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HOMODERIVATIONS AND COMMUTATIVITY OF *-PRIME RINGS

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Abstract

In this paper, we prove the commutativity of $*$-prime rings admitting homoderivations which commute with $*$ and satisfy certain conditions on $*$-ideals.

1 Introduction

Throughout this paper R represents a ring with center $Z(R)$. For any $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$, while the the anticommutator $xy + yx$ will be denoted by $x \circ y$. An additive mapping $* : R \to R$ is called an involution on R if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring R equipped with an involution $*$ is called a ring with an involution $*$ or a $*$-ring. The set of symmetric and skew elements of R will be denoted by $S_*(R) = \{x \in R | x^* = \pm x\}$ (see [3]). An ideal I of R is a $*$-ideal if $I^* = I$. A ring R with an involution $*$ is $*$-prime if $xRy = 0 = xRy^*$ implies that $x = 0$ or $y = 0$ (or equivalently $xRy = 0 = x^*Ry$ implies that $x = 0$ or $y = 0$). Clearly, every prime ring having an involution $*$ is $*$-prime but the converse is not true in general. However, if R is a $*$-prime ring such that $x \in R$ and $xRx = 0$, then $xRxRx^* = 0$. By $*$-primeness of R, it follows that $x = 0$ or $xRx^* = 0$. If $xRx^* = 0$, then $xRx = 0 = xRx^*$. Since R is $*$-prime, we have $x = 0$. Hence, every $*$-prime ring is a semiprime ring.

An additive mapping $h : R \to R$ is called a homoderivation on R if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in R$. An example of such mapping is to let $h(x) = f(x) - x$ for all $x \in R$ where f is an endomorphism on R.

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For $S \subseteq R$, a mapping $f : R \to R$ is said to be centralizing on S if $[x, f(x)] \in Z(R)$ for all $x \in S$; and f is called zero-power valued on S if $f(S) \subseteq S$ and if for each $x \in S$, there exists a positive integer $n(x) > 1$ such that $f^{n(x)}(x) = 0$.

Ashraf and Siddeeque [1] and Oukhtite and Salhi [4] proved the commutativity of $*$-prime rings under suitable differential conditions. In this paper, we prove commutativity theorems analogous to some of the results presented in [1, 4] using the concept of homoderivations. In particular, under some restrictions, we prove the commutativity of $*$-prime rings satisfying any of the following conditions on $*$-ideals:

i $[h(x), x] \in Z(R)$,

ii $[ah(x), x] = 0$ where $0 \neq a \in S_*(R)$,

iii $h([x, y]) = 0$,

iv $h(x \circ y) = 0$,

v $h([x, y]) = [x, y]$, or

vi $h(x \circ y) = x \circ y$.

2 Preliminary Results

We start with the following lemma which is essential for proving our results.

Lemma 2.1 ([4], Lemma 1). *Let R be a $*$-prime ring and let I be a nonzero $*$-ideal of R. If $x, y \in R$ are such that $xIy = 0 = xIy^*$, then $x = 0$ or $y = 0$ (or equivalently $xIy = 0 = x^*Iy$, then $x = 0$ or $y = 0$).*

Now we prove the following lemmas which will frequently be used in developing the proofs of our main results.

Lemma 2.2. *Let R be a $*$-prime ring, I a nonzero $*$-ideal of R, and h a nonzero homoderivation on R which commutes with $*$. If $[x, R]Ih(x) = 0$ for all $x \in I$, then R is commutative.*

Proof. By hypothesis, we have

$$[x, R]Ih(x) = 0 \qquad \text{for all} \;\; x \in I. \tag{1}$$

For any $x \in I$, we have $t = x - x^* \in I$. It follows by (1) that $[t, r]Ih(t) = 0$ for all $r \in R$. Since $t^* = (x - x^*)^* = x^* - x = -t$, we find that

$$\begin{aligned}([t,r])^* Ih(t) &= (tr - rt)^* Ih(t)\\ &= (r^*t^* - t^*r^*)Ih(t)\\ &= (-r^*t + tr^*)Ih(t)\\ &= ([t,r^*])Ih(t)\\ &= 0.\end{aligned}$$

