(y + y^*) = 0.$  (14)

#### Case1:

Let  $h(y + y^*) = 0$ . Then,  $h(y) = -h(y^*) = -(h(y))^*$ . Therefore, by (13), we have  $0 = [a, y]aIh(y) = [a, y]aI(h(y))^*$  and by Lemma 2.1, it follows that [a, y]a = 0 or h(y) = 0.

### Case2:

Let  $[a, y + y^*]a = 0$ . Since  $(y - y^*) \in I \cap S_*(R)$ , by the above observation we have either  $h(y - y^*) = 0$  or  $[a, y - y^*]a = 0$ . If  $h(y - y^*) = 0$ , then by a similar approach to Case 1, we get [a, y]a = 0 or h(y) = 0. If  $[a, y - y^*]a = 0$ , then  $[a, y - y^*]a + [a, y + y^*]a = 0$  which can be reduced to 2[a, y]a = 0. Since char  $R \neq 2$ , [a, y]a = 0.

Thus, both cases in (14) imply that for each  $y \in I$ ,

$$[a, y]a = 0$$
 or  $h(y) = 0$ .

Notice that the sets of  $y \in I$  for which these two conditions hold are additive subgroups of I whose union is I; but since a group cannot be the union of two of its proper subgroups, we have either

$$[a, I]a = 0$$
 or  $h(I) = 0.$  (15)

If h(I)=0, then h(x)=0 for all  $x\in I$ . Then, for arbitrary  $r\in R$ , 0=h(rx)=h(r)h(x)+h(r)x+rh(x)=h(r)x. Thus, h(r)I=0 for all  $r\in R$ ; and  $h(r)RI=0=h(r)RI^*$  for all  $r\in R$ . By \*-primeness of R, h=0 which is a contradiction.

Consequently, we must have [a, I]a = 0. Then, [a, x]a = 0 for all  $x \in I$ . Replacing x by xy yields [a, x]ya = 0 for all  $x, y \in I$ . Thus,

$$[a, x]Ia = 0$$
 for all  $x \in I$ .

As  $a \in S_*(R)$ , then

$$0 = [a, x]Ia = [a, x]Ia^*$$
 for all  $x \in I$ .

By Lemma 2.1, a centralizes I or a=0. By Lemma 2.4,  $a \in Z(R)$  or a=0. If  $0 \neq a \in Z(R)$ , then by (9), 0 = [ah(x), x] = a[h(x), x] + [a, x]h(x) = a[h(x), x]. Since  $a \in Z(R)$ , aR[h(x), x] = 0. Since  $a \in S_*(R)$ ,

$$0 = aR[h(x), x] = a^*R[h(x), x] \quad \text{for all } x \in I.$$

As  $a \neq 0$ , then \*-primeness of R implies that [h(x), x] = 0 for all  $x \in I$ . It follows from Lemma 2.3 that R is commutative.

Ashraf and Siddeeque [1] studied the commutativity of \*-prime rings admitting nonzero derivations which commute with \* and satisfy any one of the following identities on \*-ideals:

1. 
$$h([x, y]) = 0$$
,

- 2.  $h(x \circ y) = 0$ ,
- 3. h([x,y]) = [x,y], or
- 4.  $h(x \circ y) = x \circ y$ .

Investigating these identities on homoderivations, we obtain the following two results.

**Theorem 3.3.** Let R be a \*-prime ring, I a nonzero \*-ideal of R, and h a nonzero homoderivation on R which commutes with \*. If h satisfies either

- 1. h([x,y]) = 0 for all  $x, y \in I$ , or
- 2.  $h(x \circ y) = 0$  for all  $x, y \in I$ ,

then R is commutative.

**Proof.** (i) By hypothesis, we have

$$h([x,y]) = 0 \qquad \text{for all } x, y \in I. \tag{16}$$

Replacing y by yx yields 0 = h([x, yx]) = h([x, y]x) = h([x, y])h(x) + h([x, y])x + [x, y]h(x) for all  $x, y \in I$ . Applying (16), we get

$$[x, y]h(x) = 0 \qquad \text{for all } x, y \in I. \tag{17}$$

Replacing y by ry for arbitrary  $r \in R$  gives [x, ry]h(x) = 0 for all  $x, y \in I$ . Expanding this and using (17), we get

$$[x, r]yh(x) = 0$$
 for all  $x, y \in I, r \in R$ .

