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AN OPERATOR APPROACH TO THREE q-HERMITE POLYNOMIALS IN THE SPIRIT OF CIGLER

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Abstract

The purpose of this article is to define and then prove generating functions, operational formulas, power series representations, recurrences, tables, vector forms, determinant expressions, alternative operator formulas, q-Nielsen and Rodriguez formulas for three q-Hermite polynomials. Some of these polynomials have previously been investigated by Cigler, Kirschenhofer and Désarménien. Then we prove new q-orthogonality relations by q-integrals with finite integral limits for one of these polynomials. It turns out that a prerequisite, which is not sufficient, for this is that the polynomial is of q-Appell form. Therefore, we briefly outline q-Appell, and pseudo-q-Appell polynomials. We also investigate some pseudo-q-Hermite polynomials, q-analogues of x^{ν}, whose orthogonality with q-integration limits ± 1 is equivalent to our q-orthogonality by a simple change of variables in the q-integral.

Keywords: q-Hermite polynomial; q-Appell polynomial; q-Rodriguez formula ; q-orthogonality; q-integral; q-operator formula
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1 Introduction

Based on the articles [9], [11], [13] and [22], we shall study three different types of q-Hermite polynomials

$$\mathrm{H}_{\nu,s,q}(x), \psi_{\nu,s,q}(x) \text{ and } \mathrm{K}_{\nu,s,q}(x)$$

from the point of view of generating functions and operational formulas. Two of these polynomials are of q-Appell nature, and the remaining one is a pseudo-q-Appell polynomial. These polynomials depend on an extra parameter s. By simple methods we find q-difference equations and q-hypergeometric expressions for these polynomials. Rodriguez formulas for the three polynomials are obtained; one of these formulas is intimately connected with the orthogonality relation.

The paper is organized as follows: In section 2 we repeat the necessary q-calculus from [15]. Then, generalizations of the three q-Hermite type of polynomials are studied from the point of view of generating functions and operational formulas. These polynomials depend on a parameter s. Section 4 gives a brief introduction to the theory of these polynomials with corresponding operator formula definitions.

Section 5 examines recurrences, alternative operator formulas, and q-difference equations for the q-Hermite polynomials. Section 6 outlines the first q-real numbers, which are important for our umbral calculus. The q-Appell polynomials, which contain two of our q-Hermite polynomials, are also outlined. Furthermore, an umbral inverse formula between two q-Hermite polynomials is proved. Section 6.1 presents determinant expressions for the three polynomials.

In section 8 three operator formulas are introduced in connection with Rodriguez formulas. In section 9, subsections 9.1, 9.2 we introduce q-hypergeometric and power series representations, Rodriguez formulas, tables, vector forms, alternative operator formulas and q-Nielsen formulas for the q-Hermite polynomials. The Rodriguez formula is a prerequisite for the q-orthogonality proofs. Section 10 presents a new q-orthogonality for one of the q-Hermite polynomials with a square q-exponential function as weight function. In section 11, finally, some linear functionals and computations of moments are briefly outlined.

2 Basic q-calculus definitions

We now repeat some notations from our book [15]. Throughout, $\equiv$ denotes a definition.

Definition 1. Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}\,(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The power function is defined by

$$q^a \equiv e^{a\log(q)}. \tag{1}$$

Definition 2. The following notation is often used when we have long exponents.

$$\mathrm{QE}(x) \equiv q^x. \tag{2}$$

Definition 3. [15, p.19] The q-analogues of a complex number a, a natural number n the factorial and the semifactorial are defined as follows:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{0,1\}, \tag{3}$$

$$\{n\}_q \equiv \sum_{k=1}^{n} q^{k-1},\ \{0\}_q = 0,\ q \in \mathbb{C}\backslash\{0,1\}, \tag{4}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}\backslash\{0,1\}. \tag{5}$$

$$\{2n-1\}_q!! \equiv \prod_{k=1}^{n} \{2k-1\}_q,\ \{-1\}_q!! \equiv 1,\ q \in \mathbb{C}\backslash\{0,1\}, \tag{6}$$

$$\{2n\}_q!! \equiv \prod_{k=1}^{n} \{2k\}_q,\ q \in \mathbb{C}\backslash\{0,1\}. \tag{7}$$

Definition 4. The q-shifted factorial [15] is defined by

$$\langle a;q\rangle_n \equiv \prod_{m=0}^{n-1}(1-q^{a+m}). \tag{8}$$

There are three other types of q-shifted factorials [15]: In the equations (12) to (17) we assume that $(m,l)=1$. In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod $\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R}\times e^{2\pi i\theta},\ \theta \in \mathbb{R}$. The operator

$$\sim: \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by the 2-torsion

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{9}$$

By (9) it follows that

$$\widetilde{\langle a;q\rangle_n} = \prod_{m=0}^{n-1}(1+q^{a+m}), \tag{10}$$

where this time the tilde denotes an involution which changes a minus sign to a plus sign in all the n factors of $\langle a;q\rangle_n$. Furthermore we define

$$\widetilde{\langle a;q\rangle}_n \equiv \langle\widetilde{a};q\rangle_n. \tag{11}$$

The generalized tilde operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l\log q}. \tag{12}$$

We also need another generalization of the tilde operator.

$${}_k\widetilde{\langle a;q\rangle}_n \equiv \prod_{m=0}^{n-1}(\sum_{i=0}^{k-1} q^{i(a+m)}). \tag{13}$$

Formula (13) is used in (18).

The following, simple congruence rules [15] follow from (12).

Theorem 1.

$$\widetilde{\tfrac{m}{l}} a \pm b \equiv \widetilde{\tfrac{m}{l}(a \pm b)} \left(\mathrm{mod}\ \frac{2\pi i}{\log q}\right), \tag{14}$$

$$\sum_{k=1}^{n} \widetilde{\tfrac{1}{n} \pm a_k} \equiv \sum_{k=1}^{n} \pm a_k \left(\mathrm{mod}\ \frac{2\pi i}{\log q}\right), \tag{15}$$

$$\frac{m}{l} \times \widetilde{a} \equiv \widetilde{\tfrac{m}{2l}\frac{am}{l}} \left(\mathrm{mod}\ \frac{2\pi i}{\log q}\right), \tag{16}$$

$$\mathrm{QE}(\widetilde{\tfrac{m}{l} a}) = \mathrm{QE}(a)e^{\frac{2\pi i m}{l}}, \tag{17}$$

where the second equation is a consequence of the fact that we work $\mathrm{mod}\frac{2\pi i}{\log q}$.

Definition 5.

$$\langle\lambda;q\rangle_{kn} \equiv \langle\triangle(q;k;\lambda);q\rangle_n \equiv \prod_{m=0}^{k-1}\langle\frac{\lambda+m}{k};q\rangle_n \times_k \langle\widetilde{\frac{\lambda+m}{k}};q\rangle_n. \tag{18}$$

We also use the notation $\triangle(q;k;\lambda)$ as a parameter in q-hypergeometric functions.

If λ is a vector, we mean the corresponding product of vector elements. If λ is replaced by a sequence of numbers, separated by commas, we mean the corresponding product, as in the case of q-factorials.

The last factor in (18) corresponds to k^{nk}.

