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LIE OPERATORS FOR TRUST DYNAMICS ON THE UNIT SPHERE IN COMPLEX NETWORKS

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Abstract

This paper is to introduce a geometric framework for trust modeling in complex networks in which trust states are embedded as unit vectors in a Hilbert space. Trust evolution is formulated as a continuous dynamical system generated by Lie operators acting on the unit sphere, ensuring norm preservation and stability of the representation. We show that network interactions naturally induce skew-adjoint operators, leading to flows that correspond to rotations on the trust sphere. This geometric structure guarantees that trust propagation follows geodesic trajectories, providing both mathematical consistency and interpretability. The proposed framework establishes a connection between graph-based interactions, Lie algebraic dynamics, and geometric deep learning. It offers a principled foundation for modeling trust propagation with stability, invariance, and continuous-time dynamics in complex networked systems.

1 Introduction

Trust is a fundamental mechanism underlying cooperation, recommendation, and decision-making in distributed systems, including social networks, peer-to-

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peer platforms, and online marketplaces. In such environments, agents must assess the reliability of other participants based on partial, uncertain, and potentially conflicting information. Consequently, the problem of modeling trust propagation in complex networks has attracted significant attention in social computing and network science [4, 8].

Early computational approaches to trust modeling were largely based on graph ranking and reputation aggregation mechanisms. The PageRank algorithm introduced a global ranking paradigm for measuring node importance in large-scale networks [1]. Building on this idea, the EigenTrust algorithm was proposed to compute reputation scores in peer-to-peer systems, with the goal of mitigating malicious behavior [2]. Similarly, TrustRank propagates trust from a set of reliable seed nodes to detect and suppress web spam [3]. Recent work has also explored hybrid trust models combining structural and semantic information in social networks [5]. These methods demonstrate that network topology plays a central role in the evaluation and propagation of trust.

More recent developments have incorporated tools from spectral graph theory and machine learning. Spectral methods characterize trust propagation through the eigenstructure of graph operators [6], while graph signal processing provides a framework for analyzing diffusion processes on networks [7]. In parallel, graph neural networks (GNNs) have enabled data-driven approaches to trust representation learning via message-passing architectures [10, 11]. These models have achieved strong empirical performance in tasks such as node classification, recommendation, and link prediction.

Despite these advances, most existing approaches represent trust states in Euclidean spaces. Such representations may suffer from several limitations. First, iterative propagation mechanisms can lead to uncontrolled growth or attenuation of trust values, resulting in instability. Second, Euclidean embeddings do not intrinsically enforce normalization constraints, making the interpretation of trust magnitudes dependent on scale. These issues can reduce both the robustness and interpretability of trust dynamics.

Recent progress in geometric deep learning suggests that relational data is often more naturally modeled in non-Euclidean spaces, such as manifolds, spheres, or hyperbolic spaces [9]. These representations incorporate intrinsic geometric constraints that can improve stability and expressiveness. In particular, the unit sphere provides a natural normalized state space, ensuring bounded representations and preventing divergence of state magnitudes during propagation.

Motivated by these observations, we propose a geometric framework for trust propagation based on Lie operator dynamics. In this formulation, trust states are embedded as unit vectors in a Hilbert space, forming a unit sphere that serves as the state space of the system. Trust interactions are modeled by skew-adjoint linear operators, which generate continuous-time flows preserving the norm constraint. As a consequence, trust evolution is described by rotationlike dynamics along geodesics on the sphere, yielding stable and interpretable trajectories.

The main contributions of this paper are summarized as follows:

- We introduce a geometric representation of trust states as unit vectors in a Hilbert space, thereby defining a normalized and bounded state space for trust dynamics.

- We formulate trust propagation as a continuous-time dynamical system generated by skew-adjoint operators, which guarantees invariance of the norm and constrains the evolution to the unit sphere.

- We show that network interactions naturally induce Lie algebraic structures, whose associated flows generate geodesic trajectories on the unit sphere.

