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SOME SECOND-ORDER SUFFICIENT CONDITIONS FOR NON-SMOOTH CONSTRAINED MULTI-OJECTIVE OPTIMIZATION

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Abstract

This paper investigates nonsmooth multiobjective optimization problems with set constraints, and equality-inequality constraints. Using approximations as generalized derivatives we establish second-order sufficient conditions for strict local minima of order-2. The mappings involved in our problems are assumed only to be continuously differentiable of order-1. Examples are provided to show advantages of our results over recent ones in the literature.

1. Introduction

Classical optimality conditions for differentiable programming problems with constraints are basic results in many fields, such as optimization theory, control theory, the study of stability and sensitivity in mathematical programming, the convergence of algorithms, the best approximation problems, etc.

It should be noted that a vast range of generalized differentiability constructions have been developed for studying optimization-related problems in general and optimality conditions in particular. First and second-order necessary optimality conditions for programs in abstract spaces, with $\mathbb{R}$-valued or vector-valued functions, have been provided by many authors. Among those we may refer to Ben-Tal and Zowe [2], Cominetti [6],... Of these, only in [2] second-order sufficient conditions are established for differentiable programs.

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Borwein [5] also established sufficient conditions for differentiable programs with equality and set constraints. The reason for us to utilize second-order approximations is that their definitions are very simple and even discontinuous mappings may have second-order approximations. Although these approximations may not be the most powerful tools for nonsmooth analysis, our motivation in this respect lies in the advantages of our results over several recent papers in many situations provided by examples. The purpose of the present paper is to establish second-order optimality conditions for constraints multiobjective problem. We also establish the sufficient conditions for existence of support function, which is concerned in [14, 15, 16].

The paper is organized as follows: in Section 2, we introduce the notions related to vector optimization problem; in Section 3, we establish the condition for existence of support function; in Section 4, we investigate the optimality conditions for problem with set constraints; and Section 5 is devoted to the case where the constrained set is given explicitly by a system of equations and inequalities with functions.

2. Preliminaries

Let X be a Banach space and M be a nonempty subset of X. As usual we denote by $B_X(\overline{x}, r)$ the open ball centered at $\overline{x}$ and radius $r > 0$ in X, by $\text{int}M$ ($\text{cl}M$, respectively) the interior (closure, respectively) the of the set M, by $\text{conv}M$ the convex hull of M, by $\text{cone}M$ the cone generated by M and by X^* the topological dual space to X. For $\lambda \in X^*$ and $x \in X$, we will use λx instead of $\lambda(x)$ or the also usual $\langle \lambda, x \rangle$. The positive polar cone to M is

$$M^+ := \{\lambda \in X^* : \lambda x \geq 0, \forall x \in M \setminus \{0\}\},$$

and the strictly positive polar cone to M is

$$M^{++} := \{\lambda \in X^* : \lambda x > 0, \forall x \in M \setminus \{0\}\}.$$

For Banach spaces X and Y, $L(X, Y)$ denotes the space of the continuous linear mappings from X to Y, and $B(X, X, Y)$ denotes the space of the continuous bilinear mappings from $X \times X$ to Y. In the next definition, the notations of the first-order and second-order approximation in [1, 17] will be recalled.

Definition 2.1. *Let $\overline{x} \in X$ and $g : X \to Y$ be a mapping.*

(i) The set $A_g(\overline{x}) \subset L(X, Y)$ is said to be a first-order approximation of g at $\overline{x}$ if there exists a neighborhood U of $\overline{x}$ such that,

$$g(x) - g(\overline{x}) \in A_g(\overline{x})(x - \overline{x}) + o(\|x - \overline{x}\|), \forall x \in U,$$

where $\dfrac{o(\|x - \overline{x}\|)}{\|x - \overline{x}\|}$ tends to 0, as $x \to \overline{x}$.

(ii) A pair $(A_g(\overline{x}), B_g(\overline{x}))$, with $A_g(\overline{x}) \subset L(X, Y)$ and $B_g(\overline{x}) \subset B(X, X, Y)$, is said to be a second-order approximation of g at $\overline{x}$ if

(a) $A_g(\overline{x})$ *is a first-order approximation of* g *at* $\overline{x}$;

(b) $g(x) - g(\overline{x}) \in A_g(\overline{x})(x - \overline{x}) + B_g(\overline{x})(x - \overline{x}, x - \overline{x}) + o(\|x - \overline{x}\|^2)$.

Note that if g is twice Fréchet differentiable at $\overline{x}$, then $(\nabla g(\overline{x}), \frac{1}{2}\nabla^2 g(\overline{x}))$ is the second-order approximation of g at $\overline{x}$, where $\nabla g(\overline{x})$ and $\nabla^2 g(\overline{x})$ are the first and the second Fréchet derivative of g at $\overline{x}$, respectively. Various properties and examples of the first-order and the second-order approximation could be found in [1, 17, 18, 19].

The Calrke generalized Jacobian and the Clarke generalized Hessian in [7], [8] will be stated in the following definition.

Definition 2.2. *(i) Let* $g : \mathbb{R}^m \to \mathbb{R}^n$ *be a mapping of class* $C^{1,0}$ *(i.e. the set of functions whose gradient mappings is calm) . The Clarke generalized Jacobian of* g *at* $\overline{x} \in \mathbb{R}^m$*, denoted by* $\partial_C g(\overline{x})$*, is defined by*

$$\partial_C g(\overline{x}) := \mathrm{cl\,conv}\Big\{\lim_{x_i \to \overline{x}} \nabla g(x_i) : \nabla g(x_i) \quad \textit{exists}\Big\}.$$

(ii) Let $g : \mathbb{R}^m \to \mathbb{R}^n$ *be a mapping of class* $C^{1,1}$ *(i.e. the set of functions whose gradient mappings is locally Lipschitz). The Clarke generalized Hessian of* g *at* $\overline{x} \in \mathbb{R}^m$*, denoted by* $\partial^2_C g(\overline{x})$*, is defined by*

$$\partial^2_C g(\overline{x}) := \mathrm{cl\,conv}\Big\{\lim_{x_i \to \overline{x}} \nabla^2 g(x_i) : \nabla^2 g(x_i) \quad \textit{exists}\Big\}.$$

We recall the following relaxed compactness in [19] which will be needed for establishing optimality conditions in the sequel.