Definition 6. We shall define a q-hypergeometric series by

$$
{}_{p+p'}\phi_{r+r'}\left[\begin{array}{c}\hat{a}_1,\ldots,\hat{a}_p\\ \hat{b}_1,\ldots,\hat{b}_r\end{array}|q;z||\begin{array}{c}\prod_i f_i(k)\\ \prod_j g_j(k)\end{array}\right]\equiv
\sum_{k=0}^{\infty}\frac{\langle\hat{a}_1;q\rangle_k\ldots\langle\hat{a}_p;q\rangle_k}{\langle 1,\hat{b}_1;q\rangle_k\ldots\langle\hat{b}_r;q\rangle_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+r+r'-p-p'} z^k\frac{\prod_i f_i(k)}{\prod_j g_j(k)}, \tag{19}
$$

where

$$
\widehat{a}\equiv a\vee\widetilde{a}\vee\widetilde{\tfrac{m}{l}a}\vee_k\widetilde{a}\vee\triangle(q;l;\lambda). \tag{20}
$$

In case of $\triangle(q;l;\lambda)$ the index is adjusted accordingly. It is assumed that the denominator contains no zero factors, i.e. $\widehat{b_k}\neq -l+\frac{2m\pi i}{logq},\ k=1,\ldots,r,l,m\in\mathbb{N}$. We assume that the $f_i(k)$ and $g_j(k)$ contain p' and r' factors of the form $\langle\widehat{a(k)};q\rangle_k$ or $(s(k);q)_k$ respectively. In a few cases the parameter $\widehat{a}$ in (19) will be the real plus infinity
$(0<|q|<1)$. They correspond to multiplication by 1.

With this notation, q-hypergeometric function- and hypergeometric function equations become very similar.

Definition 7. The q-derivative is defined by

$$
(\mathrm{D}_q\varphi)(x)\equiv\begin{cases}\dfrac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{when } q\in\mathbb{C}\backslash\{1\},\ x\neq 0;\\ \dfrac{d\varphi}{dx}(x), & \text{when } q=1;\\ \dfrac{d\varphi}{dx}(0), & \text{when } x=0.\end{cases} \tag{21}
$$

Definition 8. The Jackson q-integral is defined by

$$
\int_a^b f(t,q)\,d_q(t)\equiv\int_0^b f(t,q)\,d_q(t)-\int_0^a f(t,q)\,d_q(t),\ \ a,b\in\mathbb{R}, \tag{22}
$$

where

$$
\int_0^a f(t,q)\,d_q(t)\equiv a(1-q)\sum_{n=0}^{\infty}f(aq^n,q)q^n,\ \ 0<|q|<1,\ \ a\in\mathbb{R}. \tag{23}
$$

Definition 9. If $|q|>1\vee 0<|q|<1, |z|<|1-q|^{-1}$, the q-exponential function $\mathrm{E}_q(z)$ is defined by [15]

$$
\mathrm{E}_q(z)\equiv\sum_{k=0}^{\infty}\frac{1}{\{k\}_q!}z^k. \tag{24}
$$

By the Euler equation, the meromorphic continuation of $\mathrm{E}_q(z)$ is given by

$$\mathrm{E}_q(z) = \frac{1}{(z(1-q);q)_\infty}. \tag{25}$$

Thus the meromorphic function $\frac{1}{(z(1-q);q)_\infty}$, with simple poles at $\frac{q^{-k}}{1-q}$, $k \in \mathbb{N}$ is a good substitute for $\mathrm{E}_q(z)$ in the whole complex plane. We shall however continue to designate this function $\mathrm{E}_q(z)$, since it plays an important role in the operator theory.

The q-difference for $\mathrm{E}_q(z)$ is

$$\mathrm{D}_q \ \mathrm{E}_q(az) = a \ \mathrm{E}_q(az). \tag{26}$$

Definition 10. The automorphism ϵ on $\mathbb{R}[x]$ is defined by

$$\epsilon f(x) \equiv f(qx). \tag{27}$$

Definition 11. Let the Gaussian q-binomial coefficients be defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k \langle 1;q\rangle_{n-k}}, k = 0, 1, \ldots, n. \tag{28}$$

Definition 12. [15, p.212]. The multiplication operator $\mathbb{C}[[x]] \to \mathbb{C}[[x]]$, i.e. multiplication with x, is denoted by $\mathbf{x}$. We have skipped the I for the unit operator to the far right in all q-difference equations.

3 Something about the history of the Hermite polynomials

We now turn to Hermite polynomials. There are two kinds of Hermite polynomials, $\mathrm{H}_n(x)$, and $\mathrm{He}_n(x)$. The first one is defined by [28]

$$e^{2xt-t^2} = \sum_{k=0}^{\infty} \frac{\mathrm{H}_n(x)t^n}{n!}. \tag{29}$$

The second one $\mathrm{He}_n(x)$, which specifies the probabilistic Hermite polynomials, is defined by

$$e^{xt-\frac{t^2}{2}} = \sum_{k=0}^{\infty} \frac{\mathrm{He}_n(x)t^n}{n!}. \tag{30}$$

The relation between the two polynomials is

$$\mathrm{He}_n(x) = 2^{-n/2}\mathrm{H}_n(x/\sqrt{2}). \tag{31}$$

The formulas for the two kinds of Hermite polynomials are quite similar; sometimes a formula only occurs in the literature in the shape of one of these polynomials. One example is Burchnall's formula [4, p. 9]. We shall, however, try to make the best of it and treat these two kinds of Hermite polynomials as a unity. When we refer to an equation with an Hermite polynomial of type (29), we denote this with a phrase like: almost a q-analogue of. The following exposition shows that the roots of the probabilists Hermite polynomials can be traced back to early nineteenth century.

Sturm in 1829 [29] and 1835 [30], proved that all the zeros of the Hermite polynomials are real.

Chebyshev in 1859 [8] studied a very general form of Hermite polynomial via the Rodriguez formula. Lagrange and Markoff used Hermite polynomials in the form $\mathrm{He}_n(x)$ in early probability theory.

In 1864 Hermite [21] presented these polynomials in the form $\mathrm{H}_n(x)$ with the Rodriguez formula, differential equation, and orthogonality. The generating function (29) had already been given by Laplace in 1811 in connection with the potential in his famous work about celestial mechanics.

Halphen [20] studied the related Appell polynomials in his own way. This made the way for the three following crucial contributions.

Laguerre [25] studied $\mathrm{He}_n(x)$ and concluded that the Hermite polynomials give the successive derivatives of $e^{\frac{x^2}{2}}$ via the Rodriguez formula. In 1880 Appell [2, p. 122] gave a modern interpretation of Hermite polynomials in terms of Appell polynomials. Unfortunately, these polynomials have fallen into oblivion, as well as their q-analogues. W. Hahn [19] studied the polynomials in the form $\mathrm{He}_n(x)$ and made a thorough study of their zeros.

The following formula and its inverse occur regularly in the literature. One of these was published in different form by Nielsen [27, p. 32].

Theorem 2. *Nielsen's formula [4, p. 10].*

$$\mathrm{He}_{n+r}(x) = \sum_{m=0}^{\min(r,n)} (-1)^m \binom{r}{m}\binom{n}{m} \mathrm{He}_{r-m}(x)\mathrm{He}_{n-m}(x) m!. \tag{32}$$

This formula has been very nicely proved by induction by Chatterjea [6, p. 53].

