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SEVERAL ITERATIVE SCHEMES FOR THE SPLIT FEASIBILITY WITH MULTIPLE OUTPUT SETS AND APPLICATIONS

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Abstract

In this paper, for solving the split feasibility problem with multiple output sets, defined by demiclosed strongly quasi-nonexpansive operators on Hilbert spaces, we propose some block-iterative schemes, using the extrapolated Landweber-type operators. The strong convergence is proved without the boundedly regular condition as well as the closedness property of the range of the transformation operators, assumed recently in the literature for the similar problems. We give a necessary and sufficient condition which ensures that a kth iterate is a solution. We also give an application of our results to solve the multiple-sets split feasibility problem (MSSFP) with multiple output level sets with computational experiments for illustration.

1 Introduction

In this paper we denote by I, $\langle \cdot, \cdot \rangle$, and $\| \cdot \|$, respectively, the identity operator, an inner product, and the corresponding norm in any Hilbert space $\mathcal{H}$. We also denote by $\mathrm{Fix}\,(T)$ the fixed point set of any operator T on $\mathcal{H}$, i.e., $\mathrm{Fix}\,(T) = \{x \in \mathcal{H} : Tx = x\}$.

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We consider the split feasibility problem with multiple output sets, formulated as finding a point

$$p \in C := \cap_{i \in \mathcal{N}} \mathrm{Fix}(U_i),\ A_j p \in \mathrm{Fix}(V_j)\ \forall j \in \mathcal{M}, \tag{1}$$

where

- U_i is a demiclosed and $\rho_{1,i}$-strongly quasi-nonexpansive operator on a real Hilbert space $\mathcal{H}_0$ for $i \in \mathcal{N} = \{1, ..., r\}$ with any positive integer r such that $\rho_1 = \inf_{i \in \mathcal{N}} \rho_{1,i} > 0$.
- V_j is a demiclosed and $\rho_{2,j}$-strongly quasi-nonexpansive operator on a real Hilbert space $\mathcal{H}_j$ for $j \in \mathcal{M} = \{1, ..., q\}$ where q is any positive integer with $\rho_2 = \inf_{j \in \mathcal{M}} \rho_{2,j} > 0$.
- A_j is a bounded linear operator from $\mathcal{H}_0$ into $\mathcal{H}_j$ with the spectral norm $\|A_j\| > 0$.

Throughout this paper, we assume that the solution set of (1), denoted by Γ, is non-empty.

This problem with demicontractive operators U_i and V_j has been very recently studied in [1]. To solve it, they suggested a Halpern method in combination with an Armijo-line search. With the additional assumptions: $p \in \Omega$, a closed and convex subset in $\mathcal{H}_0$, and $\mathcal{H}_j = \mathcal{K}$ for $j \in \mathcal{M}$, where $\mathcal{K}$ is another Hilbert space, this problem is called the multiple-operator split common fixed point problem, introduced and studied firstly by Brooke et al. [4]. To solve it, by assuming that $r = q$, designing an operator $R_j = U_j T_j$, where $T_j = I - \gamma_j A_j^*(V_j - I)A_j$ with $\gamma_j \in (0, 1/\|A_j\|^2)$, and using the general iterative scheme, introduced in [26], they suggested weakly convergent dynamic string-averaging CQ-methods.

Problem (1), when $\mathcal{H}_j = \mathcal{K}$ and $A_j = A$ for all $j \in \mathcal{M}$ with an additional requirement $Ap \in \cap_{j \in \mathcal{M}} \mathrm{Fix}\,(V_j)$, is called the multiple split common fixed point problem, that has been first studied by Censor and Segal [16] and developed by Wang with Xu [28] and Tang with Liu [27]. When r and q are sufficiently large, by using the block-iterative scheme [2], Cegielski [11] suggested a general iterative method, a block-iterative variant of Censor and Segal's method. It is well known that any sequence generated by one of these methods, does not converge strongly in general and depends on the value $\|A\|$, that is difficult to be calculated. One important approach for overcoming the difficulty in computation of $\|A\|$ is to use the extrapolated Landweber-type operator, introduced and studied in [12, 14, 8, 9] and references therein. The extrapolated Landweber-type operator related to V_j is determined by

$$\mathcal{L}_{\sigma_j}\{V_j\}x = x + \sigma_j(x)(\mathcal{L}\{V_j\}x - x),\ x \in \mathcal{H}_0, \tag{2}$$

where the extrapolation function $\sigma_j : \mathcal{H}_0 \to [1, \infty)$ is bounded from above by τ_j, defined by

$$\tau_j(x) = \left(\frac{\|A\|.\|(V_j - I)Ax\|}{\|A^*(V_j - I)Ax\|} \right)^2 \tag{3}$$

if $Ax \notin \mathrm{Fix}(V_j)$ and 1, otherwise, where $A^* : \mathcal{K} \to \mathcal{H}_0$ is the adjoint operator to A, and

$$\mathcal{L}\{V_j\} = I + \frac{1}{\|A\|^2} A^*(V_j - I)A. \tag{4}$$

Obviously, $\tau_j(x) \geq 1$, since $\|A^*(V_j - I)Ax)\| \leq \|A\| \|(V_j - I)Ax\|$ and thus $\mathcal{L}_{\sigma_j}\{V_j\}$ is well defined. Note that $\mathcal{L}_{\tau_j}\{V_j\}$ does not depend on $\|A\|$. Further, in order to obtain a strong convergence sequence, Cegielski and Al-Musallam [8] combined the above approach with the steepest descent method as follows. Put $U_{j+r} = \mathcal{L}_{\sigma_j}\{T_j\}$ and $\mathcal{I} = \{1, ..., m\}$, where $m = r + q$. Then, $\Gamma = \cap_{i \in \mathcal{I}} \mathrm{Fix}\,(U_i)$. Cegielski and Al-Musallam proposed a hybrid steepest-descent method, generating $x^{k+1} = (I - t_k F)S^k x^k, x^0 \in \mathcal{H}_0$, where t_k satisfies the condition
(t) $t_k \in (0, 1)$ for all $k \geq 0$, $\lim_{k \to \infty} t_k = 0$ and $\sum_{k \geq 0} t_k = \infty$,
F is η-strongly monotone and l-Lipschitz continuous on $\mathcal{H}_0$ with $l \geq \eta > 0$, and $S^k = I + \gamma_k(U_{i_k} - I)$ with $i_k \in \mathcal{I}$. They showed that any sequence, generated by this scheme, converges strongly to a point $p_* \in \mathcal{H}_0$, solving the variational inequality