Definition 2.3. *(i) The sequence* $(M_k) \subset L(X, Y)$ *is said to pointwisely converge to* $M \in L(X, Y)$ *and wirtten as* $M_k \xrightarrow{p} M$ *or* $M = \mathrm{p}\text{-}\lim M_k$ *if* $\lim M_k(x) = M(x)$ *for all* $x \in X$*. A similar definition is adopted for sequence* $(N_k) \subset B(X, X, Y)$ *and* $N \in B(X, X, Y)$*.*

(ii) A subset $A \subset L(X, Y)$ *(*$B \subset B(X, X, Y)$*, respectively) is called (sequentially) asymptotically pointwisely compact, or (sequentially) asymptotically* p*-compact if*

(a) each norm bounded sequence $(M_k) \subset A$ *(*$\subset B$*, respectively) has a subsequence* (M_{k_j}) *and* $M \in L(X, Y)$ *(*$M \in B(X, X, Y)$*, respectively) such that* $M = \mathrm{p}\text{-}\lim M_{k_j}$*,*

(b) for each sequence $(M_k) \subset A$ *(*$\subset B$*, respectively) with* $\lim \|M_k\| = \infty$*, the sequence* $\dfrac{M_k}{\|M_k\|}$ *has a subsequence which pointwisely converges to some* $M \in L(X, Y) \setminus \{0\}$ *(*$M \in B(X, X, Y) \setminus \{0\}$*, respectively).*

(iii) If in (ii), pointwise convergence, i.e. $\mathrm{p}\text{-}\lim$*, is replaced by convergence, i.e.* $\lim$*, a subset* $A \subset L(X, Y)$ *(or* $B \subset B(X, X, Y)$*) is called (sequentially) asymptotically compact.*

In the sequel we omit the term "sequentially" for short. Note that the asymptotical p-compactness in Definition 2.3 is equivalent to the relative p-compactness and the asymptotical p-compactness together is defined in [18].

For $A \subset L(X,Y)$ and $B \subset B(X,X,Y)$ we adopt the notations:

$$\text{p-cl}A := \{M \in L(X,Y) : \exists (M_k) \subset A, M = \text{p-}\lim M_k\}, \tag{1}$$

$$\text{p-cl}B := \{N \in B(X,X,Y) : \exists (N_k) \subset B, N = \text{p-}\lim N_k\}, \tag{2}$$

$$A_\infty := \{M \in L(X,Y) : \exists (M_k) \subset A, \exists t_k \downarrow 0, M = \lim t_k M_k\}, \tag{3}$$

$$\text{p-}A_\infty := \{M \in L(X,Y) : \exists (M_k) \subset A, \exists t_k \downarrow 0, M = \text{p-}\lim t_k M_k\}, \tag{4}$$

$$\text{p-}B_\infty := \{N \in B(X,X,Y) : \exists (N_k) \subset B, \exists t_k \downarrow 0, N = \text{p-}\lim t_k N_k\}. \tag{5}$$

The sets (3), (4) are pointwise closures; (5) is just the definition of the recession cone of A. So (**??**), (**??**) are pointwise recession cones.

Subsequently, some useful notes in [18] are recalled for illustrating the characteristics of asymptotically p-compact sets and asymptotically compact sets.

Remark 2.4. *(i) If X and Y are finite dimensional, a convergence occurs if and only if the corresponding pointwise convergence does, but in general the "if" does not hold.*

(ii) If X and Y are finite dimensional, every subset is asymptotically p-*compact and asymptotically compact but in general the asymptotical compactness is stronger.*

In the next definition, the contingent tangent set will be restated.

Definition 2.5. *Let S be a subset of X and $\overline{x}$ be an element of S. The contingent (or Bouligand) cone of S at $\overline{x}$ is*

$$T(S,\overline{x}) := \{v \in X : \exists t_n \downarrow 0, \exists v_n \to v \quad \textit{such that} \quad \overline{x} + t_n v_n \in M, \forall n \in \mathbb{N}\}.$$

Let f be a mapping from X to Y. In this paper, we are interested in the general vector optimization problem:

$$D - \min f(x), \quad \text{subject to} \quad x \in M, \tag{6}$$

where $D \subset Y$ is a closed, convex, pointed cone with nonempty interior and the partial order in Y is defined by D through the relation $y_1 \leq_D y_2$ if and only if $y_1 - y_2 \in -D$.

Let us recall that the point $\overline{x} \in M$ is said to be a local minimum for problem 6, denoted by $\overline{x} \in \text{lmin}(f,M)$ if there exists a neighborhood U of $\overline{x}$ such that,

$$\big(f(M \cap U) - f(\overline{x})\big) \cap (-D) = \{0\}.$$

Let m be a positive integer. The point $\overline{x} \in M$ is said to be a strict local minimum of order m for problem 6, denoted by $\overline{x} \in \text{strl}(m,f,M)$, if there exist a neighborhood U of $\overline{x}$ and a positive real number α such that,

$$\big(f(x) + D\big) \cap B_Y\big(f(\overline{x}), \alpha\|x - \overline{x}\|^m\big) = \emptyset, \forall x \in M \cap U \setminus \{\overline{x}\}.$$

For $j \geq m \geq 1$, it is clear that

$$\mathrm{strl}(m, f, M) \subset \mathrm{strl}(j, f, M) \subset \mathrm{lmin}(f, M)$$

Therefore, necessary conditions for the right-most term hold true also for the others and sufficient conditions for the left-most term are valid for the others as well.

Instead of "strict local minimum", other term like "firm efficient", "strict efficient" or "isolated efficient" are also used in the literature [9, 18, 19].

In finite dimensional spaces, two linearized cones of function f at $\overline{x}$, the first one open and the second one closed , is defined by:

$$C_0(f, \overline{x}) := \{v \in X : \nabla f(\overline{x})v \in -\mathrm{int}D\},$$

and

$$C(f, \overline{x}) := \{v \in X : \nabla f(\overline{x})v \in -D\}.$$

More characteristic of those cones could be found in [14, 15].

3. Support functions

Let f be a mapping which is introduced in Section 2. Subsequently, we will recall the notation of support function to general vector optimization problem in [16].

Definition 3.1. *Let $F : X \to \mathbb{R}$ be a differentiable function at $\overline{x} \in M \subset X$ and $\lambda \in D^+$. We will say that the pair (λ, F) is a (lower) local support for f at $\overline{x}$ on M if these following conditions hold:*

(i) $F(x) \leq \lambda f(x), \forall x \in M \cap B_X(\overline{x}, \delta)$ for some $\delta > 0$;

(ii) $F(\overline{x}) = \lambda f(\overline{x})$;

(iii) $\nabla F(\overline{x}) = 0$;

(iv) $\lambda \neq 0$.