Thus, $[t,r]Ih(t) = 0 = ([t,r])^* Ih(t)$ for all $r \in R$. According to Lemma 2.1, we have

$$[t,r] = 0 \qquad \text{or} \qquad h(t) = 0. \tag{2}$$

Therefore, for each $x \in I$, we have either

$$[x,r] = [x^*,r] \qquad \text{or} \qquad h(x) = h(x^*). \tag{3}$$

Suppose that $h(x) = h(x^*)$. Since h commutes with $*$, $h(x) = (h(x))^*$. Therefore, $0 = [x,r]Ih(x) = [x,r]I(h(x))^*$ for all $r \in R$. Thus, $h(x) = 0$ or $[x,r] = 0$, by Lemma 2.1.
Now suppose that $[x,r] = [x^*,r]$. Observe that

$$\begin{aligned}([x,r])^* Ih(x) &= (xr - rx)^* Ih(x)\\ &= (r^*x^* - x^*r^*)Ih(x)\\ &= [r^*,x^*]Ih(x)\\ &= [r^*,x]Ih(x)\\ &= -[x,r^*]Ih(x)\\ &= 0.\end{aligned}$$

Therefore, $[x,r]Ih(x) = 0 = ([x,r])^* Ih(x)$ for all $r \in R$ and so $h(x) = 0$ or $[x,r] = 0$, by Lemma 2.1.
Hence, both conditions in (3) imply that for each $x \in I$, either

$$h(x) = 0 \qquad \text{or} \qquad x \in Z(R).$$

Notice that the sets of $x \in I$ for which these two conditions hold are additive subgroups of I whose union is I; but since a group cannot be the union of two of its proper subgroups, we have either

$$h(I) = 0 \qquad \text{or} \qquad I \subseteq Z(R). \tag{4}$$

If $h(I) = 0$, then $h(x) = 0$ for all $x \in I$. Therefore, for all $r \in R$, $0 = h(xr) = h(x)h(r) + h(x)r + xh(r) = xh(r)$. Hence, $Ih(r) = 0$ for all $r \in R$. This implies that

$$IRh(r) = 0 = I^*Rh(r) \qquad \text{for all } \ r \in R.$$

By $*$-primeness of R, $h = 0$ which is a contradiction. From (4), it follows that $I \subseteq Z(R)$. Let $r, s \in R$ and $x \in I$. Then, $rsx = rxs = srx$ and so $[r, s]x = 0$. Thus, $[r, s]I = 0$ and

$$[r, s]RI = 0 = [r, s]RI^* \qquad \text{for all } \ r, s \in R.$$

By $*$-primeness of R, $[r, s] = 0$ for all $r, s \in R$. Hence, R is commutative. □

Lemma 2.3. *Let R be a $*$-prime ring, I a nonzero $*$-ideal of R, and h a nonzero homoderivation on R which commutes with $*$. If h is zero-power valued on I and $[h(x), x] = 0$ for all $x \in I$, then R is commutative.*

Proof. By hypothesis, we have

$$[h(x), x] = 0 \qquad \text{for all } \ x \in I. \tag{5}$$

Linearizing (5), we obtain

$$[h(x), y] + [h(y), x] = 0 \qquad \text{for all } \ x, y \in I. \tag{6}$$

Replacing y by yx, we get $[h(x), yx] + [h(yx), x] = 0$. Expanding this and using (5), we get

$$[h(x), y]x + [h(y), x]h(x) + [h(y), x]x + [y, x]h(x) = 0 \qquad \text{for all } \ x, y \in I.$$

Applying (6), we get

$$[h(y) + y, x]h(x) = 0 \qquad \text{for all } \ x, y \in I.$$

Since h is zero-power valued on I, we can replace y by $y - h(y) + h^2(y) + \cdots + (-1)^{n(y)-1}h^{n(y)-1}(y)$ to get

$$[x, y]h(x) = 0 \qquad \text{for all } \ x, y \in I.$$

Replacing y by ry for arbitrary $r \in R$, we obtain $0 = [x, ry]h(x) = [x, r]yh(x)$ for all $x, y \in I$. Hence, $[x, R]Ih(x) = 0$ for all $x \in I$. By Lemma 2.2, R is commutative. □

Lemma 2.4. *Let R be a $*$-prime ring and let I be a nonzero $*$-ideal of R. If $x \in R$ and x centralizes I, then $x \in Z(R)$.*

Proof. Let $x \in R$. Suppose that $[x, u] = 0$ for all $u \in I$. Then, for arbitrary $r \in R$, we have $0 = [x, ru] = [x, r]u$ for all $u \in I$. That is, $[x, R]I = 0$. Thus,

$$[x, R]RI = 0 = [x, R]RI^*.$$

Since R is $*$-prime, we conclude that $[x, R] = 0$ and hence $x \in Z(R)$. □

3 The Main Results

The study of centralizing mappings and commutativity of certain rings began in the 1950's. Posner [5] established the commutativity of prime rings admitting nonzero centralizing derivations. El Sofy [2] proved an analogous result concerning homodervations. More recently, Oukhtite and Salhi [4] proved the commutativity of $*$-prime rings applying Posner's conditions on $*$-ideals. Motivated by this work, we explore the commutativity of $*$-prime rings admitting centralizing homoderivations and we prove the following theorem.