$$p_* \in \Gamma : \ \langle Fp_*, p_* - p \rangle \leq 0 \ \forall p \in \Gamma, \tag{5}$$

under some conditions, two of which are that U_i for each $i \in \mathcal{I}$ is approximately shrinking and the family $\mathcal{F} := \{\mathrm{Fix}\,(U_i) : i \in \mathcal{I}\}$ is boundedly regular. Recently, in order to obtain a strong convergence result, Cegielski et al. [9] suggested the outer approximation method. They proved strong convergence of any sequence $\{x^k\}$, generated by this method, if $ImA, \mathrm{Fix}(V_1), ..., \mathrm{Fix}\,(V_q)\}$, $\{A^{-1}(Q), \mathrm{Fix}(U_1), ..., \mathrm{Fix}\,(U_r)\}$ are boundedly regular, and ImA is closed, where $Q = \cap_{j \in \mathcal{M}} \mathrm{Fix}(V_j)$.

In the case that $q = r$, problem (1) can be formulated in the following equivalent form: Find a point

$$p \in C_i := \mathrm{Fix}(U_i), \ A_i p \in \mathrm{Fix}(V_i) \ \forall i \in \mathcal{N}.$$

When U_i and V_j are cutters ($\rho_{1,i} = \rho_{2,j} = 1$), the authors of [22] proposed a weak convergent iterative method, $x^{k+1} = U_{i_k,\mu_k} T_{i_k,\lambda_k} x^k$, where the parameters $\mu_k, \lambda_k \in [\varepsilon, 2 - \varepsilon]$, $U_{i,\mu}$ is the μ-relaxation of U_i, $T_{i,\lambda}$ is λ-relaxation of $\mathcal{L}\{V_i\}$, and $\{i_k\}_{k=0}^{\infty}$ is an almost cyclic control (cf. [11], Definition 5.6.10).

Recently, in the case that $r = 1$ and U_1, T_j are demicontractive, in order to avoid the difficulties mentioned above, Wang [29] proposed an iterative method, by using the strategies of self-adaptive stepsizes in combination with the viscosity approximation method [23].

In this paper, still by using the extrapolated Landweber-type operators related to V_j in combination with the steepest-descent method, we introduce some new block-iterative schemes, strong convergence of any sequence generated by one of which is proved without computation of any $\|A_j\|$. Moreover, we give a necessary and sufficient condition which confirms that kth iterate is a solution.

We organize the rest of this paper as follows. In Section 2, we list some necessary terminologies and related facts, that will be used in the proof of our results. In Section 3, we first describe our main block-iterative schemes and then show our main results with some modifications and particularities. An application to solving the split feasibility problem with multiple output level sets and numerical experiments are given in Section 4 for illustration.

2 Preliminaries and useful lemmas

We remember that an operator T in $\mathcal{H}$ is called (see, [10]):

- contractive if $\|Tx - Ty\| \leq a\|x - y\|$ with $a \in (0, 1)$ for any $x, y \in \mathcal{H}$.
- quasi-nonexpansive, if $\mathrm{Fix}\,(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in \mathcal{H}$ and $p \in \mathrm{Fix}(T)$.
- ρ-strongly quasi-nonexpansive (ρ-SQNE), if $\mathrm{Fix}\,(T) \neq \emptyset$ and, for all $x \in \mathcal{H}$ and $p \in \mathrm{Fix}(T)$, $\|Tx-p\|^2 \leq \|x-p\|^2 - \rho\|Tx-x\|^2$ where the real number $\rho \geq 0$.
- τ-demicontractive, if $\mathrm{Fix}\,(T) \neq \emptyset$ and it satisfies the condition

  $$\|Tx - p\|^2 \leq \|x - p\|^2 + \tau\|Tx - x\|^2 \ \forall x \in \mathcal{H}, p \in \mathrm{Fix}(T)$$

  with $\tau \in (0, 1)$.
- approximately shrinking (boundedly regular [17]) on a subset $D \subseteq \mathcal{H}$, if for any sequence $\{x^k\} \subseteq D$ the following implication holds

  $$\lim_{k\to\infty} \|Tx^k - x^k\| = 0 \Longrightarrow \lim_{k\to\infty} d(x^k, \mathrm{Fix}\,(T)) = 0, \tag{6}$$

  where $d(x, C) = \inf_{y\in C} \|x - y\|$ for any $x \in \mathcal{H}$ and $C \subset \mathcal{H}$.
- demiclosed, i.e., it satisfies the demiclosedness principle: if for any sequence $\{x^k\} \subset \mathcal{H}$ it holds

  $$w - \lim_{k\to\infty} x^k = x,\ \lim_{k\to\infty} \|(I - T)x^k\| = 0) \Longrightarrow Tx = x. \tag{7}$$

Clearly, a ρ-SQNE operator T with $\rho = 0$ is quasi-nonexpansive and a nonexpansive operator T with $\mathrm{Fix}\,(T) \neq \emptyset$ is quasi-nonexpansive. If an operator T is quasi-nonexpansive or τ-demicontractive on H, then the relaxation operator, defined by $\overline{T} = I + \beta(T - I)$, is ρ-SQNE with $\rho = \beta(1-\beta)$ for any fixed $\beta \in (0,1]$ or $\rho = \beta(1 - \tau - \beta)$ for $\beta \in (0, 1-\tau)$, respectively, and $\mathrm{Fix}\,(T) = \mathrm{Fix}\,(\overline{T})$ (see, [25, 24]). In this case $\mathrm{Fix}\,(T)$ is closed and convex. It is well known that (6) is only a necessary condition for implication (7) (see, [17]).