4 Basic properties of the three q-Hermite polynomials

The generating functions for the q-Hermite polynomials corresponding to (30) use a special q-exponential function with quadratic argument. We often use the generating function technique to define polynomials. The following q-analogue of [26] is a special kind of q-Appell polynomial:

Definition 13. The β_q polynomials of degree ν and order n are given by

$$\frac{t^n}{(\mathrm{E}_q(t)-1)^n}g(t)\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu\beta_{\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{33}$$

From (33), with $g(t)=\frac{1}{\mathrm{E}_{q^2}(\frac{qt^2}{\{2\}_q})}$, we conclude that $\mathrm{H}_{\nu,q}(x)$ can be regarded as β_q polynomials of order 0,

$$\beta_{\nu,q}^{(0)}(x)\equiv\mathrm{H}_{\nu,q}(x). \tag{34}$$

With this defintion, q-Hermite polynomials of order n are given by

$$\frac{t^n}{(\mathrm{E}_q(t)-1)^n}\frac{\mathrm{E}_q(xt)}{\mathrm{E}_{q^2}(\frac{qt^2}{\{2\}_q})}=\sum_{\nu=0}^{\infty}\frac{t^\nu\mathrm{H}_{\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{35}$$

The corresponding q-Hermite numbers of order n are given by

$$\frac{t^n}{(\mathrm{E}_q(t)-1)^n}\frac{1}{\mathrm{E}_{q^2}(\frac{qt^2}{\{2\}_q})}=\sum_{\nu=0}^{\infty}\frac{t^\nu\mathrm{H}_{\nu,q}^{(n)}}{\{\nu\}_q!}. \tag{36}$$

We will now define three q-Hermite polynomials, which have been given before by Cigler [9], [11], by Désarménien [12] ($\mathrm{H}_{n,q}(x)$), ($s=1$), Kirschenhofer [24] ($\mathrm{H}_{n,q}(x),\psi_{n,s,q}(x)$), and the undersigned for $s=1$ [13]. Another q-Hermite polynomial has been introduced by Hounkonnou, Arjika, and Chung [22], which is obtained by substituting $s\to\frac{s}{q}$. These three q-Hermite polynomials can be q-Appell polynomials, or pseudo-q-Appell polynomials. One can see from the generating function which of these two classes the polynomial belongs to. The various formulas are are quite similar and will be presented in blocks to give a better overview. The following notation is very useful.

Definition 14. The multiplication operator $\mathbb{C}[[x]]\to\mathbb{C}[[x]]$, i.e. multiplication with $x,\ 0\le i\le n$, is denoted by $\mathbf{x}$.

We first define our polynomials by operator product expressions. All these formulas operate from the left side on the constant function $f(x)=1$.

Definition 15. The Cigler q-Hermite polynomials [9] are defined by

$$\mathrm{H}_{n,s,q}(x)\equiv(\mathbf{x}-q^{n-1}s\mathrm{D}_q)(\mathbf{x}-q^{n-2}s\mathrm{D}_q)\cdots(\mathbf{x}-s\mathrm{D}_q)1. \tag{37}$$

The generalized Kirschenhofer q-Hermite polynomials [24, p. 292] are given by

$$\psi_{n,s,q}(x)\equiv(\mathbf{x}-s\epsilon\mathrm{D}_q)^n1. \tag{38}$$

The polynomials $\mathrm{K}_{n,s,q}(x)$ introduced by Cigler [9] are given by

$$\mathrm{K}_{n,s,q}(x)\equiv(\mathbf{x}\epsilon-s\mathrm{D}_q)^n1. \tag{39}$$

5 Generating functions, recurrences and q-difference equations

Theorem 3. *The first of the following generating functions is found in [9, p. 42]. Formula (42) gives an example of a pseudo-q-Appell polynomial.*

$$\mathrm{E}_q(xt)\mathrm{E}_{q^{-2}}(\tfrac{-sqt^2}{\{2\}_q}) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\mathrm{H}_{\nu,s,q}(x), \tag{40}$$

$$\mathrm{E}_q(xt)\mathrm{E}_{q^2}(\tfrac{-st^2}{\{2\}_q}) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\psi_{\nu,s,q}(x), \tag{41}$$

$$\mathrm{E}_{\frac{1}{q}}(xt)\mathrm{E}_{q^2}(\tfrac{-st^2}{\{2\}_q}) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\mathrm{K}_{\nu,s,q}(x), \tag{42}$$

Corollary 4. *These q-Hermite polynomials are even for ν even and odd for ν odd.*

The following generating functions have no hypergeometric origin. For all three q-Hermite polynomials with s replaced by $s(1-q)$, the limit $q \to 1^-$ equals x^ν.

Corollary 5. *Cigler [11] generating functions:*

$$\frac{(-xt;q)_\infty}{(-st^2;q^2)_\infty} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\langle 1;q\rangle_\nu}\mathrm{K}_{\nu,(1-q)s,q}(x), \tag{43}$$

$$\frac{1}{(xt;q)_\infty(-st^2;q^2)_\infty} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\langle 1;q\rangle_\nu}\psi_{\nu,(1-q)s,q}(x). \tag{44}$$

Proof. Put $s \to (1-q)s$ and $t \to \frac{t}{1-q}$ in (42) and (41). □

Theorem 6. *An operator representation from Cigler [11, (3.13)].*

$$\mathrm{H}_{\nu,\frac{1-q}{q},q}(x) = q^{\binom{\nu}{2}-\nu}(q-1)^\nu \frac{1}{(q^2x^2;q^2)_\infty}(\epsilon^{-1}\mathrm{D}_q)^\nu(q^2x^2;q^2)_\infty. \tag{45}$$

Let's return to our ordinary q-Hermite polynomials.

Theorem 7. *We have the following recurrences, polynomials with index* 0 *are equal to* 1.

$$[9, \text{p. } 43\ (4)]\ \mathrm{H}_{\nu+1,s,q}(x) = x\mathrm{H}_{\nu,s,q}(x) - sq^\nu\{\nu\}_q\mathrm{H}_{\nu-1,s,q}(x), \tag{46}$$

$$[24, \text{p. } 292]\ \psi_{\nu+1,s,q}(x) = x\psi_{\nu,s,q}(x) - s\{\nu\}_q\psi_{\nu-1,s,q}(qx), \tag{47}$$

$$\mathrm{K}_{\nu+1,s,q}(x) = x\mathrm{K}_{\nu,s,q}(qx) - s\{\nu\}_q\mathrm{K}_{\nu-1,s,q}(qx), \tag{48}$$

Proof. The formulas (47) and (48) follow from (38) and (39). □

Remark 1. The recurrence (48) has two functions of qx, because $\mathrm{K}_{\nu,s,q}$ is a pseudo q-Appell polynomial.

Theorem 8. *The polynomial* $\mathrm{f}_{\nu,1,q}(x)$ *corresponds to Exton's* $\mathrm{H}(\nu,q;x)$ *[18, p. 197] by the substitution*

$$\mathrm{f}_{\nu,1,q}(x) = \{2\}_q^{-\frac{\nu}{2}} \mathrm{H}\left(\nu, q; \frac{x}{\sqrt{\{2\}_q}}\right), \tag{49}$$

like in formula (31).

Theorem 9. *Two operator representations from Cigler [9, p. 45], [11, p. 14] and Ernst [13].*

$$\mathrm{H}_{\nu,s,q}(x) = q^{\binom{\nu}{2}}(\mathbf{x}\epsilon^{-1} - qs\mathrm{D}_q\epsilon^{-1})^{\nu}1, \tag{50}$$

$$\mathrm{H}_{\nu,s,q}(x) = (\mathbf{x} - qs\mathrm{D}_q)(\mathbf{x} - q^3 s\mathrm{D}_q)\cdots(\mathbf{x} - q^{2\nu-1}s\mathrm{D}_q)1. \tag{51}$$

Proof. Use the recurrences, q-Pascal triangle and the definition of q-Appell polynomials. □

Theorem 10. *Assume initial values* $f(0) = 1$. *The* q*-difference equations are*

$$(sq^{\nu-1}\mathrm{D}_q^2 - x\mathrm{D}_q + \{\nu\}_q)\mathrm{H}_{\nu,s,q}(x) = 0, \tag{52}$$

$$(sq^{-2}\mathrm{D}_q^2\epsilon - x\mathrm{D}_q + \{\nu\}_q)\psi_{\nu,s,q}(x) = 0, \tag{53}$$

$$(sq\mathrm{D}_q^2\epsilon^{-1} - x\mathrm{D}_q + \{\nu\}_q)\mathrm{K}_{\nu,s,q}(x) = 0, \tag{54}$$

Proof. Use the recurrences and the definition of q-Appell polynomials. □

6 Survey of q-real numbers, q-Appell polynomials and a q-analogue of the Rota formula

The q-real numbers give a convenient notation for q-additions in formal power series, in particular for q-exponential and q-trigonometric functions. There is a one-to-one correspondence between the convergence regions of the two q-Lauricella functions $\Phi_{\mathrm{A}}^{(n)}$ and $\Phi_{\mathrm{C}}^{(n)}$ [14], and the existence of q-real numbers with n letters (or variables). We shall only discuss the first q-real numbers, the commutative monoid $\mathbb{R}_{\oplus_q}$.