We will say that (λ, F) is a (global) support if condition (i) satisfied for all $x \in M$, and we will say that it is a weak local support if conditions (i)$-$(iii) are satisfied.

The scalarization process in the previous definition is going to allow us to follow a parallel path to the scalar case and apply the results to the scalar programming.

Now we recall some useful notes in [12, 14, 15].

Remark 3.2. *Suppose that $X = \mathbb{R}^m, Y = \mathbb{R}^n$ and, $D = \mathbb{R}^n_+$.*

(i) The Definition (3.1) is equivalent to a staement that F is a support (in the Hestenes sense) for the scalar function $\langle \lambda, f\rangle = \sum_{i=1}^{n} \lambda_i f_i$;

(ii) If $n = 1$ then, saying that (λ, F) is a support of f at $\overline{x}$ is equivalent to

$$\lambda > 0, F(x) \leq \lambda f(x), \forall x \in M, F(\overline{x}) = \lambda f(\overline{x}) \quad \text{and} \quad \nabla F(\overline{x}) = 0.$$

Hence, calling $\widetilde{F} = \dfrac{F}{\lambda}$, it can be concluded that $\widetilde{F}$ is a support for f in usual sense: $\widetilde{F}(x) \leq f(x), \forall x \in M, \widetilde{F}(\overline{x}) = f(\overline{x}) \quad \text{and} \quad \nabla \widetilde{F}(\overline{x}) = 0$.

Remark 3.3. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}^k$ and $h : \mathbb{R}^m \to \mathbb{R}^p$ be differentiable mappings, $D = \mathbb{R}^n_+, K = \mathbb{R}^k_+$ and M is given by*

$$M := \{x \in \mathbb{R}^m : g(x) \in -K, h(x) = 0\}.$$

If $\overline{x} \in M$ and the Fritz John conditions for the set M are satisfied, that is, there exist $\lambda \in \mathbb{R}^n, \mu \in \mathbb{R}^k, \nu \in \mathbb{R}^p$ all nonzero such that,

$$\begin{aligned} &\lambda \in D^+, \mu \in K^+, \mu g(\overline{x}) = 0, \\ &\lambda \nabla f(\overline{x}, v) + \mu \nabla g(\overline{x}, v) + \nu \nabla h(\overline{x}, v) \geq 0, \forall v \in T(M, \overline{x}), \end{aligned}$$

then, letting F be the Lagrangian function,

$$F = \lambda f + \mu g + \nu h,$$

we have that (λ, F) is a support for f at $\overline{x}$ on M. And, if the Kuhn Tucker conditions, i.e. $\lambda \neq 0$, for the set M hold then (λ, F) is a support.

The ϵ-normal cone in [20] will be stated in order to establish conditions for the existence of support function.

Definition 3.4. *Let S be a subset of X and $\overline{x} \in S$.*

(i) A vector $v \in X^*$ is said to be a ϵ-normal to S at $\overline{x}$ if

$$\limsup_{u \xrightarrow{S} \overline{x}} \frac{\langle v, u - \overline{x} \rangle}{\|u - \overline{x}\|} \leq \epsilon,$$

where $u \xrightarrow{S} \overline{x}$ means that $u \to \overline{x}$ and $u \in S$. The set of all ϵ-normals to S at $\overline{x}$ is called the ϵ-normal cone to S at $\overline{x}$ and denoted by $\widehat{N}_\epsilon(S, \overline{x})$. When $\epsilon = 0$, this set is called the Fréchet normal cone to S at $\overline{x}$ and denoted by $\widehat{N}(S, \overline{x})$.

(ii) The limiting Fréchet normal cone to S at $\overline{x}$ is defined by

$$\begin{aligned} N_F(S, \overline{x}) := \{v \in X^* : \exists \epsilon_n \downarrow 0, x_n \xrightarrow{S} \overline{x}, v_n^* \xrightarrow{w^*} v, \\ \text{with} \quad v_n^* \in \widehat{N}_{\epsilon_n}(S, x_n) \quad \text{for all } n \in \mathbb{N}\}. \end{aligned}$$

Note also that the Fréchet normal cone is obviously convex and closed in the norm topology of X^*. When X is finite dimensional, the Fréchet normal cone to S at $\overline{x}$ is the polar of the contingent cone, that is,

$$v \in \widehat{N}(S, \overline{x}) \Longleftrightarrow \langle v, z \rangle \leq 0, \forall z \in T(S, \overline{x}).$$

Remark 3.5. *Suppose that X is finite dimensional. Then a vector v is a Fréchet normal to subset S at $\overline{x}$ if and only if*

$$\langle v, x - \overline{x} \rangle \leq o(\|x - \overline{x}\|), \forall x \in S,$$

where $\dfrac{o(\|x - \overline{x}\|)}{\|x - \overline{x}\|} \to 0$ *when* $\|x - \overline{x}\| \to 0.$

When X is finite dimensional, the Fréchet normal cone $\widehat{N}(S, \overline{x})$ has also been referred to as the regular normal cone in literature (for details see for Rockafellar and Wets [22]).

For $\Phi : X \to 2^Y$ a multifunction, let $\mathrm{Gr}\Phi$ denote the graph of Φ, that is,

$$\mathrm{Gr}\Phi := \{(x, y) \in X \times Y : y \in \Phi(x)\}.$$

For $x \in X$ and $y \in \Phi(x)$, let $\widehat{D}^*\Phi(x, y) : Y^* \to 2^{X^*}$ and $\widehat{D}^*_F(x, y) : Y^* \to 2^{X^*}$ respectively denote Fréchet and limiting Fréchet coderivatives of Φ at (x, y) in Mordukhovich's sense, that is,

$$\widehat{D}^*\Phi(x, y)(y^*) := \{x^* \in X^* : (x^*, -y^*) \in \widehat{N}(\mathrm{Gr}\Phi, (x, y))\}, \quad \text{for all} \quad y^* \in Y^*,$$

and

$$\widehat{D}^*_F\Phi(x, y)(y^*) := \{x^* \in X^* : (x^*, -y^*) \in N_F(\mathrm{Gr}\Phi, (x, y))\} \quad \text{for all} \quad y^* \in Y^*.$$

When Φ is singular-value, we denote Fréchet and limiting Fréchet coderivatives by $\widehat{D}^*\Phi(x)$ and $\widehat{D}^*_F\Phi(x)$, respectively.