For each $i \in \mathcal{I} := \{1, ..., m\}$, let C_i be a closed convex subset in $\mathcal{H}$ with $C := \cap_{i \in K} C_i \neq \emptyset$. We say that the family $\mathcal{C} = \{C_i : i \in \mathcal{I}\}$ is boundedly regular [17] if for any bounded sequence $\{x^k\} \subset \mathcal{H}$, there holds the following implication:

$$\lim_{k\to\infty} \max_{i \in K} d(u^k, C_i) = 0 \Longrightarrow \lim_{k\to\infty} d(u^k, C) = 0.$$

A mapping $F : \mathcal{H} \to \mathcal{H}$ is said to be η-strongly monotone and l-Lipschitz continuous, if it satisfies, respectively,

$$\langle Fx - Fy, x - y\rangle \geq \eta \|x - y\|^2, \ \|Fx - Fy\| \leq l\|x - y\| \ \forall x, y \in \mathcal{H}$$

with $l \geq \eta > 0$. It is well known that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \ \forall x, y \in \mathcal{H}.$$

Lemma 2.1 ([31]) *Let $\mathcal{H}$ be a real Hilbert space and let F be an η-strongly monotone and l-Lipschitz continuous operator on $\mathcal{H}$ with some positive constants $l \geq \eta > 0$. Let $T^{t,\mu} = I - t\mu F$. Then, for a fixed number $\mu \in (0, 2\eta/l^2)$ and any $t \in (0,1)$, $T^{t,\mu}$ is a contractions with coefficient $1 - t\beta$, where $\beta = (1/2)\mu(2\eta - \mu l^2)$.*

Lemma 2.2 ([30]) *Let $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ be sequences of real numbers such that, for all $k \geq 0$, $a_{k+1} \leq (1 - b_k)a_k + b_k c_k$; $a_k \geq 0$; b_k satisfies a condition of type (t); and either $\sum_{k=1}^{\infty} b_k |c_k| < \infty$ or $\limsup_{k\to\infty} c_k \leq 0$. Then, $\lim_{k\to\infty} a_k = 0$.*

Lemma 2.3 ([20]) *Let $\{c_k\}$ be a sequence of real numbers with a subsequence $\{l_k\}$ of $\{k\}$ such that $c_{l_k} < c_{l_k+1}$. Then, there exists a non-decreasing sequence $\{m_k\} \subseteq \{k\}$ such that $m_k \to \infty, c_{m_k} \leq c_{m_k+1}$ and $c_k \leq c_{m_k+1}$ for all (sufficiently large) numbers $k \geq 0$. In fact, $m_k = \max\{l \leq k : c_l \leq c_{l+1}\}$.*

Lemma 2.4 ([15]) *Let $T_i : \mathcal{H} \to \mathcal{H}$ be a ρ_i-SQNE operator with $\rho_i > 0$ for all $i \in \mathcal{L} = \{1, ..., m\}$ for some integer $m \geq 1$ such that $\cap_{i\in\mathcal{L}} \mathrm{Fix}\,(T_i) \neq \emptyset$, $T := \sum_{i\in\mathcal{L}} \omega_i T_i$, where $\omega_i \geq 0$ for all $i \in \mathcal{L}$ and $\sum_{i\in\mathcal{L}} \omega_i = 1$, and let $\tilde{T} := T_m ... T_1$. Then, for any $x \in \mathcal{H}$ and $p \in \cap_{i\in\mathcal{L}} \mathrm{Fix}\,(T_i)$, we have*

$$\frac{1}{2R} \sum_{i\in\mathcal{L}} \omega_i \rho_i \|(T_i - I)x\|^2 \leq \|(T - I)x\|^2 \tag{8}$$

and

$$\frac{1}{2R}\sum_{i\in\mathcal{L}}\rho_i\|(T^i-T^{i-1})x\|^2\le\|(\tilde{T}-I)x\|^2, \tag{9}$$

where $T^i=T_i...T_1, i\in L, T^0=I$ *and* $R\ge\|x-p\|$.

3 Main results

First of all, we formulate our block-iterative schemes.
Block-iterative schemes:

1. Choose any two points x^{-1} and x^0 in $\mathcal{H}_0$ such that $x^{-1}\neq x^0$ and an integer $s\ge 1$. Set $k:=0$.

2. Set $x^{k,0}=x^k$, the current iterate. For $t=1,2,\cdots,s$, let $\mathcal{N}_t^k$ and $\mathcal{M}_t^k$ be two ordered non-empty subsets of $\mathcal{N}$ and $\mathcal{M}$, respectively, such that $\mathcal{N}=\mathcal{N}_1^k\cup\cdots\cup\mathcal{N}_s^k$ and $\mathcal{M}=\mathcal{M}_1^k\cup\cdots\cup\mathcal{M}_s^k$ and let define $x^{k,t}$ by the rule:

$$x^{k,t}=U_t^ky^{k,t},\ y^{k,t}=V_t^kx^{k,t-1}, \tag{10}$$

where either

$$U_t^k=\sum_{i^k(t)\in\mathcal{N}_t^k}\lambda^k_{i^k(t)}U_{i^k(t)}$$

or

$$U_t^k=U_{i^k_{|\mathcal{N}_t^k|}(t)}...U_{i^k_1(t)}$$

and either

$$V_t^k=\sum_{j^k(t)\in\mathcal{M}_t^k}\theta^k_{j^k(t)}\mathcal{L}_{\sigma_{j^k(t)}}\{V_{j^k(t)}\}$$

or

$$V_t^k=\mathcal{L}_{\sigma_{j^k_{|\mathcal{M}_t^k|}(t)}}\{V_{j^k_{|\mathcal{M}_t^k|}(t)}\}...\mathcal{L}_{\sigma_{j^k_1(t)}}\{V_{j^k_1(t)}\},$$