Theorem 3.1. *Let R be a $*$-prime ring with characteristic different from two, I a nonzero $*$-ideal of R, and h a nonzero homoderivation on R which commutes with $*$. If h is centralizing and zero-power valued on I, then R is commutative.*

Proof. By hypothesis, we have

$$[h(x), x] \in Z(R) \qquad \text{for all} \;\; x \in I. \tag{7}$$

Linearizing (7), we obtain

$$[h(x), y] + [h(y), x] \in Z(R) \qquad \text{for all} \;\; x, y \in I.$$

Replacing y by x^2, we get $[h(x), x^2] + [h(x^2), x] \in Z(R)$ which can be extended to

$$x[h(x), x]+[h(x), x]x+h(x)[h(x), x]+[h(x), x]h(x)+[h(x), x]x+x[h(x), x] \in Z(R)$$

for all $x \in I$. Applying (7) yields

$$(4x + 2h(x))[h(x), x] \in Z(R) \qquad \text{for all} \;\; x \in I.$$

Since char $R \neq 2$,

$$(2x + h(x))[h(x), x] \in Z(R) \qquad \text{for all} \;\; x \in I.$$

Thus, for arbitrary $r \in R$, we have $[(2x+h(x))[h(x), x], r] = 0$. Expanding this and using (7) yields

$$[2x + h(x), r][h(x), x] = 0 \qquad \text{for all} \;\; x \in I, r \in R.$$

In particular, $[2x + h(x), x][h(x), x] = 0$ for all $x \in I$. This can be simplified to

$$[h(x), x]^2 = 0 \qquad \text{for all} \;\; x \in I. \tag{8}$$

Since every $*$-prime ring is semiprime and since the center of semiprime rings contains no nonzero nilpotent elements, we find that $[h(x), x] = 0$ for all $x \in I$ and hence by Lemma 2.3, R is commutative. □

Also, Oukhtite and Salhi [4] proved that if a $*$-prime ring R with characteristic different from two has a nonzero derivation d which commutes with $*$ and satisfies $[ad(x), x] = 0$ on $*$-ideals, then $a = 0$ or R is commutative. Our next result will provide an analogous conclusion using the concept of homoderivations.

Theorem 3.2. *Let R be a $*$-prime ring with characteristic different from two, I a nonzero $*$-ideal of R, and h a nonzero homoderivation on R which commutes with $*$. If h is zero-power valued on I and $a \in S_*(R)$ such that $[ah(x), x] = 0$ for all $x \in I$, then $a = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$[ah(x), x] = 0 \qquad \text{for all } \ x \in I. \tag{9}$$

Linearizing (9), we obtain

$$[ah(x), y] + [ah(y), x] = 0 \qquad \text{for all } \ x, y \in I. \tag{10}$$

Replacing y by yx, we get $[ah(x), yx]+[ah(y)h(x), x]+[ah(y)x, x]+[ayh(x), x] = 0$ which is equivalent to $y[ah(x), x]+[ah(x), y]x+ah(y)[h(x), x]+[ah(y), x]h(x)+[ah(y), x]x + ay[h(x), x] + a[y, x]h(x) + [a, x]yh(x) = 0$. Applying (9) and (10), yields

$$ah(y)[h(x), x]+[ah(y), x]h(x)+ay[h(x), x]+a[y, x]h(x)+[a, x]yh(x) = 0 \qquad \text{for all } \ x, y \in I.$$

This can be written as

$$a(h(y)+y)[h(x), x]+[a, x](h(y)+y)h(x)+a[h(y)+y, x]h(x) = 0 \qquad \text{for all } \ x, y \in I.$$

Since h is zero-power valued on I,

$$ay[h(x), x] + [a, x]yh(x) + a[y, x]h(x) = 0 \qquad \text{for all } \ x, y \in I. \tag{11}$$

Replacing y by ay, we get

$$a^2y[h(x), x] + [a, x]ayh(x) + a^2[y, x]h(x) + a[a, x]yh(x) = 0 \qquad \text{for all } \ x, y \in I. \tag{12}$$