The multifunction Φ is said to be regular normal at $(x, y) \in X \times Y$ if

$$\widehat{D}^*\Phi(x, y)(y^*) = \widehat{D}^*_F\Phi(x, y)(y^*), \forall y^* \in Y^*.$$

The set S is said to be regular normal at $\overline{x}$ if

$$\widehat{N}(S, \overline{x}) = N_F(S, \overline{x}).$$

The following propositions establish sufficient conditions for the existence of support function.

Proposition 3.6. *If there exist $\lambda \in Y^*$ and $y \in X^*$ such that $(y, -\lambda) \in \widehat{N}(\mathrm{Gr}f, (\overline{x}, f(\overline{x})))$ then there exists a function $F : X \to \mathbb{R}$, which is differentiable at $\overline{x}$, such that* *(i)* $F(x) \leq \lambda f(x),\ \forall x \in M \cap B(\overline{x}, \delta)$ *for some* $\delta > 0$*;*

(ii) $F(\overline{x}) = \lambda f(\overline{x})$*;*

(iii) $\nabla F(\overline{x}) = y$*.*

Proof. Let $h(t) := \sup\{\langle (y, -\lambda), (x, \alpha) - (\overline{x}, f(\overline{x}))\rangle, (x, \alpha) \in \mathrm{Gr}f \cap B((\overline{x}, f(\overline{x})), t)\}$, for all $t \in \mathbb{R}^+$. Then $h(.)$ is a nondecreasing function on $[0, \infty)$ and $h(t) \geq 0, \forall t \in \mathbb{R}^+$. Since $(y, -\lambda) \in \widehat{N}(\mathrm{Gr}f, (\overline{x}, f(\overline{x})))$, we deduce that $t^{-1}h(t)$ tends to 0 as $t \downarrow 0$. The function

$$g(x, \alpha) := \langle (y, -\lambda), (x, \alpha) - (\overline{x}, f(\overline{x}))\rangle - h(\|(x, \alpha) - (\overline{x}, f(\overline{x}))\|), \tag{1}$$

is, therefore, differentiable at $(\overline{x}, f(\overline{x}))$. From (1) and the definition of h, we have $g(x, \alpha) \leq 0$, for all $(x, \alpha) \in \mathrm{Gr}f$. Since $(x, f(x)) \in \mathrm{Gr}f$, we get

$$g(x, f(x)) = \langle y, x - \overline{x}\rangle + \langle \lambda, f(\overline{x})\rangle - \langle \lambda, f(x)\rangle - h(\|(x, f(x)) - (\overline{x}, f(\overline{x}))\|) \leq 0. \tag{2}$$

Let $F(x) := \langle y, x - \overline{x}\rangle + \langle \lambda, f(\overline{x})\rangle - h(\|(x, f(x)) - (\overline{x}, f(\overline{x}))\|)$. From (2) we deduce that

(i) $F(x) \leq \lambda f(x)$;

(ii) $F(\overline{x}) = \lambda f(\overline{x})$.

Furthermore $F(x)$ is differentiable at $\overline{x}$ and $\nabla F(\overline{x}) = y$, since $(y, -\lambda) \in \widehat{N}(\mathrm{Gr}f, (\overline{x}, f(\overline{x})))$. □

The two next propositions are quoted from [23].

Proposition 3.7. *Suppose that* $\overline{x}$ *is a local minimum for problem 6. Then there exists* $c^* \in C^+$ *with* $\|c^*\| = 1$ *such that*

$$0 \in \widehat{D}^*_F f(\overline{x})(c^*) + N_F(M, \overline{x}).$$

Proposition 3.8. *Suppose that* $\overline{x}$ *is a local minimum for problem 6,* f *is a regular normal mapping and* M *is a regular normal subset. Then there exists* $c^* \in C^+$ *with* $\|c^*\| = 1$ *such that*

$$0 \in \widehat{D}^* f(\overline{x})(c^*) + \widehat{N}(M, \overline{x}).$$

Using the Proposition 3.6, Proposition 3.8 and the definition of Fréchet coderivative of f, we get the sufficient condition for the existence of support function.

Proposition 3.9. *Suppose that* $\overline{x}$ *is a local minimum for problem 6,* f *is a regular normal mapping and* M *is a regular normal subset. Then there exist* $\lambda \in C^+$ *with* $\|\lambda\| = 1$ *and a function* $F : X \to \mathbb{R}$*, which is differentiable at* $\overline{x}$*, such that* (λ, F) *is a local support for* f *at* $\overline{x}$ *on* M*.*

The proof is easy, so it is omitted.

4. Problem with set constraints

Consider the problem 6 stated in Section 2. Throughout this paper, we assume that X and Y are finite dimensional spaces and f is differentiable at $\overline{x}$. The following proposition states the basic properties which are verified if a support function for f at $\overline{x}$ on M exists. Note that the first property is also the first-order necessary condition.

Theorem 4.1. *Assume that $(\nabla F(\overline{x}), B_F(\overline{x}))$ is a asymptotically* p-*compact second-order approximation of F at $\overline{x}$. Then we have,*

(i) If (λ, F) is a local support for f at $\overline{x}$ on M then

$$T(M,\overline{x}) \cap C_0(f,\overline{x}) = \emptyset; \tag{1}$$

(ii) If (λ, F) is a weak local support for f at $\overline{x}$ on M and there exists $v \in T(M,\overline{x})$ such that $N(v,v) > 0, \forall N \in \text{p-cl}B_F(\overline{x}) \cup \text{p-}B_F(\overline{x})_\infty \backslash \{0\}$ then $\lambda \neq 0$, that is (λ, F) is a local support.

Proof. (i) The proof is similar to that of Proposition 3.1 in [14] and therefore omitted.

(ii) For $v \in T(M,\overline{x})$, there exist some sequence $(v_n) \subset X, v_n \to v$ and $t_n \subset \mathbb{R}, t_n \to 0^+$ such that $x_n = \overline{x} + t_n v_n \in M$, for all $n \in \mathbb{N}$.

If $\lambda = 0$ then we have

$$F(x) \leq F(\overline{x}) = 0, \forall x \in M \cap B(\overline{x}, \delta),$$

for some $\delta > 0$. By the definition of the second-order approximation, for n sufficiently large,

$$\begin{aligned} 0 \geq F(\overline{x} + t_n v_n) - F(\overline{x}) &= t_n \nabla F(\overline{x})(v_n) + t_n^2 N_n(v_n, v_n) + o(t_n^2) \\ &= t_n^2 N_n(v_n, v_n) + o(t_n^2), \end{aligned} \tag{2}$$

for some $N_n \in B_F(\overline{x})$.