$\mathcal{N}_t^k=\{i_1^k(t),...,i^k_{|\mathcal{N}_t^k|}(t)\}$, $\mathcal{M}_t^k=\{j_1^k(t),...,j^k_{|\mathcal{M}_t^k|}(t)\}$, the parameters $\lambda^k_{i^k(t)}$ and $\theta^k_{j^k(t)}\ge\theta$ satisfy

$$\lambda^k_{i^k(t)},\theta^k_{j^k(t)}\ge\theta>0,\ \sum_{i^k(t)\in\mathcal{N}_t^k}\lambda^k_{i^k(t)}=\sum_{j^k(t)\in\mathcal{M}_t^k}\theta^k_{j^k(t)}=1,$$

$\mathcal{L}_{\sigma_{j^k(t)}}\{V_{j^k(t)}\}$ is defined by (2)-(4) with $j=j^k(t)$ and $A=A_{j^k(t)}$, a function $\sigma_{j^k(t)}:\mathcal{H}_0\to[1,\infty)$ is an extrapolation functional bounded from above by $\tau_{j^k(t)}$, defined by (3) with $V_{j^k(t)}$ and $A_{j^k(t)}$ instead of V_j and A, respectively, i.e.,

$$\mathcal{L}_{\sigma_{j^k(t)}}\{V_{j^k(t)}\}x = x + \sigma_{j^k(t)}(x)(\mathcal{L}\{V_{j^k(t)}\}x - x),\ x \in \mathcal{H}_0,$$

$$\mathcal{L}\{V_{j^k(t)}\} = I + \frac{1}{\|A_{j^k(t)}\|^2}A^*_{j^k(t)}(V_{j^k(t)} - I)A_{j^k(t)},$$

$$\tau_{j^k(t)}(x) = \left(\frac{\|A_{j^k(t)}\|\|(V_{j^k(t)} - I)A_{j^k(t)}x\|}{\|A^*_{j^k(t)}(V_{j^k(t)} - I)A_{j^k(t)}x)\|}\right)^2$$

if $A_{j^k(t)}x \notin \mathrm{Fix}(V_{j^k(t)})$ and 1, otherwise.

If $x^{k,s} = x^k$, then stop. x^k is a solution of (1). Otherwise, go to the next step.

3. Compute

$$x^{k+1} = (1-\xi_k)(I - t_k\mu_k F)x^k + \xi_k x^{k,s}, \tag{11}$$

where $\xi_k \in [\xi, 1-\xi]$ with a sufficiently small real number $\xi > 0$, F is an η-strongly monotone and l-Lipschitz continuous operator on $\mathcal{H}_0$ with $l \geq \eta > 0$, t_k satisfies (t), and

$$\mu_k = \frac{\langle Fx^k - Fx^{k-1}, x^k - x^{k-1}\rangle}{\|Fx^k - Fx^{k-1}\|^2},$$

if $x^k \neq x^{k-1}$ and μ_{k-1}, otherwise.

Set $x^{k-1} := x^k$, $x^k := x^{k+1}$ and $k := k+1$. Return to step 2.

Further, we need the following lemmas.

Lemma 3.1 *There holds the following inequality*

$$\|(I - V_j)A_j x\|^2 \leq \frac{2}{\rho_2 + 1}\langle A_j^*(I - V_j)A_j x, x - p\rangle,\ \forall x \in \mathcal{H}_0, \tag{12}$$

for any $p \in \Gamma$.

Proof This is followed from Lemmas 2.5, 2.6 in [14] and Fact 2.7, (i) in [13].

Lemma 3.2 *For any sequence* $\{x^k\}$*, generated by one of our block-iterative schemes, we have:*

(i)

$$\|x^{k,s} - p\|^2 \leq \|x^k - p\|^2 - \tilde{\rho}\sum_{t=1}^{s}\big[\|(V_t^k - I)x^{k,t-1}\|^2 + \|(U_t^k - I)y^{k,t}\|^2\big], \tag{13}$$

where p *is any point in* Γ, $\tilde{\rho} = \min\{\rho_1/r, \rho_2/q, \rho_2/(qa)\}$, *and* $a = \max_{1\leq j\leq q}\|A_j\|^2$.

(ii) x^k *is a solution of (1) if and only if* $x^{k,s} = x^k$.

Proof (i) First, we consider the following case, when

$$U_t^k = \sum_{i^k(t)\in\mathcal{N}_t^k} \lambda^k_{i^k(t)} U_{i^k(t)},\ V_t^k = \sum_{j^k(t)\in\mathcal{M}_t^k} \theta^k_{j^k(t)} \mathcal{L}_{\sigma_{j^k(t)}}. \tag{14}$$

Using the definition of $y^{k,t}$ in (10), (12), the properties of V_j, and the definition of $\sigma_{k,t}, \theta^k_{j^k(t)}$, we get

$$\begin{aligned}
\|y^{k,t} - p\|^2 \quad & = \|V_t^k x^{k,t-1} - p\|^2 \le \sum_{j^k(t)\in\mathcal{M}_t^k} \theta^k_{j^k(t)} \|\mathcal{L}_{\sigma_{j^k(t)}} x^{k,t-1} - p\|^2 \\
& = \sum_{j^k(t)\in\mathcal{M}_t^k} \theta^k_{j^k(t)} \|x^{k,t-1} - p \\
& \qquad + \sigma_{j^k(t)}(x^{k,t-1})(\mathcal{L}\{V_{j^k(t)}\} - I)x^{k,t-1}\|^2 \\
& = \sum_{j^k(t)\in\mathcal{M}_t^k} \theta^k_{j^k(t)} \Big[\|x^{k,t-1} - p\|^2 \\
& \quad -2\frac{\sigma_{k,t}(x^{k,t-1})}{\|A_{j^k(t)}\|^2} \langle A^*_{j^k(t)}(I - V_{j^k(t)})A_{j^k(t)}x^{k,t-1}, x^{k,t-1} - p\rangle \\
& \quad + \frac{\sigma^2_{k,t}(x^{k,t-1})}{\|A_{j^k(t)}\|^4} \|A^*_{j^k(t)}(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1}\|^2 \Big] \\
& \le \sum_{j^k(t)\in\mathcal{M}_t^k} \theta^k_{j^k(t)} \Big[\|x^{k,t-1} - p\|^2 \\
& \qquad -((\rho_2/q) + 1)\frac{\sigma_{j^k(t)}(x^{k,t-1})}{\|A_{j^k(t)}\|^2} \|(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1}\|^2 \\
& \qquad + \frac{\sigma_{k,t}(x^{k,t-1})}{\|A_{j^k(t)}\|^2} \frac{\tau_{k,t}(x^{k,t-1})}{\|A_{j^k(t)}\|^2} \|A^*_{j^k(t)}(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1}\|^2 \Big] \\
& = \sum_{j^k(t)\in\mathcal{M}_t^k} \theta^k_{j^k(t)} \Big[\|x^{k,t-1} - p\|^2 \\
& \qquad -((\rho_2/q) + 1)\frac{\sigma_{j^k(t)}(x^{k,t-1})}{\|A_{j^k(t)}\|^2} \|(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1}\|^2 \\
& \quad + \frac{\sigma_{k,t}(x^{k,t-1})}{\|A_{j^k(t)}\|^2} \|(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1}\|^2 \Big] \\
& = \sum_{j^k(t)\in\mathcal{M}_t^k} \theta^k_{j^k(t)} \Big[\|x^{k,t-1} - p\|^2 \\
& \qquad -(\rho_2/q)\frac{\sigma_{j^k(t)}(x^{k,t-1})}{\|A_{j^k(t)}\|^2} \|(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1}\|^2 \Big].
\end{aligned}$$