- We establish a unified framework that connects graph-based trust modeling, functional analysis, and geometric deep learning, providing a principled foundation for trust propagation in complex networks.

The proposed approach offers a principled geometric perspective on trust propagation in complex networks and opens new directions for integrating continuous-time dynamics and symmetry-preserving operators into modern learning architectures.

2 Trust State Representation on the Unit Sphere

The proposed framework is based on a three-stage abstraction process:

- *Trust features*: raw observations and interaction data (e.g., ratings, transaction histories, or behavioral signals),

- *Trust states*: vector representations obtained by embedding these features into a Hilbert space,

- *Unit trust states*: normalized representations obtained by projecting trust states onto the unit sphere.

This abstraction separates the representation of data from the imposition of geometric constraints, thereby enabling the formulation of trust dynamics as norm-preserving transformations on a structured state space. In particular, trust modeling is realized through the mapping

$$\text{features} \;\longrightarrow\; x \in H \;\longrightarrow\; \hat{x} = \frac{x}{\|x\|} \in S(H),$$

where the normalization step enforces a unit-norm constraint. As a result, all admissible trust states lie on a geometrically constrained manifold, which is well suited for invariant dynamical modeling.

Let H be a real Hilbert space equipped with the inner product

$$\langle \cdot, \cdot \rangle$$

and the induced norm

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Within this framework, the trust state of an agent is represented as a unit vector in H.

Definition 1 (Trust State)**.** *A trust state is a vector $x \in H$ satisfying*

$$\|x\| = 1.$$

Definition 2 (Unit Trust Sphere)**.** *The set of all admissible trust states is the unit sphere*

$$S(H) := \{x \in H \mid \|x\| = 1\}.$$

The space $S(H)$ serves as the geometric state space for trust propagation. In this representation, the direction of a vector encodes the qualitative configuration of trust, while the unit-norm constraint ensures boundedness and eliminates scale ambiguity. Consequently, trust dynamics are intrinsically constrained to evolve on a nonlinear manifold.

Proposition 1. *Suppose H be a real Hilbert space and let*

$$S(H) := \{x \in H \mid \|x\| = 1\}$$

be the unit sphere. Then $S(H)$ is a smooth submanifold of H, and for each $x \in S(H)$, the tangent space at x is given by

$$T_x S(H) = \{v \in H \mid \langle x, v \rangle = 0\}.$$

Proof. Let $x(t)$ be a differentiable curve[1] in $S(H)$ such that $x(0) = x$. Since $\|x(t)\| = 1$ for all t, we have

$$\langle x(t), x(t) \rangle = 1.$$

Differentiating with respect to t yields

$$\frac{d}{dt}\langle x(t), x(t) \rangle = 2\langle x(t), \dot{x}(t) \rangle = 0.$$

[1] Differentiability is understood in the sense of Fréchet differentiability for mappings from an interval in $\mathbb{R}$ into H. The normalization mapping $x \mapsto x/\|x\|$ is smooth on $H \setminus \{0\}$; see, e.g., [12].

Evaluating at $t = 0$, we obtain

$$\langle x, \dot{x}(0) \rangle = 0.$$

Thus any tangent vector $v = \dot{x}(0)$ satisfies

$$\langle x, v \rangle = 0.$$

Conversely, let $v \in H$ such that $\langle x, v \rangle = 0$. Define

$$x(t) = \frac{x + tv}{\|x + tv\|}.$$

Since $\|x\| = 1$ and $\langle x, v \rangle = 0$, we have

$$\|x + tv\|^2 = 1 + t^2 \|v\|^2,$$

which is nonzero for all sufficiently small t. Hence $x(t)$ is well-defined and satisfies $x(t) \in S(H)$.