If N_n is norm bounded, assume that $N_n \to N \in \text{p-cl}B_F(\overline{x})$, then we obtain

$$N(v,v) = \lim_{n\to\infty} \frac{F(\overline{x} + t_n v_n) - F(\overline{x})}{t_n^2} \leq 0,$$

which is contradiction.

If N_n is norm unbounded, one can assume that $\|N_n\| \to +\infty$ and $\dfrac{N_n}{\|N_n\|} \to N \in \text{p-}B_F(\overline{x})_\infty$. Dividing (2) by $t_n^2\|N_n\|$ and letting $n \to \infty$, we obtain $N(v,v) \leq 0$, a contradiction. □

Next we recall a result about the necessary and sufficient conditions to strict local minimum of order 1 in [14].

Theorem 4.2.

$$T(M,\overline{x}) \cap C(f,\overline{x}) = \{0\} \Longleftrightarrow \overline{x} \in \mathrm{strl}(1,f,M). \tag{3}$$

Theorem 4.3. *If for every $v \in T(M,\overline{x}) \cap C(f,\overline{x})\backslash\{0\}$ there exists (λ, F), a weak local support for f at $\overline{x}$ on M and $N(v,v) > 0$ for all $N \in$ p-cl$B_F(\overline{x}) \cup$ p-$B_F(\overline{x})_\infty\backslash\{0\}$ then $\overline{x} \in \mathrm{strl}(2,f,M)$, where $(\nabla F(\overline{x}), B_F(\overline{x}))$ is a asymptotically* p-*compact second-order approximation of F at $\overline{x}$.*

Proof. Suppose that $\overline{x} \notin \mathrm{strl}(2,f,M)$. Then exist sequences $x_n \in M \cap B\big(\overline{x}, \frac{1}{n}\big)\backslash\{0\}$ and $d_n \in D$ such that

$$f(x_n) - f(\overline{x}) + d_n = c_n \in B\big(0, \frac{1}{n}t_n^2\big), \tag{4}$$

where $t_n = \|x_n - x_0\|$. Since X is finite dimension space, we assume that

$$\lim_{n\to\infty} \frac{x_n - \overline{x}}{t_n} = v \in T(M,\overline{x}), \|v\| = 1. \tag{5}$$

From (5), we have $\nabla f(\overline{x}) \in -D$ and therefore $v \in T(M,\overline{x}) \cap C(f,\overline{x})\backslash\{0\}$. Since $N(v,v) > 0$ for all $N \in$ p-cl$B_F(\overline{x}) \cup$ p-$B_F(\overline{x})_\infty\backslash\{0\}$, by Theorem 4.1, we have $\lambda \neq 0$. Thus, applying to (4) the continuous linear function we have

$$\begin{aligned} & \lambda f(x_n) - \lambda f(\overline{x}) + \lambda d_n = \lambda c_n \\ \Leftrightarrow\quad & \big[\lambda f(x_n) - F(x_n)\big] + F(x_n) - F(\overline{x}) - \big[\lambda f(\overline{x}) - F(\overline{x})\big] + \lambda d_n = \lambda c_n \\ \Leftrightarrow\quad & F(x_n) - F(\overline{x}) + \big[\lambda f(x_n) - F(x_n)\big] + \lambda d_n = \lambda c_n. \end{aligned} \tag{6}$$

By the definition of the second-order approximation with $\nabla F(\overline{x}) = 0$, we get

$$N_n(x_n - \overline{x}, x_n - \overline{x}) + o(t_n^2) - \big[\lambda f(x_n) - F(x_n)\big] + \lambda d_n = \lambda c_n,$$

where $N_n \in B_F(\overline{x})$.

If N_n is norm bounded, assume that $N_n \to N \in$ p-cl$B_F(\overline{x})$. Dividing by t_n^2 and taking the limit we obtain

$$\lim_{n\to\infty} \frac{N_n(x_n - \overline{x}, x_n - \overline{x}) + o(t_n^2)}{t_n^2} + \lim_{n\to\infty} \frac{\big[\lambda f(x_n) - F(x_n)\big] + \lambda d_n}{t_n^2} = 0.$$

The first limit exists and equal to $N(v,v)$, for some $N \in$ p-cl$B_F(\overline{x})$. Thus the second limit exists and since $\lambda f(x_n) - F(x_n) \geq 0$ and $\lambda d_n \geq 0$. This follows $N(v,v) \leq 0$, which is a contradiction.

If N_n is norm unbounded, one can assume that $\|N_n\| \to +\infty$ and $\dfrac{N_n}{\|N_n\|} \to N \in$ p-$B_F(\overline{x})_\infty$. Dividing (6) by $t_n^2\|N_n\|$ and letting $n \to \infty$, by an argument similar to that for the above boundedness case, we obtain $N(v,v) \leq 0$, for some $N \in$ p-$B_F(\overline{x})_\infty$, a contradiction. □

Remark 4.4. *In this theorem we only concern the case* $T(M,\overline{x})\cap C(f,\overline{x})\neq\{0\}$ *because if* $T(M,\overline{x})\cap C(f,\overline{x})=\{0\}$*, by Theorem 4.2,* $\overline{x}\in\mathrm{strl}(1,f,M)$ *and thus* $\overline{x}\in\mathrm{strl}(2,f,M)$.

Corollary 4.5. *Assume that* $(\nabla f(\overline{x},B_f(\overline{x}))$ *is asymptotically* p-*compact second-order approximation of* f *at* $\overline{x}$. *If* $v\in T(M,\overline{x})\cap C(f,\overline{x})\backslash\{0\},\exists\lambda\in D^+:\lambda\nabla f(\overline{x})=0$ *and* $\lambda N(v,v)>0,\forall N\in\text{p-cl}B_f(\overline{x})\cup\text{p-}B_f(\overline{x})_\infty\backslash\{0\}$*, then* $\overline{x}\in\mathrm{strl}(2,f,M)$.