From this, $\sigma_{k,t} \geq 1$, $a \geq \|A_{j^k(t)}\|^2$, $\|A_{j^k(t)}\| = \|A^*_{j^k(t)}\|$, and

$$\|A^*_{j^k(t)}\|\|(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1}\| \geq \|A^*_{j^k(t)}(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1}\|$$

we deduce that

$$\begin{aligned}
\|y^{k,t} - p\|^2 &\leq \|x^{k,t-1} - p\|^2 \\
&\quad - \frac{\rho_2}{qa} \sum_{j^k(t)\in\mathcal{M}_t^k} \theta^k_{j^k(t)} \frac{\sigma_{j^k(t)}(x^{k,t-1})}{\|A_{j^k(t)}\|^2} \|A^*_{j^k(t)}(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1}\|^2 \\
&\leq \|x^{k,t-1} - p\|^2 \\
&\quad - \frac{\rho_2}{qa} \left\| \sum_{j^k(t)\in\mathcal{M}_t^k} \theta^k_{j^k(t)} \frac{\sigma_{j^k(t)}(x^{k,t-1})}{\|A_{j^k(t)}\|^2} A^*_{j^k(t)}(V_{j^k(t)} - I)A_{j^k(t)}x^{k,t-1} \right\|^2 \\
&= \|x^{k,t-1} - p\|^2 - \frac{\rho_2}{qa}\|(V_t^k - I)x^{k,t-1}\|^2.
\end{aligned}$$

Then,

$$\begin{aligned}
\|x^{k,t} - p\|^2 &\leq \sum_{i^k(t)\in\mathcal{N}_t^k} \lambda^k_{i^k(t)} \|U_{i^k(t)}y^{k,t} - p\|^2 \\
&\leq \|y^{k,t} - p\|^2 - \sum_{i^k(t)\in\mathcal{N}_t^k} \lambda^k_{i^k(t)}\rho_{i^k(t)}\|(U_{i^k(t)} - I)y^{k,t}\|^2 \\
&\leq \|y^{k,t} - p\|^2 - \rho_1 \sum_{i^k(t)\in\mathcal{N}_t^k} \lambda^k_{i^k(t)}\|(U_{i^k(t)} - I)y^{k,t}\|^2 \\
&\leq \|y^{k,t} - p\|^2 - \rho_1 \left\| \sum_{i^k(t)\in\mathcal{N}_t^k} \lambda^k_{i^k(t)}(U_{i^k(t)} - I)y^{k,t} \right\|^2 \\
&= \|y^{k,t} - p\|^2 - \rho_1\|(U_t^k - I)y^{k,t}\|^2.
\end{aligned}$$

This together with (15) implies that

$$\begin{aligned}
\|x^{k,t} - p\|^2 &\leq \|x^{k,t-1} - p\|^2 - \rho_1\|(U_t^k - I)y^{k,t}\|^2 \\
&\quad - \tfrac{\rho_2}{qa}\|(V_t^k - I)x^{k,t-1}\|^2.
\end{aligned} \tag{15}$$

Now, we consider the case when U_t^k and V_t^k are defined by

$$\begin{aligned}
U_t^k &= U_{i^k_{|\mathcal{N}_t^k|}(t)} ... U_{i^k_1(t)}, \\
V_t^k &= \mathcal{L}_{\sigma_{j^k_{|\mathcal{M}_t^k|}(t)}}\{V_{j^k_{|\mathcal{M}_t^k|}(t)}\} ... \mathcal{L}_{\sigma_{j^k_1(t)}}\{V_{j^k_1(t)}\}.
\end{aligned} \tag{16}$$

Then, the ρ_1-SQNE property of U_i yields

$$\begin{aligned}\|x^{k,t}-p\|^2 \qquad &= \|U_t^k y^{k,t}-p\|^2 = \|U_{i^k_{|L_t^k|}(t)} U_{i^k_{|L_t^k|-1}(t)} ... U_{i_1^k(t)} y^{k,t}-p\|^2 \\ &\leq \|U_{i^k_{|L_t^k|-1}(t)} ... U_{i_1^k(t)} y^{k,t}-p\|^2 \\ &\quad -\rho_1 \|U_{i^k_{|L_t^k|}(t)} U_{i^k_{|L_t^k|-1}(t)} ... U_{i_1^k(t)} y^{k,t} - U_{i^k_{|L_t^k|-1}(t)} ... U_{i_1^k(t)} y^{k,t}\|^2 \\ &\leq \|y^{k,t}-p\|^2 - \rho_1 \sum_{h=1}^{|\mathcal{N}_t^k|} \|(U^{i_h^k(t)} - U^{i_{h-1}^k(t)}) y^{k,t}\|^2 \\ &\leq \|y^{k,t}-p\|^2 - \frac{\rho_1}{r} \|(U_t^k - I) y^{k,t}\|^2\end{aligned}$$

where $U^{i_h^k(t)}$ being a product of the first h operators along the string $\mathcal{N}_t^k$ and $U^{i_0^k(t)} = I$, because $|\mathcal{N}_t^k| \leq r$ and

$$\|(U_t^k - I) y^{k,t}\|^2 \leq |\mathcal{N}_t^k| \sum_{h=1}^{|\mathcal{N}_t^k|} \|(U^{i_h^k(t)} - U^{i_{h-1}^k(t)}) y^{k,t}\|^2.$$