Differentiating at $t = 0$, we obtain

$$\dot{x}(0) = v - \langle x, v \rangle x = v,$$

since $\langle x, v \rangle = 0$. Therefore, every vector orthogonal to x arises as the tangent vector of a curve on $S(H)$ at x. □

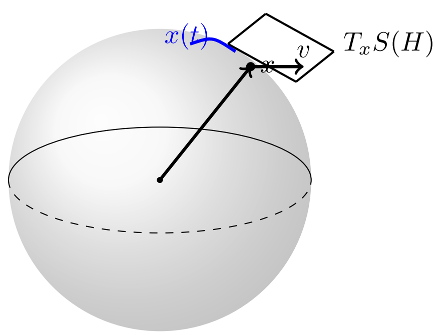


Figure 1: Geometric structure of the unit trust sphere. The trust state $x \in S(H)$ evolves along trajectories $x(t)$ constrained to the sphere. Admissible directions belong to the tangent space $T_x S(H)$ and are orthogonal to x, ensuring preservation of the unit-norm constraint.

The above characterization shows that admissible directions of trust evolution are precisely those orthogonal to the current state. Therefore, the dynamics cannot alter the magnitude of the trust state, but only its direction. In geometric terms, trust propagation corresponds to motion along the surface of the unit sphere, which guarantees normalization of trust states throughout the evolution.

3 Lie Operator Formulation of Trust Dynamics

Trust propagation in complex networks can be naturally modeled as a continuous-time dynamical process. In many areas of network science and control theory, the evolution of states on a graph is described by linear differential equations because they provide a mathematically tractable representation of interaction-driven dynamics [6, 8]. Linear operators capture how the influence of neighboring nodes modifies the current state of an agent while maintaining important analytical properties such as stability and spectral structure.

3.1 Skew-Adjoint Operators

Definition 3 (Skew-Adjoint Operator)**.** *Let H be a real Hilbert space and let $L : H \to H$ be a bounded linear operator. The operator L is said to be* skew-adjoint *if*

$$L^* = -L,$$

where L^* denotes the adjoint of L.

Proposition 2. *Let H be a real Hilbert space and let $L : H \to H$ be a bounded linear operator. Then the following statements are equivalent:*

(i) L is skew-adjoint, i.e., $L^* = -L$;

(ii) $\langle Lx, y\rangle = -\langle x, Ly\rangle, \quad \forall\, x, y \in H$.

Proposition 3. *Let $L : H \to H$ be a skew-adjoint operator. Then for every $x \in H$,*

$$\langle x, Lx\rangle = 0.$$

In particular,

$$Lx \in \{x\}^{\perp},$$

that is, Lx is orthogonal to x.

Proof. Using the definition of the adjoint, we have

$$\langle Lx, x\rangle = \langle x, L^*x\rangle.$$

Since $L^* = -L$, it follows that

$$\langle Lx, x\rangle = -\langle x, Lx\rangle.$$

Hence

$$\langle x, Lx\rangle + \langle Lx, x\rangle = 0,$$

which implies

$$2\langle x, Lx\rangle = 0.$$

Therefore $\langle x, Lx \rangle = 0$. □

The above proposition shows that the action of a skew-adjoint operator is orthogonal to the current state. Consequently, if $x \in S(H)$, then $Lx \in T_x S(H)$, and the induced motion is tangential to the unit sphere. In geometric terms, skew-adjoint operators generate infinitesimal rotations: they modify the direction of a state vector without changing its norm.

3.2 Trust Evolution Equation

Trust states are represented as normalized vectors

$$x(t) \in S(H), \qquad S(H) := \{x \in H : \|x\| = 1\},$$

where H is a real Hilbert space.

Definition 4 (Trust Dynamics)**.** *Let $L : H \to H$ be a bounded linear operator. The evolution of a trust state is described by the differential equation*

$$\frac{dx(t)}{dt} = Lx(t), \quad t \geq 0,$$

with initial condition $x(0) \in S(H)$.

The operator L encodes the interaction structure of the network and determines how trust propagates among agents.