Proof. Let $F(x)=\lambda f(x)$. It is easy to check that (λ,F) is weak local support for f at $\overline{x}$ on M. Indeed, from the definition of F, the condition $F(x)\leq\lambda f(x),\forall x\in M\cap B(x,\delta)$ and $F(\overline{x})=\lambda f(\overline{x})$ are held. On the other hand, $\nabla F(\overline{x})=\lambda\nabla f(\overline{x})=0$ and therefore (iii) of Definition 3.1 is held. Moreover, one can check that $B_F(\overline{x})=\lambda B_f(\overline{x})$, where $(\nabla F(\overline{x}),B_F(\overline{x}))$ is asymptotically p-compact second-order approximation of F at $\overline{x}$. Thus, we may apply Theorem 4.3 to obtain above results. □

We can see that Theorem 4.3 is general and Corollary 4.4 is more simple to apply. In the following results we address other sufficient conditions in which the support function does not change with the vector.

Proposition 4.6. *Assume that* $(\nabla f(\overline{x}),B_f(\overline{x}))$ *is an asymptotically* p-*compact second-order approximation of* f *at* $\overline{x}$.

(i) If $\exists\lambda\in D^+:\lambda\nabla f(\overline{x})=0$ *and* $\lambda N(v,v)>0,\forall v\in T(M,\overline{x})\cap C(f,\overline{x})\backslash\{0\}$, $\forall N\in\text{p-cl}B_f(\overline{x})\cup\text{p-}B_f(\overline{x})_\infty\backslash\{0\}$*, then* $\overline{x}\in\mathrm{strl}(2,f,M)$.

(ii) If $\exists\lambda\in D^{++}:\lambda\nabla f(\overline{x})=0$ *and* $\lambda N(v,v)>0,\forall v\in T(M,\overline{x})\cap\mathrm{Ker}\nabla f(\overline{x})\backslash\{0\}$, $\forall N\in\text{p-cl}B_f(\overline{x})\cup\text{p-}B_f(\overline{x})_\infty\backslash\{0\}$*, then* $\overline{x}\in\mathrm{strl}(2,f,M)$.

Proof. (i) It follows Corollary 4.5.

(ii) One can check that $T(M,\overline{x})\cap\mathrm{Ker}\nabla f(\overline{x})\subset T(M,\overline{x})\cap C(f,\overline{x})$. Therefore, based on case (i) we only need to prove that

$$T(M,\overline{x})\cap C(f,\overline{x})\subset T(M,\overline{x})\cap\mathrm{Ker}\nabla f(\overline{x}).$$

For arbitrary $v\in T(M,\overline{x})\cap C(f,\overline{x})$, if $\nabla f(\overline{x})v\neq 0,\lambda\nabla f(\overline{x})v>0$ which contradicts to $\lambda\nabla f(\overline{x})=0$. Therefore, $\nabla f(\overline{x})v=0$. This follows $T(M,\overline{x})\cap C(f,\overline{x})\subset T(M,\overline{x})\cap\mathrm{Ker}\nabla f(\overline{x})$. □

Remark 4.7. *The results in Theorem 4.3, Corollary 4.5 and Proposition 4.6 are more applicable then those in [14] because it does not require* f *is twice differentiable. We can see that in the following example*

Example 4.8. *1. Let* $X=\mathbb{R},Y=\mathbb{R}^2,D=\mathbb{R}^2_+,M=\mathbb{R}_+,\overline{x}=0$ *and* $f(x)=\left(\|x\|^{\frac{4}{3}},2x^2\right)$. *Then* $\nabla f(\overline{x})=(0,0),B_f(\overline{x})=\{(a,4):a>1\},\text{p-cl}B_f(\overline{x})=\{(a,4):a\geq 1\},\text{p-}B_f(\overline{x})_\infty=\{(a,0):a\geq 0\},\mathrm{Ker}\nabla f(\overline{x})=\mathbb{R}$ *and* $T(M,\overline{x})=\mathbb{R}_+$. *Choose* $\lambda=(1,1)\in D^{++}$. *One can check that all conditions in (ii) of Proposition 4.6 are held. Thus,* $\overline{x}\in\mathrm{strl}(2,f,M)$.

2. *Let* $X = \mathbb{R}, Y = \mathbb{R}^2, D = \mathbb{R}^2_+, M = \mathbb{R}_+, \overline{x} = 0$ *and*

$$f(x) = \begin{cases} (|x|^{\frac{4}{3}}, x^2), & \text{if } x \geq 0, \\ (|x|^{\frac{4}{3}}, -x^2), & \text{otherwise.} \end{cases}$$

Then $\nabla f(\overline{x}) = (0,0), B_f(\overline{x}) = \{(a,b) : a > 1, b \in [-2,2]\}, \text{p-cl}B_f(\overline{x}) = \{(a,b) : a \geq 1, b \in [-2,2]\}, \text{p-}B_f(\overline{x})_\infty = \{(a,0) : a \geq 0\}$*,* $C(f,\overline{x}) = \mathbb{R}$ *and* $T(M,\overline{x}) = \mathbb{R}_+$*. It is easy to see that we cannot apply (ii) of Proposition 4.6. However, if we choose* $\lambda = (1,0) \in D^+$*, one can check that all conditions in (i) of Proposition 4.6 are held. Thus,* $\overline{x} \in \text{strl}(2,f,M)$*.*

We can see that in both above examples f is not twice differentiable at $\overline{x}$ (f is also not of class $C^{1,1}$) and then results in [14] cannot be applied.

In the case $X = \mathbb{R}^m, Y = \mathbb{R}^n$ and $f : X \to Y$ is one of class $C^{1,1}$, from above results, by a simple implication we have following corollaries.

Corollary 4.9. *Assume that* $f \in C^{1,1}$*. If* $\forall v \in T(M,\overline{x}) \cap C(f,\overline{x})\backslash\{0\}, \exists \lambda \in D^+ : \lambda \nabla f(\overline{x}) = 0$ *and* $\lambda N(v,v) > 0, \forall N \in \partial^2_C f(\overline{x})$*, then* $\overline{x} \in \text{strl}(2,f,M)$*, where* $\partial^2_C f(\overline{x})$ *is Clarke generalized Hessian of* f *at* $\overline{x}$*.*

Corollary 4.10. *Assume that* $f \in C^{1,1}$*.*

(i) If $\exists \lambda \in D^+ : \lambda \nabla f(\overline{x}) = 0$ *and* $\lambda N(v,v) > 0, \forall v \in T(M,\overline{x}) \cap C(f,\overline{x})\backslash\{0\}, \forall N \in \partial^2_C f(\overline{x})$*, then* $\overline{x} \in \text{strl}(2,f,M)$*.*