Similarly, we get

$$\begin{aligned}\|y^{k,t}-p\|^2 \qquad &= \|\mathcal{L}_{\sigma_{j^k_{|\mathcal{M}_t^k|}(t)}} \{V_{j^k_{|\mathcal{M}_t^k|}(t)}\} ... \mathcal{L}_{\sigma_{j_1^k(t)}} \{V_{j_1^k(t)}\} x^{k,t-1} - p\|^2 \\ &\leq \|\mathcal{L}_{\sigma_{j^k_{|\mathcal{M}_t^k|-1}(t)}} \{V_{j^k_{|\mathcal{M}_t^k|}(t)}\} ... \mathcal{L}_{\sigma_{j_1^k(t)}} \{V_{j_1^k(t)}\} x^{k,t-1} - p\|^2 \\ &\quad -\rho_2 \big\| \big(\mathcal{L}_{\sigma_{j^k_{|\mathcal{M}_t^k|}(t)}} \{V_{j^k_{|\mathcal{M}_t^k|}(t)}\} ... \mathcal{L}_{\sigma_{j_1^k(t)}} \{V_{j_1^k(t)}\} \\ &\quad -\mathcal{L}_{\sigma_{j^k_{|\mathcal{M}_t^k|-1}(t)}} \{V_{j^k_{|\mathcal{M}_t^k|}(t)}\} ... \mathcal{L}_{\sigma_{j_1^k(t)}} \{V_{j_1^k(t)}\} x^{k,t-1}\big) x^{k,t-1} \big\|^2 \\ &\leq \|x^{k,t-1}-p\|^2 - \rho_2 \sum_{h=1}^{|\mathcal{M}_t^k|} \|(\mathcal{L}^{i_h^k(t)} - \mathcal{L}^{i_{h-1}^k(t)}) x^{k,t-1}\|^2, \qquad (17)\end{aligned}$$

where $\mathcal{L}^{i_h^k(t)}$ is defined similarly as $U^{i_h^k(t)}$. Since

$$\|(V_i^k - I) x^{k,t-1}\|^2 \leq q \sum_{h=1}^{|\mathcal{M}_t^k|} \|(\mathcal{L}^{i_h^k(t)} - \mathcal{L}^{i_{h-1}^k(t)}) x^{k,t-1}\|^2,$$

from (17) we obtain (15) with $a = 1$. Therefore,

$$\begin{aligned}\|x^{k,t}-p\|^2 \qquad &\leq \|x^{k,t-1}-p\|^2 - \rho_1 \sum_{h=1}^{|L_t^k|} \|(U^{i_h^k(t)} - U^{i_{h-1}^k(t)}) y^{k,t}\|^2 \\ &\quad -\frac{\rho_2}{q} \|(V_t^k - I) x^{k,t-1}\|^2. \qquad (18)\end{aligned}$$

Noting $x^{k,0} = x^k$ and summing each (16) and (18) with $t = 1, ..., s$, we obtain (13). It is easy to see that in the rest cases, when U_t^k, V_t^k are given by

$$U_t^k = \sum_{i^k(t) \in \mathcal{N}_t^k} \lambda_{i^k(t)}^k U_{i^k(t)},\ V_t^k = \mathcal{L}_{\sigma_{j^k_{|\mathcal{M}_t^k|}(t)}}\{V_{j^k_{|\mathcal{M}_t^k|}(t)}\}...\mathcal{L}_{\sigma_{j_1^k(t)}}\{V_{j_1^k(t)}\}, \tag{19}$$

or

$$U_t^k = U_{i^k_{|\mathcal{N}_t^k|}(t)}...U_{i_1^k(t)},\ V_t^k = \sum_{j^k(t) \in \mathcal{M}_t^k} \theta_{j^k(t)}^k \mathcal{L}_{\sigma_{j^k(t)}}, \tag{20}$$

we also obtain the same results.
(ii) It is obvious that if $x^k \in \Gamma$ then $x^{k,s} = x^k$. Inversely, we have to prove that if $x^{k,s} = x^k$ then $x^k \in \Gamma$. From (13) and $x^{k,s} = x^k$ we deduce that

$$\|y^{k,t} - x^{k,t-1}\| = \|(V_t^k - I)x^{k,t-1}\| = 0,\ \|(U_t^k - I)y^{k,t}\| = 0, \tag{21}$$

for $t = 1, ..., s$. On the other hand, by (10),

$$\|x^{k,t} - x^{k,t-1}\| = \|U_t^k y^{k,t} - x^{k,t-1}\| \leq \|(U_t^k - I)y^{k,t}\| + \|(V_t^k - I)x^{k,t-1}\|,$$

for $t = 1, ..., s$. Then, by (21) $\|x^{k,t-1} - x^k\| = 0$. This and (21) yield

$$(U_t^k - I)x^k = 0,\ (V_t^k - I)x^k = 0, \tag{22}$$

for $t = 1, ..., s$.