Applying (11) to (12) yields $[a, x]ayh(x) = 0$ for all $x, y \in I$. That is,

$$[a, x]aIh(x) = 0 \qquad \text{for all } \ x \in I. \tag{13}$$

Observe that for $x \in I \cap S_*(R)$ we have $x^* = \pm x$. Thus, since h commutes with $*$, we have $(h(x))^* = h(x^*) = \pm h(x)$. So, by (13), $[a, x]aIh(x) = 0 = [a, x]aI(h(x))^*$ and by Lemma 2.1, it follows that $[a, x]a = 0$ or $h(x) = 0$.
Now consider $y \in I$. Since $(y + y^*) \in I \cap S_*(R)$, by the above observation we have

$$[a, y + y^*]a = 0 \qquad \text{or} \qquad h(y + y^*) = 0. \tag{14}$$

Case1:
Let $h(y+y^*)=0$. Then, $h(y)=-h(y^*)=-(h(y))^*$. Therefore, by (13), we have $0=[a,y]aIh(y)=[a,y]aI(h(y))^*$ and by Lemma 2.1, it follows that $[a,y]a=0$ or $h(y)=0$.
Case2:
Let $[a,y+y^*]a=0$. Since $(y-y^*)\in I\cap S_*(R)$, by the above observation we have either $h(y-y^*)=0$ or $[a,y-y^*]a=0$. If $h(y-y^*)=0$, then by a similar approach to Case 1, we get $[a,y]a=0$ or $h(y)=0$. If $[a,y-y^*]a=0$, then $[a,y-y^*]a+[a,y+y^*]a=0$ which can be reduced to $2[a,y]a=0$. Since char $R\neq 2$, $[a,y]a=0$.
Thus, both cases in (14) imply that for each $y\in I$,

$$[a,y]a=0 \qquad \text{or} \qquad h(y)=0.$$

Notice that the sets of $y\in I$ for which these two conditions hold are additive subgroups of I whose union is I; but since a group cannot be the union of two of its proper subgroups, we have either

$$[a,I]a=0 \qquad \text{or} \qquad h(I)=0. \tag{15}$$

If $h(I)=0$, then $h(x)=0$ for all $x\in I$. Then, for arbitrary $r\in R$, $0=h(rx)=h(r)h(x)+h(r)x+rh(x)=h(r)x$. Thus, $h(r)I=0$ for all $r\in R$; and $h(r)RI=0=h(r)RI^*$ for all $r\in R$. By $*$-primeness of R, $h=0$ which is a contradiction.
Consequently, we must have $[a,I]a=0$. Then, $[a,x]a=0$ for all $x\in I$. Replacing x by xy yields $[a,x]ya=0$ for all $x,y\in I$. Thus,

$$[a,x]Ia=0 \qquad \text{for all} \;\; x\in I.$$

As $a\in S_*(R)$, then

$$0=[a,x]Ia=[a,x]Ia^* \qquad \text{for all} \;\; x\in I.$$

By Lemma 2.1, a centralizes I or $a=0$. By Lemma 2.4, $a\in Z(R)$ or $a=0$. If $0\neq a\in Z(R)$, then by (9), $0=[ah(x),x]=a[h(x),x]+[a,x]h(x)=a[h(x),x]$. Since $a\in Z(R)$, $aR[h(x),x]=0$. Since $a\in S_*(R)$,

$$0=aR[h(x),x]=a^*R[h(x),x] \qquad \text{for all} \;\; x\in I.$$

As $a\neq 0$, then $*$-primeness of R implies that $[h(x),x]=0$ for all $x\in I$. It follows from Lemma 2.3 that R is commutative. □

Ashraf and Siddeeque [1] studied the commutativity of $*$-prime rings admitting nonzero derivations which commute with $*$ and satisfy any one of the following identities on $*$-ideals:

1. $h([x,y])=0$,

2. $h(x \circ y) = 0$,
3. $h([x, y]) = [x, y]$, or
4. $h(x \circ y) = x \circ y$.

Investigating these identities on homoderivations, we obtain the following two results.

Theorem 3.3. *Let R be a $*$-prime ring, I a nonzero $*$-ideal of R, and h a nonzero homoderivation on R which commutes with $*$. If h satisfies either*

1. $h([x, y]) = 0$ for all $x, y \in I$, or
2. $h(x \circ y) = 0$ for all $x, y \in I$,

then R is commutative.