Definition 16. [17] Let $a, b \in \mathbb{R}$. Then the first q-real numbers $\mathbb{R}_{\oplus_q}$ are defined by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \; n = 0, 1, 2, \ldots, \; a \oplus_q b \in \mathbb{R}_{\oplus_q}. \tag{55}$$

In particular, $(a \oplus_q b)^0 \equiv 1$.

Definition 17. Assume that $\sim$ means equality on $\mathbb{R}[[x]]$ [15, p. 101], [17].
There is a certain linear functional
$v: \mathbb{R}[[x]] \times \mathbb{R}_{\oplus_q} \mapsto \mathbb{R}$, with $v(f, 0) = a_0 \in \mathbb{R}$, called the evaluation.

Theorem 11. *For $\alpha, \beta, \gamma \in \mathbb{R}_{\oplus_q}$ we note [15]:*
Commutativity:

$$\alpha \oplus_q \beta \sim \beta \oplus_q \alpha. \tag{56}$$

Associativity

$$(\alpha \oplus_q \beta) \oplus_q \gamma \sim \alpha \oplus_q (\beta \oplus_q \gamma). \tag{57}$$

We will now describe the q-Appell polynomials [15].

Definition 18. [15, p. 114]. For every power series $f_n(t)$, with $f_n(0) \neq 0$, the Φ_q polynomials of degree ν and order n have the following generating function:

$$f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi_{\nu,q}^{(n)}(x). \tag{58}$$

By putting $x = 0$, we have

$$f_n(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi_{\nu,q}^{(n)}, \tag{59}$$

where $\Phi_{\nu,q}^{(n)}$ is called a Φ_q number of degree ν and order n. It will be convenient to fix the value for $n = 0$ and $n = 1$:

$$\Phi_{\nu,q}^{(1)}(x) \equiv \Phi_{\nu,q}(x);\ \Phi_{\nu,q}^{(0)} \equiv 0. \tag{60}$$

Theorem 12. *Rules for q-derivatives and q-integrals of q-Appell polynomials.*

$$\mathrm{D}_q \Phi_{\nu,q}^{(n)}(x) = \{\nu\}_q \Phi_{\nu-1,q}^{(n)}(x). \tag{61}$$

$$\int_a^x \Phi_{\nu,q}^{(n)}(t)\, d_q(t) = \frac{\Phi_{\nu+1,q}^{(n)}(x) - \Phi_{\nu+1,q}^{(n)}(a)}{\{\nu+1\}_q}. \tag{62}$$

Definition 19. For every power series $f_n(t)$, the pseudo-q-Appell polynomials or Φ^q polynomials of degree ν and order n have the following generating function

$$f_n(t)\mathrm{E}_{\frac{1}{q}}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi_{\nu}^{(n;q)}(x). \tag{63}$$

Theorem 13. *Cigler [11] proved the following umbral formula, which shows that the sequence $\{\Psi_{-qs,q}\}$ is the umbral inverse of the sequence $\{\mathrm{H}_{s,q}\}$.*

$$(x \oplus_q y)^n = \sum_{k=0}^{n} \binom{n}{k}_q \mathrm{H}_{n-k,s,q}(x)\Psi_{k,-qs,q}(y) \equiv (\mathrm{H}_{s,q}(x) \oplus_q \Psi_{-qs,q}(y))^n. \tag{64}$$

Proof. Use the generating functions and equate coefficients of t^n. □

6.1 Determinant expressions for the three q-Hermite polynomials

The recurrences (46)– (48) imply the following determinant expressions, where we give the formulas for $\nu = 4$. The first formula is from Kirschenhofer [24, p. 293].

Theorem 14.

$$\mathrm{H}_{4,s,q}(x) = \begin{vmatrix} x & \{1\}_q qs & 0 & 0 \\ 1 & x & \{2\}_q q^2 s & 0 \\ 0 & 1 & x & \{3\}_q q^3 s \\ 0 & 0 & 1 & x \end{vmatrix}. \tag{65}$$

$$\psi_{4,s,q}(x) = \begin{vmatrix} x & \{1\}_q s & 0 & 0 \\ 1 & x & \{2\}_q s & 0 \\ 0 & 1 & x & \{3\}_q s \\ 0 & 0 & 1 & x \end{vmatrix}. \tag{66}$$

$$\mathrm{K}_{4,s,q}(x) = \begin{vmatrix} x & \{1\}_q s & 0 & 0 \\ 1 & qx & \{2\}_q s & 0 \\ 0 & 1 & q^2 x & \{3\}_q s \\ 0 & 0 & 1 & q^3 x \end{vmatrix}. \tag{67}$$

Remark 2. The diagonal elements in (67) are typical for pseudo-q-Appell polynomials.

7 Pseudo q-Hermite polynomials and a fourth q-Hermite polynomial

There are also so-called pseudo q-Hermite polynomials, which do not have proper limits $q \to 1^-$. Some of these have applications in combinatorics [10], [23], [12]. We call such an object (equation, function etc.) a pseudo q-analogue. Assume that we have a recurrence of the form

$$h_{0,s,q}(x) = 1,\ h_{1,s,q}(x) = x,\ h_{n+1,s,q}(x) = xh_{\nu,s,q}(x) - sq^a\{n\}_q h_{n-1,s,q}(x). \tag{68}$$

By the substitution

$$h_{n,s,q}(x) = \left(\frac{q}{1-q}\right)^{\frac{n}{2}} \mathrm{F}_{n,s,q}\left(x\sqrt{\tfrac{1-q}{q}}\right),\ q \neq 1,\ y = x\sqrt{\tfrac{1-q}{q}}, \tag{69}$$

formula (68) takes the form

$$\mathrm{F}_{n+1,s,q}(y) = y\mathrm{F}_{n,s,q}(y) - sq^{a-1}(1-q^n)\mathrm{F}_{n-1,s,q}(y). \tag{70}$$

The case $a = n$ in (68) and (70) corresponds to Carlitz-AlSalam polynomials ($s = 1$) [1] and $\mathrm{H}_{n,s,q}(x)$ in (37). The cases $a = n+1$ and $a = n-1$ in (68) corresponds to formulas in [13] and [22]. The case $a = 1$ in formula (70) corresponds to a bivariate q-Hermite polynomial [10].

Finally, the case $a = 0$ in formula (68) gives a fourth q-Hermite polynomial $\mathrm{h}_{n,s,q}(x)$ with the following values:

1
x
$x^2 - s$
$x^3 - (2+q)sx$
$x^4 - (3+2q+q^2)sx^2 + \{3\}_q s^2$
$x^5 - sx^3(q^3+2q^2+3q+4) + (3+4q+4q^2+3q^3+q^4)s^2x$

Table 1: A fourth q-Hermite polynomial $\mathrm{h}_{n,s,q}(x)$

In [23] a generating function for $\mathrm{h}_{n,1,q}(x)$ is given.