3.3 Tangential Structure of the Dynamics

Lemma 1. *Let $x(t)$ be a solution of*

$$\frac{dx(t)}{dt} = Lx(t).$$

If $x(t) \in S(H)$, then

$$Lx(t) \in T_{x(t)} S(H).$$

Proof. Since $\|x(t)\| = 1$, we have

$$\langle x(t), x(t) \rangle = 1.$$

Differentiating yields

$$\langle \dot{x}(t), x(t) \rangle + \langle x(t), \dot{x}(t) \rangle = 0.$$

Substituting $\dot{x}(t) = Lx(t)$ gives

$$\langle Lx(t), x(t) \rangle + \langle x(t), Lx(t) \rangle = 0.$$

Hence $\langle x(t), Lx(t)\rangle = 0$, which implies

$$Lx(t) \in T_{x(t)}S(H).$$

□

This lemma shows that the evolution is intrinsically tangential to the sphere, and therefore modifies only the direction of the trust state.

3.4 Norm Preservation and Invariance

Proposition 4. *If L is skew-adjoint, then the flow generated by*

$$\frac{dx(t)}{dt} = Lx(t)$$

preserves the unit sphere $S(H)$.

Proof. We compute

$$\frac{d}{dt}\|x(t)\|^2 = \langle \dot{x}(t), x(t)\rangle + \langle x(t), \dot{x}(t)\rangle.$$

Substituting $\dot{x}(t) = Lx(t)$ gives

$$\frac{d}{dt}\|x(t)\|^2 = \langle Lx(t), x(t)\rangle + \langle x(t), Lx(t)\rangle.$$

If $L^* = -L$, then

$$\langle Lx, x\rangle = \langle x, L^*x\rangle = -\langle x, Lx\rangle,$$

hence

$$\frac{d}{dt}\|x(t)\|^2 = 0.$$

Therefore $\|x(t)\|$ is constant, and $x(t) \in S(H)$ for all t. □

3.5 Lie Flow of Trust States

The solution to the evolution equation is given by

$$x(t) = e^{tL}x_0,$$

where e^{tL} denotes the operator exponential.

Proposition 5. *If L is skew-adjoint, then e^{tL} is a unitary operator for all $t \in \mathbb{R}$.*

Proof. Using $(e^{tL})^* = e^{tL^*}$ and $L^* = -L$, we obtain

$$(e^{tL})^{=} e^{-tL}.$$

Thus

$$e^{tL}(e^{tL})^{=} e^{tL} e^{-tL} = I,$$

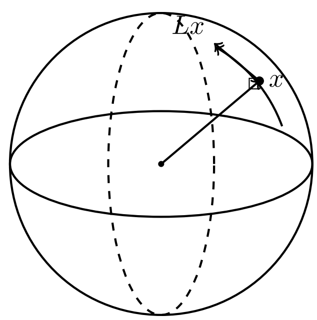
which shows that e^{tL} is unitary. □

Theorem 1. *Let L be a skew-adjoint operator on H. Then the flow*

$$x(t) = e^{tL} x_0$$

defines a one-parameter group of isometries on $S(H)$. In particular, trust dynamics correspond to a rotation-type flow on the unit sphere.

The evolution generated by a skew-adjoint operator is tangential and norm-preserving, and therefore corresponds to motion along the surface of the unit sphere. Such flows can be interpreted as continuous rotations, ensuring that trust states remain bounded and evolve in a geometrically consistent manner.



Trust sphere $S(H)$

Figure 2: Geometric interpretation of trust dynamics on the unit sphere. The trust state x lies on the sphere $S(H)$ and is represented by a normalized vector. A skew-adjoint operator generates a tangent vector Lx orthogonal to x, producing rotational motion along the surface of the sphere while preserving the norm of the trust state.

4 Trust Propagation in Networks

Let a trust network be represented by a directed graph

$$G = (V, E),$$

where $|V| = n$, together with a weighted adjacency matrix $A \in \mathbb{R}^{n\times n}$. The entry A_{ij} encodes the strength of trust that node i assigns to node j.

We identify the global trust state of the network with a vector

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

which we regard as an element of the Hilbert space $H = \mathbb{R}^n$ equipped with the standard inner product.

4.1 Interaction Operator

To capture directional trust interactions, we extract the antisymmetric component of the adjacency matrix.