(ii) If $\exists \lambda \in D^{++} : \lambda \nabla f(\overline{x}) = 0$ *and* $\lambda N(v,v) > 0, \forall v \in T(M,\overline{x}) \cap \text{Ker}\nabla f(\overline{x})\backslash\{0\}, \forall N \in \partial^2_C f(\overline{x})$*, then* $\overline{x} \in \text{strl}(2,f,M)$*.*

5. Problem with equality-inequaility constraints

Let V, Z be Banach spaces, $g : X \to V, h : X \to Z$ two functions and $S \subset X, K \subset V$ arbitrary subsets. Let M be the set defined by

$$M := \{x \in X : g(x) \in -K, h(x) = 0\}. \tag{1}$$

We consider following problem

$$D - \min f(x), \quad \text{subject to} \quad x \in M \cap S \tag{2}$$

Setting $C(M,\overline{x}) := \{v \in X : \nabla g(\overline{x})v \in \text{clcone}(-K - g(\overline{x})), \nabla h(\overline{x})v = 0\}$. For $(\lambda, \beta, \gamma) \in D^+ \times K^+ \times Z^*$ we define Lagrangian function

$$L(x, \lambda, \beta, \gamma) = \lambda f(x) + \beta g(x) + \gamma h(x).$$

Assume that $(\nabla f(\overline{x}), B_f(\overline{x})), (\nabla g(\overline{x}), B_g(\overline{x}))$ and $(\nabla h(\overline{x}), B_h(\overline{x}))$ are asymptotically p-compact second-order approximation of f, g and h at $\overline{x}$, respectively, with $B_f(\overline{x}), B_g(\overline{x})$ and $B_h(\overline{x})$ being norm bounded. Then, we have

Theorem 5.1. *If* $\forall v \in C(M,\overline{x}) \cap T(S,\overline{x}) \cap C(f,\overline{x})\backslash\{0\}, \exists(\lambda,\beta,\gamma) \in D^+ \times K^+ \times Z^*$*:*

(i) $\beta g(\overline{x}) = 0$,

(ii) $\nabla L(\overline{x}, \lambda, \beta, \gamma) = 0$,

(iii) $(\lambda M+\beta N+\gamma P)(v, v) > 0, \forall (M, N, P) \in (\text{p-cl}B_f(\overline{x}), \text{p-cl}B_g(\overline{x}), \text{p-cl}B_h(\overline{x}))$,

(iv) $\lambda N(v, v) > 0, \forall N \in \text{p-}B_f(\overline{x})_\infty \backslash \{0\}$,

then $\overline{x} \in \text{strl}(2, f, M \cap S)$.

Proof. Let $F(x) = L(x, \lambda, \beta, \gamma)$. Then $F(x) = \lambda f(x) + \beta g(x) + \gamma h(x) \leq f(x), \forall x \in M$ and $F(\overline{x}) = \lambda f(\overline{x}) + \beta g(\overline{x}) + \gamma h(\overline{x}) = \lambda f(\overline{x})$. Besides, we have $\nabla F(\overline{x}) = \nabla L(\overline{x}, \lambda, \beta, \gamma) = 0$. Thus, $F(x)$ is a weak (local) support for f at $\overline{x}$ on $M \cap S$ and $B_F(\overline{x}) = \lambda B_f(\overline{x}) + \beta B_g(\overline{x}) + \gamma B_h(\overline{x})$. Moreover, since $B_g(\overline{x})$ and $B_h(\overline{x})$ are norm bounded, $\text{p-cl}B_F(\overline{x}) = \lambda.\text{p-cl}B_f(\overline{x}) + \beta.\text{p-cl}B_g(\overline{x}) + \gamma.\text{p-cl}B_h(\overline{x})$ and $\text{p-}B_F(\overline{x})_\infty = \lambda.\text{p-}B_f(\overline{x})_\infty$. From (iii), (iv), this implies, $\mathcal{N}(v, v) > 0, \forall \mathcal{N} \in \text{p-cl}B_F(\overline{x}) \cup \text{p-}B_F(\overline{x})_\infty \backslash \{0\}$. Thus, applying Theorem 4.3 we obtain above results with attention $T(M \cap S, \overline{x}) \subset C(M, \overline{x}) \cap T(S, \overline{x})$. □

Remark 5.2. *In the case* $C(M, \overline{x}) \cap T(S, \overline{x}) \cap C(f, \overline{x}) = \{0\}$, *since* $T(M \cap S, \overline{x}) \subset C(M, \overline{x}) \cap T(S, \overline{x})$, *we have* $T(M \cap S, \overline{x}) \cap C(f, \overline{x}) = \{0\}$. *From Theorem 4.2,* $\overline{x} \in \text{strl}(1, f, M \cap S)$ *and therefore* $\overline{x} \in \text{strl}(2, f, M \cap S)$.

In particular $S = X$, we induce the following corollary

Corollary 5.3. *If* $\forall v \in C(M, \overline{x}) \cap C(f, \overline{x}) \backslash \{0\}, \exists (\lambda, \beta, \gamma) \in D^+ \times K^+ \times Z^*$:

(i) $\beta g(\overline{x}) = 0$,

(ii) $\nabla L(\overline{x}, \lambda, \beta, \gamma) = 0$,

(iii) $(\lambda M+\beta N+\gamma P)(v, v) > 0, \forall (M, N, P) \in (\text{p-cl}B_f(\overline{x}), \text{p-cl}B_g(\overline{x}), \text{p-cl}B_h(\overline{x}))$,

(iv) $\lambda N(v, v) > 0, \forall N \in \text{p-}B_f(\overline{x})_\infty \backslash \{0\}$,

then $\overline{x} \in \text{strl}(2, f, M \cap S)$.

Corollary 5.4. *If* $\forall v \in T(M, \overline{x}) \cap C(f, \overline{x}) \backslash \{0\}, \exists (\lambda, \beta, \gamma) \in D^+ \times K^+ \times Z^*$:

(i) $\beta g(\overline{x}) = 0$,

(ii) $\nabla L(x_0, \lambda, \beta, \gamma) = 0$,

(iii) $(\lambda M+\beta N+\gamma P)(v, v) > 0, \forall (M, N, P) \in (\text{p-cl}B_f(\overline{x}), \text{p-cl}B_g(\overline{x}), \text{p-cl}B_h(\overline{x}))$,

(iv) $\lambda N(v, v) > 0, \forall N \in \text{p-}B_f(\overline{x})_\infty \backslash \{0\}$,

then $\overline{x} \in \text{strl}(2, f, M \cap S)$.