Assume that we have the generating function

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} h_{\nu,q}(x) = \mathrm{E}_q(xt)\mathrm{E}_{q^2}\left(\frac{at^2}{\{2\}_q}\right). \tag{71}$$

for a given q-Hermite polynomial $h_{\nu,q}(x)$. These polynomials are connected to certain $\mathrm{F}_{n,q}(x)$ polynomials by the following substitution. This substitution is only valid for $q \neq 1$.

$$h_{\nu,q}(x) \equiv \left(\frac{q}{1-q}\right)^{\frac{\nu}{2}} \mathrm{F}_{\nu,q}\left(x\sqrt{\frac{1-q}{q}}\right). \tag{72}$$

By using the infinite product expansion for the q-exponential function, we arrive at the following generating function for $\mathrm{F}_{\nu,q}(y)$:

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathrm{F}_{\nu,q}(y) = \mathrm{E}_q(yt)\mathrm{E}_q\left(t\sqrt{\frac{a}{q}}\right)\mathrm{E}_q\left(-t\sqrt{\frac{a}{q}}\right). \tag{73}$$

T

8 Operator formulas

In this section we give four operator formulas, which are necessary for the proofs of the Rodriguez formulas in the next section. Two of these are slightly extended versions of two formulas in [15, p. 215].

Theorem 15.

$$(\epsilon \mathrm{D}_q)^n = \mathrm{D}_q^n \epsilon^n q^{-\binom{n+1}{2}}. \tag{74}$$

Proof. For $n=1$ the formula is obvious. Using [15, (6.98)] we will prove the equivalent formula

$$(\epsilon \mathrm{D}_q)^n f(x) = (q-1)^{-n} x^{-n} q^{-\binom{n+1}{2}-\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} f(q^{2n-k}x) \quad (75)$$

by induction. Assume that the formula is true for $n=m$. Then it also applies to $n=m+1$, because

$$\begin{aligned} &\left((\epsilon \mathrm{D}_q)^{m+1}\right) f(x) \\ &\overset{\text{by}(75)}{=} (q-1)^{-m} q^{-\binom{m+1}{2}-\binom{m}{2}} \epsilon \mathrm{D}_q \left(x^{-m} \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\binom{k}{2}} f(q^{2m-k}x) \right) \\ &= \frac{(q-1)^{-m} q^{-\binom{m+1}{2}-\binom{m}{2}}}{qx(q-1)x^m} \left[q^{-2m} \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\binom{k}{2}} f(q^{2m+2-k}x) - \right. \\ &\left. - q^{-m} \sum_{k=0}^{m} \binom{m}{k}_q (-1)^k q^{\binom{k}{2}} f(q^{2m+1-k}x) \right]. \end{aligned} \quad (76)$$

By the q-Pascal identity [15, (6.90)], we can show that this is equal to

$$(q-1)^{-m-1} x^{-m-1} q^{-\binom{m+2}{2}-\binom{m+1}{2}} \sum_{k=0}^{m+1} \binom{m+1}{k}_q (-1)^k q^{\binom{k}{2}} f(q^{2m+2-k}x), \quad (77)$$

which completes the proof. □

Theorem 16.

$$\frac{1}{\mathrm{E}_{q^2}(\frac{-x^2}{s\{2\}_q})} \mathrm{D}_q \mathrm{E}_{q^2}(\frac{-x^2}{s\{2\}_q}) = \mathrm{D}_q - \frac{1}{s}\mathbf{x}\epsilon. \quad (78)$$

Theorem 17.

$$\mathrm{E}_{q^2}(\frac{x^2 q}{s\{2\}_q})(\epsilon^{-1}\mathrm{D}_q) \frac{1}{\mathrm{E}_{q^2}(\frac{x^2 q}{s\{2\}_q})} = \epsilon^{-1}\mathrm{D}_q - \frac{1}{s}\mathbf{x}\epsilon^{-1}. \quad (79)$$

Proof.

$$\begin{aligned} &\mathrm{E}_{q^2}(\frac{qx^2}{s\{2\}_q})(\epsilon^{-1}\mathrm{D}_q) \frac{1}{\mathrm{E}_{q^2}(\frac{qx^2}{s\{2\}_q})} = \epsilon^{-1} \mathrm{E}_{q^2}(\frac{q^3x^2}{s\{2\}_q}) \mathrm{D}_q \frac{1}{\mathrm{E}_{q^2}(\frac{qx^2}{s\{2\}_q})} \\ &= \epsilon^{-1}(\mathrm{D}_q \mathrm{E}_{q^2}(\frac{qx^2}{s\{2\}_q}) - qx \mathrm{E}_{q^2}(\frac{qx^2}{s\{2\}_q})) \frac{1}{\mathrm{E}_{q^2}(\frac{qx^2}{s\{2\}_q})} \\ &= \epsilon^{-1}(\mathrm{D}_q - \frac{q}{s}\mathbf{x}) = \epsilon^{-1}\mathrm{D}_q - \frac{1}{s}\mathbf{x}\epsilon^{-1}. \end{aligned} \quad (80)$$

□

Theorem 18.

$$\mathrm{E}_{q^2}(\tfrac{qx^2}{s\{2\}_q})(\epsilon\mathrm{D}_q)\frac{1}{\mathrm{E}_{q^2}(\frac{x^2}{qs\{2\}_q})} = \epsilon\mathrm{D}_q - \frac{1}{s}\mathbf{x}. \tag{81}$$

Proof.

$$\begin{aligned}\mathrm{LHS} &= \epsilon\mathrm{E}_{q^2}(\tfrac{x^2}{qs\{2\}_q})\mathrm{D}_q\frac{1}{\mathrm{E}_{q^2}(\frac{x^2}{qs\{2\}_q})}\\ &= \epsilon\left[\mathrm{D}_q\mathrm{E}_{q^2}(\tfrac{x^2}{qs\{2\}_q}) - \frac{1}{qs}\mathbf{x}\mathrm{E}_{q^2}(\tfrac{x^2}{qs\{2\}_q})\epsilon\right]\frac{1}{\mathrm{E}_{q^2}(\frac{x^2}{qs\{2\}_q})} = \mathrm{RHS}.\end{aligned} \tag{82}$$

□Conclusion: The formulas (78) and (79) have the same q-exponential. They are used to prove q-Rodriguez formulas (87) and (89). Formula (81) is used in the proof of q-Rodriguez formula (88).

9 Explicit formulas, Rodriguez formulas, tables

We have written the following formulas in two different forms.

Theorem 19. *Explicit formulas for the three polynomials.*

$$\begin{aligned}&[24, p.\ 296]\ \mathrm{H}_{n,s,q}(x) = \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}_q(-s)^k q^{k^2}\{2k-1\}_q!!x^{n-2k}\\ &\equiv x^n{}_4\phi_1\left(\triangle(q;2;-n);\widetilde{1}\middle|q;-\frac{q^{2n}s}{x^2(1-q)}\right),\end{aligned} \tag{83}$$

$$\begin{aligned}&[24,\ p.\ 296]\ \psi_{n,s,q}(x) = \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}_q(-s)^k\{2k-1\}_q!!x^{n-2k}\\ &\equiv x^n{}_6\phi_1\left(\triangle(q;2;-n),2\infty;\widetilde{1}\middle|q;-\frac{q^{2n-1}s}{x^2(1-q)}\right),\end{aligned} \tag{84}$$

$$\begin{aligned}\mathrm{K}_{n,s,q}(x) &= \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}_q(-s)^k q^{\binom{n-2k}{2}}\{2k-1\}_q!!x^{n-2k}\\ &\equiv q^{\binom{n}{2}}x^n{}_4\phi_3\left(\triangle(q;2;-n);\widetilde{1},2\infty\middle|q;-\frac{q^2s}{x^2(1-q)}\right),\end{aligned} \tag{85}$$

Proof. We prove the first formula.

$$\mathrm{LHS} = x^n \sum_i \frac{\langle 1;q\rangle_n \langle 1;q\rangle_{2i-1}(-1)^i x^{-2i} q^{i^2} s^i}{\langle 1;q\rangle_{2i}\langle 1;q\rangle_{n-2i}\langle 1;q^2\rangle_{i-1}(1-q)^i}$$

$$= x^n \sum_i \frac{\langle -n;q\rangle_{2i} q^{-\binom{2i}{2}+2in+i^2}(-1)^i x^{-2i} s^i}{\langle 1;q^2\rangle_{i-1}(1-q)^i(1-q^{2i})} = \tag{86}$$

$$x^n \sum_i \frac{\langle -n;q\rangle_{2i}(-s)^i x^{-2i} q^{-2\binom{i}{2}+2in}}{\langle 1,\widetilde{1};q\rangle_i (1-q)^i} = \mathrm{RHS}.$$

□ The q-Rodriguez formulas are q-analogues of [3, p. 334].