Therefore, [x, R]Ih(x) = 0 for all  $x \in I$ . By Lemma 2.2, R is commutative.

(ii) By hypothesis, we have

$$h(x \circ y) = 0 \qquad \text{for all } x, y \in I. \tag{18}$$

Replacing y by yx yields  $0 = h(x \circ yx) = h((x \circ y)x) = h(x \circ y)h(x) + h(x \circ y)x + (x \circ y)h(x)$  for all  $x, y \in I$ . Applying (18), we get

$$(x \circ y)h(x) = 0$$
 for all  $x, y \in I$ .

This is equivalent to

$$xyh(x) = -yxh(x)$$
 for all  $x, y \in I$ . (19)

Replacing y by ry for arbitrary  $r \in R$  gives

$$xryh(x) = -ryxh(x)$$
 for all  $x, y \in I$ . (20)

From (19) and (20), it follows that xryh(x) = rxyh(x) for all  $x, y \in I$ . Thus, [x, r]yh(x) = 0 for all  $x, y \in I$  and  $r \in R$ . Therefore, [x, R]Ih(x) = 0 for all  $x \in I$ . By Lemma 2.2, R is commutative.

**Theorem 3.4.** Let R be a \*-prime ring with characteristic different from two, I a nonzero \*-ideal of R and let h be a nonzero homoderivation on R which commutes with \*. If If h satisfies either

1. 
$$h([x,y]) = [x,y]$$
 for all  $x,y \in I$ , or

2. 
$$h(x \circ y) = x \circ y$$
 for all  $x, y \in I$ ,

then R is commutative.

**Proof.** (i) By hypothesis, we have

$$h([x,y]) = [x,y] \qquad \text{for all } x,y \in I. \tag{21}$$

Replacing y by yx, we get h([x,y]x) = [x,y]x for all  $x,y \in I$ . Thus,

$$h([x,y])h(x) + h([x,y])x + [x,y]h(x) = [x,y]x$$
 for all  $x, y \in I$ .

Applying (21), we get

$$2[x, y]h(x) = 0$$
 for all  $x, y \in I$ .

Since char  $R \neq 2$ ,

$$[x, y]h(x) = 0 for all x, y \in I. (22)$$

Replacing y by ry for arbitrary  $r \in R$  gives [x, ry]h(x) = 0 for all  $x, y \in I$ . Expanding this and using (22), we get

$$[x, r]yh(x) = 0$$
 for all  $x, y \in I, r \in R$ .

Therefore, [x, R]Ih(x) = 0 for all  $x \in I$ . By Lemma 2.2, R is commutative.

(ii) By hypothesis, we have

$$h(x \circ y) = x \circ y$$
 for all  $x, y \in I$ . (23)

Replacing y by yx, we get  $h((x \circ y)x) = (x \circ y)x$  for all  $x, y \in I$ . Thus,

$$h(x \circ y)h(x) + h(x \circ y)x + (x \circ y)h(x) = (x \circ y)x$$
 for all  $x, y \in I$ .

Applying (23), we get

$$2(x \circ y)h(x) = 0$$
 for all  $x, y \in I$ .

Since char  $R \neq 2$ ,

$$(x \circ y)h(x) = 0$$
 for all  $x, y \in I$ .

This is equivalent to

$$xyh(x) = -yxh(x)$$
 for all  $x, y \in I$ . (24)

Replacing y by ry for arbitrary  $r \in R$  gives

$$xryh(x) = -ryxh(x)$$
 for all  $x, y \in I$ . (25)

From (24) and (25), it follows that xryh(x) = rxyh(x) for all  $x, y \in I$ . Thus, [x, r]yh(x) = 0 for all  $x, y \in I$  and  $r \in R$ . Therefore, [x, R]Ih(x) = 0 for all  $x \in I$ . By Lemma 2.2, R is commutative.

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