Now, for arbitrary fixed $i \in \mathcal{N}$, $j \in \mathcal{M}$, and $k \geq 0$, from the definition of the proposed iterative schemes there exist integers $t_1^k, t_2^k \in \{1, ..., s\}$ such that $i \in \mathcal{N}_{t_1^k}^k$ and $j \in \mathcal{M}_{t_2^k}^k$. When U_t^k and V_t^k are defined by (14), from (8) and (22) we obtain

$$0 \leq \frac{\rho_{1,i}\theta}{2R}\|(U_i - I)x^k\|^2 \leq \|(U_{t_1^k}^k - I)x^k\|^2 = 0,$$
$$0 \leq \frac{\rho_{2,j}\theta}{2R}\|(\mathcal{L}_{\sigma_j}\{V_j\} - I)x^k\|^2 \leq \|(V_{t_2^k}^k - I)x^k\|^2 = 0.$$

It means that $x^k \in \mathrm{Fix}(U_i)$ and $\|(\mathcal{L}_{\sigma_j}\{V_j\} - I)x^k\|^2 = 0$, which and (4) yield $A_j^*(V_j - I)A_jx^k = 0$. The last equality and (12) yield $(V_j - I)A_jx^k = 0$. It means that $A_jx^k \in \mathrm{Fix}(V_j)$. Since i and j are arbitrary in $\mathcal{N}$ and $\mathcal{M}$, respectively, $x^k \in \Gamma$. When $U_{t_1^k}^k$ and V_t^k are given by (16) we have

$$\|(U_{t_1^k}^k - I)x^k\|^2 = \|(U_{i^k_{|\mathcal{N}_{t_1^k}^k|}(t_1^k)}...U_i...U_{i_1^k(t_1^k)} - I)x^k\|^2 = 0,$$
$$\|(V_{t_2^k}^k - I)x^k\|^2 = \|(\mathcal{L}_{\sigma_j}\{V_j\} - I)x^k\|^2 = \|(\mathcal{L}_{\sigma_{j^k_{|\mathcal{M}_{t_2^k}^k|}(t_2^k)}}\{V_{j^k_{|\mathcal{M}_{t_2^k}^k|}(t_2^k)}\}$$
$$...\mathcal{L}_{\sigma_j}\{V_j\}...\mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\} - I)x^k\|^2 = 0, \tag{23}$$

where i coincides at least with one of $\{i_1^k(t_1^k), ..., i^k_{|\mathcal{N}^k_{t_1^k}|}(t_1^k)\}$ and j is one of $\{j_1^k(t_2^k), ..., j^k_{|\mathcal{M}^k_{t_2^k}|}(t_2^k)\}$. Without loss of generality, we can write

$$\begin{aligned}\mathcal{N}^k_{t_1^k} &= \{i_1^k(t_1^k), ..., i_l^k(t_1^k), i, ..., i^k_{|\mathcal{N}^k_t|}(t_1^k)\},\\ \mathcal{M}^k_{t_2^k} &= \{j_1^k(t_2^k), ..., j_{l'}^k(t_2^k), j, ..., j^k_{|\mathcal{M}^k_t|}(t_2^k)\}.\end{aligned}$$

By using (9) with $\tilde{T} = U^k_{t_1^k}$ and $\tilde{T} = V^k_{t_2^k}$, that satisfy (23), we get, respectively,

$$\begin{aligned}0 \le \quad & e\big[\|(U_i U_{i_l^k(t_1^k)} ... U_{i_1^k(t_1^k)} - U_{i_l^k(t_1^k)} ... U_{i_1^k(t_1^k)})x^k\|^2 + \\ & \|(U_{i_l^k(t_1^k)} ... U_{i_1^k(t_1^k)} - U_{i_{l-1}^k(t_1^k)} ... U_{i_1^k(t_1^k)})x^k\|^2 + ... + \\ & \|(U_{i_2^k(t_1^k)} U_{i_1^k(t_1^k)} - U_{i_1^k(t_1^k)})x^k\|^2 + \|(U_{i_1^k(t_1^k)} - I)x^k\|^2\big] \le 0,\end{aligned}$$

where $e = \theta\rho_1/(2\tilde{R})$ and

$$\begin{aligned}0 \le \quad & e'\big[\|\big(\mathcal{L}_{\sigma_j}\{V_j\}\mathcal{L}_{\sigma_{j_{l'}^k(t_2^k)}}\{V_{j_{l'}^k(t_2^k)}\} ... \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\} \\ & \qquad - \mathcal{L}_{\sigma_{j_{l'}^k(t_2^k)}}\{V_{j_{l'}^k(t_2^k)}\} ... \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\}\big)x^k\|^2 + \\ & \|\big(\mathcal{L}_{\sigma_{j_{l'}^k(t_2^k)}}\{V_{j_{l'}^k(t_2^k)}\} ... \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\} \\ & \qquad - \mathcal{L}_{\sigma_{j_{l'-1}^k(t_2^k)}}\{V_{j_{l'-1}^k(t_2^k)}\} ... \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\}\big)x^k\|^2 + ... + \\ & \|\big(\mathcal{L}_{\sigma_{j_2^k(t_2^k)}}\{V_{j_2^k(t_2^k)}\}\mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\} - \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\}\big)x^k\|^2 \\ & + \|\big(\mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\} - I\big)x^k\|^2\big] \le 0,\end{aligned}$$

where $e' = \theta\rho_2/(2\tilde{R})$ with some constant $\tilde{R} > 0$. Hence,

$$\begin{aligned}& \|(U_i U_{i_l^k(t_1^k)} ... U_{i_1^k(t_1^k)} - U_{i_l^k(t_1^k)} ... U_{i_1^k(t_1^k)})x^k\|^2 = \\ & \|(U_{i_l^k(t_1^k)} ... U_{i_1^k(t_1^k)} - U_{i_{l-1}^k(t_1^k)} ... U_{i_1^k(t_1^k)})x^k\|^2 = ... = \\ & \|(U_{i_2^k(t_1^k)} U_{i_1^k(t_1^k)} - U_{i_1^k(t_1^k)})x^k\|^2 = \|(U_{i_1^k(t_1^k)} - I)x^k\|^2 = 0\end{aligned}$$