Theorem 20. *The q-Rodriguez formula for* $\mathrm{K}_{\nu,s,q}(x)$.

$$\mathrm{K}_{\nu,s,q}(x) = (-s)^\nu (\mathrm{E}_{q^2}(\tfrac{-x^2}{s\{2\}_q}))^{-1}\mathrm{D}_q^\nu \mathrm{E}_{q^2}(\tfrac{-x^2}{s\{2\}_q}). \tag{87}$$

Proof. Use formula (78) together with (39). □

Theorem 21. *The Rodriguez formula for* $\psi_{\nu,s,q}(x)$.

$$\psi_{\nu,s,q}(x) = (-s)^\nu \mathrm{E}_{q^2}(\tfrac{x^2 q}{s\{2\}_q})(\epsilon \mathrm{D}_q)^\nu \frac{1}{\mathrm{E}_{q^2}(\frac{x^2}{qs\{2\}_q})}. \tag{88}$$

Proof. Use formula (81) together with (38). □In the same way follows

Theorem 22. *Cigler [9, p. 45]. The Rodriguez formula for* $\mathrm{H}_{\nu,s,q}(x)$*, a q-analogue of [3, p. 334].*

$$\mathrm{H}_{\nu,s,q}(x) = (-s)^\nu q^{\binom{\nu}{2}} (\mathrm{E}_{\frac{1}{q^2}}(\tfrac{-x^2 q}{s\{2\}_q}))^{-1}\mathrm{D}_{\frac{1}{q}}^\nu \mathrm{E}_{\frac{1}{q^2}}(\tfrac{-x^2 q}{s\{2\}_q}). \tag{89}$$

Proof. Use formulas (79), (50) together with the following obvious rule:

$$\mathrm{D}_{\frac{1}{q}} = \epsilon^{-1}\mathrm{D}_q = q\mathrm{D}_q\epsilon^{-1}. \tag{90}$$

□

Theorem 23. *Another operational formula due to Cigler [9, p. 42, (3)], [12, p. 8, 6.3].*

$$\mathrm{H}_{\nu,s,q}(x) = \sum_{k=0}^{\infty} \frac{q^{k^2}(-s\mathrm{D}_q^2)^k}{\{2k\}_q!!} x^\nu. \tag{91}$$

Theorem 24. *An operational formula for* $\Psi_{\nu,s,q}(x)$.

$$\Psi_{\nu,s,q}(x) = \sum_{k=0}^{\infty} \frac{(-s\mathrm{D}_q^2)^k}{\{2k\}_q!!} x^\nu. \tag{92}$$

Remark 3. We conclude that the formulas (91) and (92) give exactly the correct q-powers in the formulas.
A similar formula exists for the third q-Hermite polynomial.

The following table lists the first five $\mathrm{H}_{n,s,q}(x)$:

1
x
$x^2 - qs$
$x^3 - q\{3\}_q sx$
$x^4 - q\binom{4}{2}_q sx^2 + q^4\{3\}_q s^2$
$x^5 - \binom{5}{2}_q sx^3 q + q^4\{3\}_q\{5\}_q s^2 x$

Table 2: The first five $\mathrm{H}_{n,s,q}(x)$

The following table lists the first five $\psi_{n,s,q}(x)$:

1
x
$x^2 - s$
$x^3 - \{3\}_q sx$
$x^4 - \binom{4}{2}_q sx^2 + \{3\}_q s^2$
$x^5 - \binom{5}{2}_q sx^3 + \{3\}_q\{5\}_q s^2 x$

Table 3: The first five $\psi_{n,s,q}(x)$

The following table lists the first five $\mathrm{K}_{n,s,q}(x)$:

1
x
$x^2 q - s$
$x^3 q^3 - \{3\}_q sx$
$x^4 q^6 - \binom{4}{2}_q sx^2 q + \{3\}_q s^2$
$x^5 q^{10} - \binom{5}{2}_q sx^3 q^3 + \{3\}_q\{5\}_q xs^2$

Table 4: The first five $\mathrm{K}_{n,s,q}(x)$

We immediately conclude the following theorem.

Theorem 25. *The two q-Hermite polynomials* $\mathrm{H}_{n,s,q}$ *and* $\psi_{n,s,q}$ *are monic.*

Proof. We put $k = 0$ in the respective definitions and get the terms x^n. $\square$

9.1 Vector forms

As all our polynomials form (pseudo)q-Appell sequences it makes sense to define their vector forms. We will use the following vector forms for these polynomials

$$\mathrm{H}_{s,q}(x) \equiv (\mathrm{H}_{0,s,q}(x), \mathrm{H}_{1,s,q}(x), \ldots, \mathrm{H}_{n-1,s,q}(x))^T. \tag{93}$$

$$\Psi_{s,q}(x) \equiv (\psi_{0,s,q}(x), \psi_{1,s,q}(x), \ldots, \psi_{n-1,s,q}(x))^T. \tag{94}$$

$$\mathrm{K}_{s,q}(x) \equiv \mathrm{K}_{0,s,q}(x), \mathrm{K}_{1,s,q}(x), \ldots, \mathrm{K}_{n-1,s,q}(x))^T. \tag{95}$$

Following our article about q-Pascal matrices [16], we let the $n \times n$ q-matrix $\mathrm{D}_{n,q}$ be given by

$$\begin{aligned} &\mathrm{D}_{n,q}(i, i-1) \equiv \{i\}_q, \ i = 1, \ldots, n-1, \\ &\mathrm{D}_{n,q}(i, j) \equiv 0, \ j \neq i-1. \end{aligned} \tag{96}$$

Then

$$\mathrm{H}_{s,q}(x) = \mathrm{P}_{n,q}(x)\mathrm{H}_{s,q}(0), \tag{97}$$

where the q-Pascal matrix $\mathrm{P}_{n,q}(t) \equiv \mathrm{E}_q(\mathrm{D}_{n,q}t)$ is given by the familiar expression

$$\mathrm{P}_{n,q}(t) \equiv \sum_{k=0}^{\infty} \frac{t^k}{\{k\}_q!} \mathrm{D}_{n,q}^k. \tag{98}$$

The same formula applies for $\Psi_{s,q}(x)$, $\mathrm{F}_{s,q}(x)$ and $\mathrm{M}_{s,q}(x)$. Finally,

$$\mathrm{K}_{s,q}(x) = \mathrm{E}_{\frac{1}{q}}(\mathrm{D}_{n,q}x)\mathrm{K}_{s,q}(0). \tag{99}$$

9.2 q-analogues of the Nielsen formula

We will now focus on q-analogues of formula (32). The proofs always look the same. First, the left hand side is rewritten using the formula [15, 4.107]. Then this expression is transformed three times by the corresponding Rodriguez formula with the values $r - m$, n and $n + r$.