Definition 5 (Trust Interaction Operator)**.** *The trust interaction operator associated with the network G is defined by*

$$L := A - A^T.$$

The operator L isolates the asymmetric component of the trust relations. Symmetric interactions do not contribute to L, whereas imbalanced trust relations generate nontrivial directional effects.

Proposition 6. *The operator $L = A - A^T$ is skew-adjoint on the Hilbert space $H = \mathbb{R}^n$.*

Proof. Taking the transpose yields

$$L^T = (A - A^T)^T = A^T - A = -L.$$

Since the adjoint coincides with the transpose in $\mathbb{R}^n$ equipped with the standard inner product, we obtain

$$L^* = L^T = -L.$$

Thus L is skew-adjoint. □

4.2 Network-Induced Trust Dynamics

The evolution of trust states is governed by the linear system

$$\frac{dx(t)}{dt} = Lx(t), \quad x(0) \in S(H).$$

For each node $i \in V$, the local dynamics are given by

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} L_{ij} x_j(t).$$

Equivalently,

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n}(A_{ij} - A_{ji})\, x_j(t),$$

which shows explicitly that only asymmetric trust relations contribute to the evolution.

4.3 Geometric Interpretation

Since L is skew-adjoint, the induced flow preserves the unit sphere $S(H)$ and evolves tangentially along it. Consequently, network-driven trust propagation admits the following interpretation:

- asymmetric trust relations act as generators of rotational dynamics on the trust sphere,
- the evolution is norm-preserving, ensuring bounded and stable trust states,
- global trust patterns emerge as collective geometric flows induced by local interactions.

5 Lie Algebraic Structure of Trust Dynamics

The geometric trust framework developed in the previous sections naturally admits a Lie-theoretic formulation. Since trust evolution is generated by skew-adjoint operators, the collection of admissible trust generators forms a Lie algebra, while the corresponding trust transformations form a unitary Lie group [15, 16, 17]. This structure provides a rigorous mathematical foundation for continuous trust propagation and interaction among multiple trust mechanisms.

5.1 Trust Lie Algebra

Let H be a real or complex Hilbert space and denote by $\mathcal{B}(H)$ the Banach algebra of bounded linear operators on H [12, 13].

Definition 6 (Trust Lie Algebra)**.** *The set*

$$\mathfrak{T} = \{L \in \mathcal{B}(H) : L^* = -L\}$$

is called the trust Lie algebra*. Elements of* $\mathfrak{T}$ *are referred to as* infinitesimal trust generators*.*

The skew-adjoint condition guarantees that generated trajectories remain on the trust sphere and is analogous to the role of infinitesimal generators of unitary flows in operator theory [14, 13]. For $L_1, L_2 \in \mathfrak{T}$, define the commutator bracket

$$[L_1, L_2] = L_1 L_2 - L_2 L_1.$$

Proposition 7. *The space $\mathfrak{T}$ is closed under the commutator operation. Consequently,*

$$[L_1, L_2] \in \mathfrak{T}$$

for all $L_1, L_2 \in \mathfrak{T}$.

Proof. Using the properties of the adjoint operator,

$$[L_1, L_2]^* = (L_1 L_2 - L_2 L_1)^* = L_2^* L_1^* - L_1^* L_2^*.$$

Since $L_1^* = -L_1$ and $L_2^* = -L_2$,

$$[L_1, L_2]^* = L_2 L_1 - L_1 L_2 = -[L_1, L_2].$$

Therefore $[L_1, L_2]$ is skew-adjoint. □

Theorem 2 (Trust Lie Algebra)**.** *The pair*

$$(\mathfrak{T}, [\cdot, \cdot])$$

forms a real Lie algebra.

Proof. The space $\mathfrak{T}$ is a real vector space and is closed under the commutator by Proposition 7. The bracket is bilinear, antisymmetric, and satisfies the Jacobi identity, which is a fundamental property of operator commutators [15, 16, 17].

$$[L_1, L_2] = -[L_2, L_1],$$

and satisfies the Jacobi identity

$$[L_1, [L_2, L_3]] + [L_2, [L_3, L_1]] + [L_3, [L_1, L_2]] = 0.$$

Hence $(\mathfrak{T}, [\cdot, \cdot])$ is a Lie algebra. □

5.2 Trust Lie Group

The Lie algebra $\mathfrak{T}$ generates a Lie group of trust transformations.