Remark 5.5. *This Corollary show the advantages of our results because it requires the weaker hypotheses than the Theorem* 4.2 *in [11].*

Example 5.6. *Let* $X = V = Z = \mathbb{R}, Y = \mathbb{R}^2, D = \mathbb{R}^2_+, K = \{0\}, S = X, \overline{x} = 0, f(x) = (|x|^{\frac{4}{3}}, 2x^2), g(x) = x^2 - 4x, h(x) = 0$. *Then,* $M = [0;4], \nabla f(\overline{x}) = (0;0), B_f(\overline{x}) = \{(a;4) : a > 1\}, \text{p-cl}B_f(\overline{x}) = \{(a;4) : a \geq 1\}, \text{p-}B_f(\overline{x})_\infty = \{(a;0) : a \geq 0\}\}, C(f,\overline{x}) = \mathbb{R}$ *and* $T(M,\overline{x}) = \mathbb{R}_+, \nabla g(\overline{x}) = -4, B_g(\overline{x}) = 0, \nabla h(\overline{x}) = 0, B_h(\overline{x}) = 0$. *It is clear to see that we can not apply results in [11], because neither* f *is belong to* $C^{1,1}$ *nor* f *is twice differentiable at* $\overline{x}$. *However, if we choose* $\lambda = (0;1), \beta = \gamma = 0$, *then we can apply Corollary 5.4. Thus* $\overline{x} \in \text{strl}(2, f, M)$.

In the rest of paper, we always assume that $X = \mathbb{R}^m, Y = \mathbb{R}^n, V = \mathbb{R}^p, Z = \mathbb{R}^q$ and $f : X \to Y, g : X \to V$ and $h : X \to Z$ are in class $C^{1,1}$.

Theorem 5.7. *If* $\forall v \in C(M,\overline{x}) \cap T(S,\overline{x}) \cap C(f,\overline{x})\backslash\{0\}, \exists(\lambda,\beta,\gamma) \in D^+ \times K^+ \times Z^*$*:*

(i) $\beta g(\overline{x}) = 0$,

(ii) $\nabla L(\overline{x},\lambda,\beta,\gamma) = 0$,

(iii) $\lambda N(v,v) > 0, \forall N \in \partial^2_C L(\overline{x},\lambda,\beta,\gamma)$,

then $\overline{x} \in \text{strl}(2, f, M \cap S)$.

The proof of the above theorem was directly deduced form the Theorem 5.1, remark 2.4, and the fact that $\partial^2_C g(\overline{x}), \partial^2_C h(\overline{x})$ are norm bounded. From above results, we have following corollaries.

Corollary 5.8. *If* $\forall v \in C(M,\overline{x}) \cap C(f,\overline{x})\backslash\{0\}, \exists(\lambda,\beta,\gamma) \in D^+ \times K^+ \times Z^*$*:*

(i) $\beta g(\overline{x}) = 0$,

(ii) $\nabla L(\overline{x},\lambda,\beta,\gamma) = 0$,

(iii) $\lambda N(v,v) > 0, \forall N \in \partial^2_C L(\overline{x},\lambda,\beta,\gamma)$,

then $\overline{x} \in \text{strl}(2, f, M)$.

Corollary 5.9. *If* $\forall v \in T(M,\overline{x}) \cap C(f,\overline{x})\backslash\{0\}, \exists(\lambda,\beta,\gamma) \in D^+ \times K^+ \times Z^*$*:*

(i) $\beta g(\overline{x}) = 0$,

(ii) $\nabla L(\overline{x},\lambda,\beta,\gamma) = 0$,

(iii) $\lambda N(v,v) > 0, \forall N \in \partial^2_C L(\overline{x},\lambda,\beta,\gamma)$,

then $\overline{x} \in \text{strl}(2, f, M)$.

To end this section, we give a sufficient condition for local minimum.

Corollary 5.10. *If there exists* $\delta > 0$ *such that, for each* $v \in M_\delta(\overline{x}) := \{\alpha(x - \overline{x}) : \alpha \geq 0, x \in M, ||x-\overline{x}|| \leq \delta\}$, *one can find a vector* $(\lambda,\beta,\gamma) \in D^{++} \times K^+ \times Z^*$ *satisfying* *(i)* $\beta g(\overline{x}) = 0,$

(ii) $\nabla L(\overline{x}, \lambda, \beta, \gamma) = 0,$

(iii) $N(v, v) \geq 0, \forall N \in \{\partial_C^2 L(x, \lambda, \beta, \gamma) : ||x - \overline{x}|| \leq \delta\}$

then $\overline{x} \in$lmin(f, M).

Proof. If $\overline{x}$ is not a local minimum, then there exists $(x_n) \subset M, x_n \to \overline{x}$ such that

$$f(x_n) - f(\overline{x}) \in -D \backslash \{0\}, \forall n \in \mathbb{N}.$$

This implies

$$\begin{aligned} 0 > &\ \langle \lambda, f(x_n) - f(\overline{x}) \rangle \\ \geq &\ L(x_n, \lambda, \beta, \gamma) - L(\overline{x}, \lambda, \beta, \gamma) \\ \geq &\ \nabla L(\overline{x}, \lambda, \beta, \gamma)(x_n - \overline{x}) + \tfrac{1}{2} N_n(x_n - \overline{x}, x_n - \overline{x}), \end{aligned}$$

for some $N_n \in$ clconv$\{\partial_C^2 L(x, \lambda, \beta, \gamma) : x \in [x_n, \overline{x}]\}$. For n sufficiently large such that $\|x_n - \overline{x}\| \leq \delta$, we conclude that

$$L(x_n, \lambda, \beta, \gamma) - L(\overline{x}, \lambda, \beta, \gamma) \geq 0.$$

A contradiction. □

If M is a convex set. By the same argument we have

Corollary 5.11. *If M is a convex set and exists $\delta > 0$ such that, for each $v \in T(M, \overline{x})$, one can find a vector $(\lambda, \beta, \gamma) \in D^{++} \times K^+ \times Z^*$ satisfying*

(i) $\beta g(\overline{x}) = 0,$

(ii) $\nabla L(\overline{x}, \lambda, \beta, \gamma) = 0,$

(iii) $N(v, v) \geq 0, \forall N \in \{\partial_C^2 L(x, \lambda, \beta, \gamma) : ||x - \overline{x}|| \leq \delta\},$

then $\overline{x} \in$lmin(f, M).

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