Theorem 26.

$$\mathrm{K}_{n+r,s,q}(x) = \sum_{m=0}^{\min(r,n)} (-s)^m \binom{r}{m}_q \binom{n}{m}_q \mathrm{K}_{r-m,s,q}(x)\mathrm{K}_{n-m,s,q}(xq^r)\{m\}_q! q^{\binom{m}{2}}. \tag{100}$$

Proof. We start with the following formula, modify it and verify that it is equal to the LHS.

$$\sum_{m=0}^{r}(-s)^m\binom{r}{m}_q \mathrm{K}_{r-m,s,q}(x)\epsilon^{r-m}\mathrm{D}_q^m\mathrm{K}_{n,s,q}(x) \overset{\text{by}(87)}{=} (-s)^r(\mathrm{E}_{q^2}(\tfrac{-x^2}{s\{2\}_q}))^{-1}$$
$$\sum_{m=0}^{r}\binom{r}{m}_q \mathrm{D}_q^{r-m}\mathrm{E}_{q^2}(\tfrac{-x^2}{s\{2\}_q})\epsilon^{r-m}\mathrm{D}_q^m\mathrm{K}_{n,s,q}(x) \overset{\text{by}[15,6.95]}{=} (-s)^r(\mathrm{E}_{q^2}(\tfrac{-x^2}{s\{2\}_q}))^{-1}$$
$$\mathrm{D}_q^r(\mathrm{E}_{q^2}(\tfrac{-x^2}{s\{2\}_q})\mathrm{K}_{n,s,q}(x)) \overset{\text{by}(87)}{=} (-s)^{n+r}(\mathrm{E}_{q^2}(\tfrac{-x^2}{s\{2\}_q}))^{-1}\mathrm{D}_q^{n+r}\mathrm{E}_{q^2}(\tfrac{-x^2}{s\{2\}_q}))$$
$$\overset{\text{by}(87)}{=} \mathrm{K}_{n+r,s,q}(x). \tag{101}$$

On the other hand, the first expression can be simplified to

$$\sum_{m=0}^{\min(r,n)}(-s)^m\binom{r}{m}_q\binom{n}{m}_q \mathrm{K}_{r-m,s,q}(x)\mathrm{K}_{n-m,s,q}(xq^r)\{m\}_q!q^{\binom{m}{2}}. \tag{102}$$

□ Cigler [9, p. 45 (15)] has proved a similar formula (for $s = 1$):

$$\mathrm{H}_{n+r,s,q}(x) = q^{nr}\sum_{m=0}^{\min(r,n)}(-s)^m\binom{r}{m}_q\binom{n}{m}_q \mathrm{H}_{r-m,s,q}(x)\mathrm{H}_{n-m,s,q}(xq^{-r})\{m\}_q!. \tag{103}$$

Proof. Similar to above.

$$\mathrm{RHS} = \sum_{m=0}^{r}(-s)^m q^{\binom{n+r}{2}-\binom{n}{2}-\binom{m}{2}-\binom{r-m}{2}+m(m-r)}\binom{n}{m}_q\binom{r}{m}_q \mathrm{H}_{r-m,s,q}(x)\epsilon^{r-m}\mathrm{H}_{n-m,s,q}(xq^{-r})\{m\}_q!$$
$$\overset{\text{by}[15,6.87,6.131],(89)}{=} (-s)^r(\mathrm{E}_{\frac{1}{q^2}}(\tfrac{-x^2q}{s\{2\}_q}))^{-1}\sum_{m=0}^{r}q^{\binom{n+r}{2}-\binom{n}{2}}\binom{r}{m}_{\frac{1}{q}}\mathrm{D}_{\frac{1}{q}}^{r-m}\mathrm{E}_{\frac{1}{q^2}}(\tfrac{-x^2q}{s\{2\}_q})\epsilon^{m-r}\mathrm{D}_{\frac{1}{q}}^{m}\mathrm{H}_{n,s,q}(x)$$
$$\overset{\text{by}[15,6.95]}{=} (-s)^r q^{\binom{n+r}{2}-\binom{n}{2}}(\mathrm{E}_{\frac{1}{q^2}}(\tfrac{-x^2q}{s\{2\}_q}))^{-1}\mathrm{D}_{\frac{1}{q}}^{r}(\mathrm{E}_{\frac{1}{q^2}}(\tfrac{-x^2q}{s\{2\}_q})\mathrm{H}_{n,s,q}(x))$$
$$\overset{\text{by}(89)}{=} (-s)^{n+r}q^{\binom{n+r}{2}}(\mathrm{E}_{\frac{1}{q^2}}(\tfrac{-x^2q}{s\{2\}_q}))^{-1}\mathrm{D}_{\frac{1}{q}}^{n+r}\mathrm{E}_{\frac{1}{q^2}}(\tfrac{-x^2q}{s\{2\}_q}) \overset{\text{by}(89)}{=} \mathrm{H}_{n+r,s,q}(x). \tag{104}$$

□ We now come to formulas which are some kind of inverse for the power series representations of q-Hermite polynomials.

Theorem 27. *Kirschenhofer [24, p. 295]. Almost a q-analogue of [5, p. 370, (5.4)].*

$$x^n = \sum_{2k \leq n} (qs)^k \{2k-1\}_q!! \binom{n}{2k}_q \mathrm{H}_{n-2k,s,q}(x). \tag{105}$$

Proof. Use the generating function (40), multiply by $\mathrm{E}_{q^2}(\frac{qt^2}{s\{2\}_q})$, and equate coefficients of t^n. □

Theorem 28. *A corrected version of Kirschenhofer [24, p. 295]. Almost a q-analogue of [5, p. 370, (5.4)].*

$$x^n = \sum_{2k \leq n} q^{2\binom{k}{2}} s^k \{2k-1\}_q!! \binom{n}{2k}_q \Psi_{n-2k,s,q}(x). \tag{106}$$

Theorem 29.

$$x^n = \sum_{2k \leq n} q^{2\binom{k}{2}-\binom{n}{2}} s^k \{2k-1\}_q!! \binom{n}{2k}_q \mathrm{K}_{n-2k,s,q}(x). \tag{107}$$

10 A new q-orthogonality

The following q-orthogonality has not been found before. This q-Hermite polynomial is of a more general form, and the weight function is easier to work with than in the previous article [13]. I thank Johann Cigler for pointing this out.

Lemma 30. *Put*

$$x_0 \equiv \frac{\sqrt{s}}{\sqrt{q-q^2}}. \tag{108}$$

Then we have

$$\int_{-x_0}^{x_0} \mathrm{E}_{\frac{1}{q^2}}\left(\frac{-qx^2}{s\{2\}_q}\right) d_q(x) = 2q\left(\frac{s(1-q)}{q}\right)^{\frac{1}{2}} \frac{\langle 1,1;q^2\rangle_\infty}{\langle 1;q\rangle_\infty}. \tag{109}$$

Proof.

$$\text{LHS} \overset{\text{by}[15,6.189]}{=} \int_{-x_0}^{x_0} \left(\frac{qx^2(1-q)}{s};q^2\right)_\infty d_q(x) \overset{\text{by}[15,6.54]}{=} 2x_0(1-q)\sum_{n=0}^{\infty} q^n \langle n;q^2\rangle_\infty$$
$$\overset{\text{by}[15,6.8]}{=} 2x_0(1-q)\langle 1;q^2\rangle_\infty \sum_{n=0}^{\infty} q^n(1-q^{2n})\frac{1}{\langle 1;q^2\rangle_n}$$
$$\overset{\text{by}[15,6.188]}{=} 2\left(\frac{s(1-q)}{q}\right)^{\frac{1}{2}} \langle 1;q^2\rangle_\infty \left[\frac{1}{\langle \frac{1}{2};q^2\rangle_\infty} - \frac{1}{\langle \frac{3}{2};q^2\rangle_\infty}\right] \overset{\text{by}[15,6.42]}{=} \text{RHS}. \tag{110}$$