Definition 7 (Trust Transformation Group)**.** *Define*

$$\mathcal{U}_T = \{U \in \mathcal{B}(H) : U^*U = UU^* = I\}.$$

The group $\mathcal{U}_T$ *is called the* trust transformation group*.*

Each element of $\mathcal{U}_T$ preserves the trust norm and therefore maps trust states to trust states.

Theorem 3 (Exponential Map)**.** *For every* $L \in \mathfrak{T}$*,*

$$\exp(L) = \sum_{k=0}^{\infty} \frac{L^k}{k!}$$

belongs to $\mathcal{U}_T$*. Moreover, the exponential map*

$$\exp : \mathfrak{T} \to \mathcal{U}_T$$

defines a local diffeomorphism near the identity [15, 16, 18].

Proof. Since $L^* = -L$,

$$(\exp L)^* = \exp(L^*) = \exp(-L).$$

Therefore

$$(\exp L)^* \exp L = \exp(-L) \exp(L) = I.$$

Hence $\exp(L)$ is unitary. The local diffeomorphism property follows from standard Lie group theory [15, 18]. □

5.3 Trust Evolution

Trust propagation is modeled by the linear evolution equation

$$\frac{dT(t)}{dt} = LT(t), \qquad T(0) = T_0, \tag{1}$$

where $L \in \mathfrak{T}$. Such equations are classical examples of linear evolution systems generated by bounded operators [19, 20].

Theorem 4 (Norm-Preserving Trust Flow)**.** *Equation* (1) *admits the unique solution*

$$T(t) = e^{tL} T_0.$$

Furthermore,

$$\|T(t)\| = \|T_0\|, \qquad t \geq 0.$$

Proof. The existence and uniqueness of the solution follow from the theory of strongly continuous semigroups generated by bounded operators [19, 20]. The solution is given by the exponential operator,

$$T(t) = e^{tL} T_0.$$

By Theorem 3, e^{tL} is unitary. Therefore

$$\|T(t)\|^2 = \langle e^{tL} T_0, e^{tL} T_0 \rangle = \langle T_0, T_0 \rangle = \|T_0\|^2.$$

Taking square roots yields the result. □

5.4 Composition of Trust Mechanisms

Suppose $L_1, L_2 \in \mathfrak{T}$ represent two trust propagation mechanisms.

Theorem 5 (Baker–Campbell–Hausdorff Formula). *There exists $L_{\mathrm{eff}} \in \mathfrak{T}$ such that*

$$e^{L_1} e^{L_2} = e^{L_{\mathrm{eff}}},$$

where

$$L_{\mathrm{eff}} = L_1 + L_2 + \frac{1}{2}[L_1, L_2] + \frac{1}{12}[L_1, [L_1, L_2]] - \frac{1}{12}[L_2, [L_1, L_2]] + \cdots .$$

The series is given by the classical Baker–Campbell–Hausdorff expansion [15, 16, 17].

The BCH expansion shows that trust interactions are generally noncommutative. The commutator term

$$\frac{1}{2}[L_1, L_2]$$

represents the first-order interaction between two trust mechanisms, while higher nested commutators encode increasingly complex dependency structures.

From a network perspective, the quantity

$$[L_1, L_2]$$

measures the discrepancy between applying one trust propagation process before the other and reversing the order. Thus, nonzero commutators quantify interaction effects, coupling strength, and structural heterogeneity in trust dynamics.

Together, Theorems 2, 3, 4, and 5 establish a complete Lie-theoretic framework for continuous trust propagation on complex networks, extending classical trust propagation models such as EigenTrust and TrustRank into a geometric dynamical systems setting [2, 3, 5].