Proof. *(i)* By hypothesis, we have

$$h([x, y]) = 0 \qquad \text{for all } \ x, y \in I. \tag{16}$$

Replacing y by yx yields $0 = h([x, yx]) = h([x, y]x) = h([x, y])h(x) + h([x, y])x + [x, y]h(x)$ for all $x, y \in I$. Applying (16), we get

$$[x, y]h(x) = 0 \qquad \text{for all } \ x, y \in I. \tag{17}$$

Replacing y by ry for arbitrary $r \in R$ gives $[x, ry]h(x) = 0$ for all $x, y \in I$. Expanding this and using (17), we get

$$[x, r]yh(x) = 0 \qquad \text{for all } \ x, y \in I, r \in R.$$

Therefore, $[x, R]Ih(x) = 0$ for all $x \in I$. By Lemma 2.2, R is commutative.

(ii) By hypothesis, we have

$$h(x \circ y) = 0 \qquad \text{for all } \ x, y \in I. \tag{18}$$

Replacing y by yx yields $0 = h(x \circ yx) = h((x \circ y)x) = h(x \circ y)h(x) + h(x \circ y)x + (x \circ y)h(x)$ for all $x, y \in I$. Applying (18), we get

$$(x \circ y)h(x) = 0 \qquad \text{for all } \ x, y \in I.$$

This is equivalent to

$$xyh(x) = -yxh(x) \qquad \text{for all } \ x, y \in I. \tag{19}$$

Replacing y by ry for arbitrary $r \in R$ gives

$$xryh(x) = -ryxh(x) \qquad \text{for all } \ x, y \in I. \tag{20}$$

From (19) and (20), it follows that $xryh(x) = rxyh(x)$ for all $x, y \in I$. Thus, $[x, r]yh(x) = 0$ for all $x, y \in I$ and $r \in R$. Therefore, $[x, R]Ih(x) = 0$ for all $x \in I$. By Lemma 2.2, R is commutative. □

Theorem 3.4. *Let R be a $*$-prime ring with characteristic different from two, I a nonzero $*$-ideal of R and let h be a nonzero homoderivation on R which commutes with $*$. If If h satisfies either*

1. $h([x, y]) = [x, y] \quad \text{for all } x, y \in I$, or
2. $h(x \circ y) = x \circ y \quad \text{for all } x, y \in I$,

then R is commutative.

Proof. *(i)* By hypothesis, we have

$$h([x, y]) = [x, y] \qquad \text{for all } \ x, y \in I. \tag{21}$$

Replacing y by yx, we get $h([x, y]x) = [x, y]x$ for all $x, y \in I$. Thus,

$$h([x, y])h(x) + h([x, y])x + [x, y]h(x) = [x, y]x \qquad \text{for all } \ x, y \in I.$$

Applying (21), we get

$$2[x, y]h(x) = 0 \qquad \text{for all } \ x, y \in I.$$

Since char $R \neq 2$,

$$[x, y]h(x) = 0 \qquad \text{for all } \ x, y \in I. \tag{22}$$

Replacing y by ry for arbitrary $r \in R$ gives $[x, ry]h(x) = 0$ for all $x, y \in I$. Expanding this and using (22), we get

$$[x, r]yh(x) = 0 \qquad \text{for all } \ x, y \in I, r \in R.$$

Therefore, $[x, R]Ih(x) = 0$ for all $x \in I$. By Lemma 2.2, R is commutative.

(ii) By hypothesis, we have

$$h(x \circ y) = x \circ y \qquad \text{for all } \ x, y \in I. \tag{23}$$

Replacing y by yx, we get $h((x \circ y)x) = (x \circ y)x$ for all $x, y \in I$. Thus,

$$h(x \circ y)h(x) + h(x \circ y)x + (x \circ y)h(x) = (x \circ y)x \qquad \text{for all } \ x, y \in I.$$

Applying (23), we get

$$2(x \circ y)h(x) = 0 \qquad \text{for all } \ x, y \in I.$$

Since char $R \neq 2$,

$$(x \circ y)h(x) = 0 \qquad \text{for all } \ x, y \in I.$$

This is equivalent to

$$xyh(x) = -yxh(x) \qquad \text{for all} \;\; x, y \in I. \tag{24}$$

Replacing y by ry for arbitrary $r \in R$ gives

$$xryh(x) = -ryxh(x) \qquad \text{for all} \;\; x, y \in I. \tag{25}$$

From (24) and (25), it follows that $xryh(x) = rxyh(x)$ for all $x, y \in I$. Thus, $[x, r]yh(x) = 0$ for all $x, y \in I$ and $r \in R$. Therefore, $[x, R]Ih(x) = 0$ for all $x \in I$. By Lemma 2.2, R is commutative. □

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