□

Theorem 31. *q-orthogonality for $\mathrm{H}_{n,s,q}(x)$. For $n \geq m$ we have*

$$\begin{aligned}&\int_{-x_0}^{x_0} \mathrm{H}_{n,s,q}(x)\mathrm{H}_{m,s,q}(x)\mathrm{E}_{\frac{1}{q^2}}\left(\frac{-qx^2}{s\{2\}_q}\right) d_q(x)\\ &= 2\delta(m,n)\mathrm{QE}\left(\binom{n+1}{2}+\frac{1}{2}\right)(1-q)^{\frac{1}{2}}\{n\}_q! s^{n+\frac{1}{2}}\frac{\langle 1,1;q^2\rangle_\infty}{\langle 1;q\rangle_\infty}.\end{aligned} \tag{111}$$

Proof. q-Integration by parts [15, (6.58)] m times gives

$$\begin{aligned}&\int_{-x_0}^{x_0} \mathrm{H}_{n,s,q}(x)\mathrm{H}_{m,s,q}(x)\mathrm{E}_{\frac{1}{q^2}}\left(\tfrac{-qx^2}{s\{2\}_q}\right) d_q(x)\\ &\overset{\text{by}(89)}{=} \int_{-x_0}^{x_0} \mathrm{H}_{m,s,q}(x)(-qs)^n q^{\binom{n}{2}}(\mathrm{D}_q\epsilon^{-1})^n \mathrm{E}_{\frac{1}{q^2}}\left(\tfrac{-qx^2}{s\{2\}_q}\right) d_q(x)\\ &= \left[H_{m,s,q}(x)(-qs)^n q^{\binom{n}{2}}\epsilon^{-1}(\mathrm{D}_q\epsilon^{-1})^{n-1}\left(\mathrm{E}_{\frac{1}{q^2}}\left(\frac{-qx^2}{s\{2\}_q}\right)\right)\right]_{-x_0}^{x_0}\\ &- \int_{-x_0}^{x_0} \mathrm{D}_q(\mathrm{H}_{m,s,q}(x))(-qs)^n q^{\binom{n}{2}}(\mathrm{D}_q\epsilon^{-1})^{n-1}\left(\mathrm{E}_{\frac{1}{q^2}}\left(\frac{-qx^2}{s\{2\}_q}\right)\right) d_q(x) = \ldots\\ &= -\sum_{l=1}^{m}\left[q^{\binom{n}{2}-\binom{n-l}{2}}(qs)^l\{m-l+2\}_{l-1,q}\mathrm{H}_{m-l+1,s,q}(x)\epsilon^{-1}\left(\mathrm{E}_{\frac{1}{q^2}}\left(\frac{-qx^2}{s\{2\}_q}\right)\mathrm{H}_{n-l,s,q}(x)\right)\right]_{-x_0}^{x_0}\\ &+ \delta(m,n)q^{\binom{n+1}{2}}\{n\}_q! s^n \int_{-x_0}^{x_0} \mathrm{E}_{\frac{1}{q^2}}\left(\frac{-qx^2}{s\{2\}_q}\right) d_q(x) \overset{\text{by}(109)}{=} \text{RHS}.\end{aligned} \tag{112}$$

We used formula [15, (6.155)] for the zeros of $\mathrm{E}_{\frac{1}{q}}$ in the end. Formula (89) was employed for the integrated parts and then changed back. The defining formula [15, (4.107)] was used for each q integral.

□

11 Linear functionals and computations of moments

We shall now prove the previous orthogonality relations by using linear functionals according to Cigler [11].
First define the linear functional in x, Λ_H by

$$\Lambda_\mathrm{H}(\mathrm{H}_{n,s,q}(x)) \equiv \delta_{n,0}. \tag{113}$$

By operating with this on the generating function (40), we obtain

$$\Lambda_\mathrm{H}(\mathrm{E}_q(xt)) = \mathrm{E}_{q^2}\left(\tfrac{sqt^2}{\{2\}_q}\right). \tag{114}$$

The moments are obtained by equating coefficients of t^k:

$$\begin{aligned}\Lambda_{\mathrm{H}}(x^{2m}) &= s^m q^m \{2m-1\}_q!!,\\ \Lambda_{\mathrm{H}}(x^{2m-1}) &= 0.\end{aligned} \tag{115}$$

Definition 20. In the same way, we define the linear functionals Λ_ψ, Λ_{K} by

$$\begin{aligned}\Lambda_\psi(\psi_{n,s,q}(x)) &\equiv \delta_{n,0},\\ \Lambda_{\mathrm{K}}(\mathrm{K}_{n,s,q}(x)) &\equiv \delta_{n,0}.\end{aligned} \tag{116}$$

By operating with these functionals on the generating functionswe obtain

$$\begin{aligned}\Lambda_\psi(\mathrm{E}_q(xt)) &= \mathrm{E}_{q^{-2}}(\tfrac{st^2}{\{2\}_q}),\\ \Lambda_{\mathrm{K}}(\mathrm{E}_{\frac{1}{q}}(xt)) &= \mathrm{E}_{q^{-2}}(\tfrac{sqt^2}{\{2\}_q}).\end{aligned} \tag{117}$$

The moments are again revealed by equating coefficients of t^k:

$$\begin{aligned}\Lambda_\psi(x^{2m}) &= s^m \mathrm{QE}\left(m^2-m\right)\{2m-1\}_q!!,\\ \Lambda_\psi(x^{2m-1}) &= 0,\end{aligned} \tag{118}$$

$$\begin{aligned}\Lambda_{\mathrm{K}}(x^{2m}) &= s^m \mathrm{QE}\left(\binom{m}{2}\right)\{2m-1\}_q!!,\\ \Lambda_{\mathrm{K}}(x^{2m-1}) &= 0.\end{aligned} \tag{119}$$

We continue to prove the previous orthogonality relation. By the recurrence (46) we have for $m \le n,\ s > 0$:

$$\begin{aligned}&\Lambda_{\mathrm{H}}(x^n \mathrm{H}_{n,s,q}(x)) \overset{\text{by}(46)}{=} \Lambda_{\mathrm{H}}(x^{n-1}(\mathrm{H}_{n+1}(x) + sq^n\{n\}_q \mathrm{H}_{n-1}(x))\\ &= sq^n\{n\}_q \Lambda_{\mathrm{H}}(x^{n-1}\mathrm{H}_{n-1,s,q}(x)).\end{aligned} \tag{120}$$

This implies by induction

$$\Lambda_{\mathrm{H}}(x^n \mathrm{H}_{n,s,q}(x) = s^n q^{\binom{n+1}{2}}\{n\}_q!, \tag{121}$$

which is equivalent to the proof of (111).
By the recurrences (47) and 48), the polynomials $\psi_{n,s,q}(x)$ and $\mathrm{K}_{n,s,q}(x)$ are not orthogonal with respect to a q-integral with base q, since we cannot use the above proof once again.

12 Conclusion

We have shown that most of the properties of Hermite polynomials have similar q-analogues. The introduction of the extra extra variable s enabled us to define several pseudo q-Hermite polynomials with limits equal to x^ν. It was possible to find a q-orthogonality for the first q-Appell polynomial by replacing the two infinity limits in the integral by finite functions of s and q. The fundamental property for the q-derivative of q-Appell polynomials was essential in the proof, and the proper formula for q-integration by parts had to be found. We had to choose the right formula for q-integration by parts. Furthermore, it was necessary to choose operational formulas starting with ϵ^{-1} and E_{q^2} in the denominator, since the two E_{q^2} in the numerator should be the same. Finally, we showed that the linear functional approach with Favards theorem gives a similar result.

13 Discussion

As was pointed out in [3, p. 331], the Hermite polynomials are special cases of superior spherical functions. Our general approach enables us to continue by constructing q-analogues of the normal distribution, Hermite functions and several number sequences.

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15 Statements and Declarations

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16 Data availability statement

The quoted papers can be found in the database or on internet.

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