6 Consensus and Equilibrium Trust States

The rotational nature of the proposed trust dynamics naturally raises the question of equilibrium and consensus. In practical trust systems, stable trust configurations correspond to states that remain invariant under propagation.

Definition 8 (Consensus Trust State)**.** *A trust state $x^* \in S(H)$ is called a consensus trust state if*

$$Lx^* = 0.$$

Theorem 6 (Consensus Characterization)**.** *Let L be a skew-adjoint trust interaction operator. A trust state x^* is stationary under the trust dynamics*

$$\frac{dx}{dt} = Lx$$

if and only if

$$x^* \in \ker(L).$$

Proof. Suppose x^* is stationary. Then

$$\frac{dx}{dt} = 0,$$

which implies

$$Lx^* = 0.$$

Hence $x^* \in \ker(L)$.

Conversely, if $Lx^* = 0$, then

$$x(t) = e^{tL}x^* = x^*.$$

Thus the state remains unchanged. □

Theorem 7 (Spectral Decomposition of Trust Dynamics)**.** *Let L be a skew-adjoint operator on a finite-dimensional Hilbert space. Then*

$$H = \ker(L) \oplus \mathcal{H}_{\text{osc}},$$

where $\mathcal{H}_{\text{osc}}$ is invariant under L.

For every initial trust state

$$x_0 = x_c + x_o,$$

with

$$x_c \in \ker(L), \qquad x_o \in \mathcal{H}_{\text{osc}},$$

the solution satisfies

$$x(t) = x_c + e^{tL} x_o.$$

7 Connections with Classical Trust Models

7.1 Connection with EigenTrust

The EigenTrust algorithm computes a global trust vector through the dominant eigenvector of a normalized trust matrix.

Suppose the trust matrix is symmetric,

$$A = A^T.$$

Then

$$L = A - A^T = 0.$$

Consequently,

$$x(t) = x_0.$$

Hence the proposed framework identifies symmetric trust configurations as equilibrium trust states.

Proposition 8. *If the network trust matrix is symmetric, then every trust state is stationary.*

7.2 Connection with TrustRank

TrustRank propagates trust from a collection of trusted seed nodes. To incorporate this mechanism, consider

$$\frac{dx}{dt} = Lx - \beta(I - P)x,$$

where

$$P$$

is the orthogonal projection onto the trusted seed subspace.

Theorem 8 (TrustRank-Type Convergence)**.** *Let $\beta > 0$. Then*

$$\|(I - P)x(t)\|$$

decreases monotonically and

$$(I - P)x(t) \to 0.$$

8 Numerical Illustration

To demonstrate the behavior of the proposed Lie-theoretic trust dynamics, we consider a simple trust network consisting of three agents. The purpose of this example is to illustrate how asymmetric trust relationships generate continuous trust evolution while preserving the normalization constraint.

8.1 Network Construction

Consider the directed trust network with adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This network represents a cyclic trust structure: agent 1 trusts agent 2, agent 2 trusts agent 3, and agent 3 trusts agent 1.

Following Definition 5, the trust interaction operator is

$$L = A - A^T.$$

Hence

$$L = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Since

$$L^T = -L,$$

the operator is skew-adjoint and therefore generates norm-preserving trust dynamics on the unit sphere.

The trust evolution equation is

$$\frac{dx(t)}{dt} = Lx(t), \qquad x(0) = x_0.$$

The solution is

$$x(t) = e^{tL}x_0.$$

By Theorem 1,

$$\|x(t)\| = \|x_0\|$$

for all $t \geq 0$.

8.2 Consensus Trust State

Consider the normalized trust state

$$x_c = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

A direct computation gives

$$Lx_c = \frac{1}{\sqrt{3}} \begin{pmatrix} 1-1 \\ -1+1 \\ 1-1 \end{pmatrix} = 0.$$

Therefore,

$$x_c \in \ker(L).$$

By Theorem 6,

$$x(t) = x_c$$

for all $t \geq 0$.

Thus the vector x_c is a consensus trust state. The trust configuration is perfectly balanced and no further propagation occurs. In this situation, every agent possesses the same trust level and the network remains in equilibrium.

8.3 Rotational Trust Dynamics

Next consider the initial trust state

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This state corresponds to a situation in which trust is initially concentrated entirely at the first agent.

Applying the trust operator yields

$$Lx_0 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

The resulting vector is orthogonal to x_0:

$$\langle x_0, Lx_0 \rangle = 0.$$

Consequently, the induced motion is tangent to the trust sphere and preserves the norm of the state.

Using the matrix exponential

$$x(t) = e^{tL} x_0,$$

the trust state evolves continuously along a spherical orbit. Table 1 reports representative values of the solution.

Table 1: Evolution of the trust state under Lie dynamics

t	$x_1(t)$	$x_2(t)$	$x_3(t)$
0.0	1.000	0.000	0.000
0.5	0.765	-0.322	0.557
1.0	0.390	-0.303	0.868
2.0	-0.275	0.749	0.603

For each time instant,

$$x_1(t)^2 + x_2(t)^2 + x_3(t)^2 = 1,$$

confirming the norm-preserving property predicted by Theorem 4.

8.4 Geometric Interpretation

The numerical experiment illustrates two fundamental properties of the proposed framework.

First, consensus trust states correspond precisely to vectors belonging to the kernel of the interaction operator. Such states remain invariant under trust propagation and represent equilibrium configurations.

Second, asymmetric trust relationships generate rotational trust dynamics. Rather than causing unbounded growth or decay, trust propagates along bounded trajectories on the unit sphere. The evolution changes only the direction of the trust state while preserving its magnitude.

This behavior contrasts with many classical propagation models where repeated iterations may amplify or attenuate trust values. The geometric Lie-operator formulation guarantees stability through intrinsic norm preservation and provides an interpretable representation of trust evolution as motion on a curved state space.

The example therefore illustrates how the proposed framework combines network interactions, geometric constraints, and Lie-theoretic dynamics within a unified mathematical model for trust propagation.

9 Conclusion

This paper developed a novel Lie-theoretic framework for trust propagation in complex networks by representing trust states as points on the unit sphere of a Hilbert space and modeling trust dynamics through skew-adjoint operators. Within this formulation, trust evolution is governed by continuous geometric flows that preserve the norm of trust states, thereby ensuring boundedness, stability, and mathematical consistency throughout the propagation process.

A central contribution of the work is the establishment of a rigorous connection between trust dynamics and the theory of Lie groups and Lie algebras. The collection of admissible trust generators was shown to form a Lie algebra under the commutator bracket, while the associated exponential map generates norm-preserving trust transformations. This structure provides a principled mathematical mechanism for describing, analyzing, and composing multiple trust propagation processes. Furthermore, the Baker–Campbell–Hausdorff expansion reveals that interactions among trust mechanisms are generally non-commutative, allowing higher-order dependencies and interaction effects to be characterized through nested commutator terms.

From a geometric perspective, the proposed framework interprets trust evolution as motion along a trust manifold rather than unrestricted movement in Euclidean space. Such a formulation naturally incorporates invariance properties and offers a mathematically robust foundation for studying trust dynamics in heterogeneous and large-scale networks. Consequently, the framework bridges ideas from operator theory, differential geometry, dynamical systems, and network science within a unified setting.

Beyond its theoretical significance, the proposed approach opens several promising research directions. Future work will investigate finite-dimensional matrix realizations of trust Lie algebras for practical network applications, connections with graph Laplacians and spectral network representations, and extensions to stochastic, time-varying, and heterogeneous trust environments. Another important direction is the integration of the proposed geometric formulation with graph neural networks, geometric deep learning architectures, and representation learning methods, where Lie operators may serve as inductive biases for modeling trust-aware interactions. Empirical validation on real-world trust and recommendation datasets will further assess the effectiveness of the framework and its potential for explainable trust modeling in modern intelligent systems.

Overall, the results establish a rigorous geometric and algebraic foundation for continuous trust propagation and provide a new perspective for the development of mathematically principled trust-aware learning models in complex networks.

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