and

$$\begin{aligned}& \|\big(\mathcal{L}_{\sigma_j}\{V_j\}\mathcal{L}_{\sigma_{j_{l'}^k(t_2^k)}}\{V_{j_{l'}^k(t_2^k)}\} ... \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\} \\ & \qquad - \mathcal{L}_{\sigma_{j_{l'}^k(t_2^k)}}\{V_{j_{l'}^k(t_2^k)}\} ... \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\}\big)x^k\|^2 = \\ & \|\big(\mathcal{L}_{\sigma_{j_{l'}^k(t_2^k)}}\{V_{j_{l'}^k(t_2^k)}\} ... \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\} \\ & \qquad - \mathcal{L}_{\sigma_{j_{l'-1}^k(t_2^k)}}\{V_{j_{l'-1}^k(t_2^k)}\} ... \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\}\big)x^k\|^2 = ... = 0, \\ & \|\big(\mathcal{L}_{\sigma_{j_2^k(t_2^k)}}\{V_{j_2^k(t_2^k)}\}\mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\} - \mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\}\big)x^k\|^2 = 0, \\ & \|\big(\mathcal{L}_{\sigma_{j_1^k(t_2^k)}}\{V_{j_1^k(t_2^k)}\} - I\big)x^k\|^2\big] = 0.\end{aligned}$$

Consequently, $\|(U_i - I)x^k\|^2 = 0$ and $\|(\mathcal{L}_{\sigma_j}\{V_j\} - I)x^k\|^2 = 0$. Therefore, $x^k \in \mathrm{Fix}(U_i)$ and $A_j x^k \in \mathrm{Fix}(T_j)$.

Evidently, in the rest cases, when U_t^k and V_t^k are defined by (19) and (20), we also get the same results. This completes the proof.

Lemma 3.3 *For any $p \in \Gamma$, the following inequality holds*

$$\begin{aligned}\|x^{k+1} \quad -p\|^2 \le [1-(1-\theta_k)\alpha_k\beta_k]\|x^k - p\|^2 + 2(1-\theta_k)\alpha_k\big[\langle Fp, p - x^k\rangle \\ +\alpha_k\|Fp\|M_1\big] - \tilde{\rho}\textstyle\sum_{t=1}^{s}\big[\|(V_t^k - I)x^{k,t-1}\|^2 + \|(U_t^k - I)y^{k,t}\|^2\big] \quad (24)\end{aligned}$$

for all $k \ge \tilde{k}$, a positive integer, where $\alpha_k = t_k\mu_k$ and $\beta_k = (1/2)(2\eta - \alpha_k l^2)$ such that $\alpha_k \in (0, \eta/l^2)$, $\eta \ge \beta_k \ge \eta/2$, and $1 - \alpha_k\beta_k \ge \tilde{c}$ for all $k \ge \tilde{k}$, and M_1, $\tilde{c}$ are positive real numbers.

Proof Clearly, from (13) it follows that $\|x^{k,s} - p\| \le \|x^k - p\|$ for all $k \ge 0$. We put

$$\eta_k = \frac{\langle Fx^k - Fx^{k-1}, x^k - x^{k-1}\rangle}{\|x^k - x^{k-1}\|^2},\ l_k = \frac{\|Fx^k - Fx^{k-1}\|}{\|x^k - x^{k-1}\|}.$$

Then,

$$\eta \le \eta_k \le \frac{\|Fx^k - Fx^{k-1}\|}{\|x^k - x^{k-1}\|} = l_k \le l,$$

and hence

$$\frac{\eta}{l^2} \le \mu_k = \frac{\eta_k}{l_k^2} \le \frac{1}{\eta_k} \le \frac{1}{\eta}. \quad (25)$$

Since $t_k \to 0$ as $k \to \infty$, by (25), there exists a positive integer $\tilde{k}$ such that $\alpha_k \in (0, \eta/l^2)$ and $\eta \ge \beta_k \ge \eta/2$ for all $k \ge \tilde{k}$. Thus, by using Lemma 2.1 and (25), we obtain

$$\begin{aligned}\|x^{k+1} - p\| &= \|(1-\theta_k)(I - t_k\mu_k F)x^k + \theta_k x^{k,s} - p\| \\ &\le (1-\theta_k)\|(I - t_k\mu_k F)x^k - p\| + \theta_k\|x^{k,s} - p\| \\ &= (1-\theta_k)\|(I - \alpha_k F)x^k - (I - \alpha_k F)p - \alpha_k Fp\| \\ &\qquad + \theta_k\|x^{k,s} - p\| \\ &\le (1-\theta_k)\big[(1-\alpha_k\beta_k)\|x^k - p\| + \alpha_k\|Fp\|\big] + \theta_k\|x^k - p\| \\ &= [1-(1-\theta_k)\alpha_k\beta_k]\|x^k - p\| + (1-\theta_k)\alpha_k\beta_k\|Fp\|/\beta_k \\ &\le [1-(1-\theta_k)\gamma_k\beta_k]\|x^k - p\| + (1-\theta_k)\gamma_k\beta_k 2\|Fp\|/\eta \\ &\le r' := \max\,\{\|x^{\tilde{k}} - p\|, 2\|Fp\|/\eta\}\ \forall k \ge \tilde{k}.\end{aligned}$$

It means that $\{x^k\}$ is bounded. Hence, $\{Fx^k\}$ is bounded. Thus, $\|Fx^k\| \le M_1$

for all $k \geq 0$ where M_1 is some positive constant. Next, again by Lemma 2.1,

$$\begin{aligned}\|x^{k+1}-p\|^2 \quad &\leq (1-\theta_k)\|(I-t_k\mu_kF)x^k-p\|^2+\theta_k\|x^{k,s}-p\|^2\\ &=(1-\theta_k)\|(I-\alpha_kF)x^k-(I-\alpha_kF)p-\alpha_kFp\|^2\\ &\qquad +\theta_k\|x^{k,s}-p\|^2\\ &\leq(1-\theta_k)\big[(1-\alpha_k\beta_k)\|x^k-p\|^2+2\alpha_k\big(\langle Fp,p-x^k\rangle\\ &\qquad +\alpha_k\|Fp\|M_1\big)\big]+\theta_k\|x^{k,s}-p\|^2. \qquad (26)\end{aligned}$$

Replacing $\|x^{k,s}-p\|^2$ in (26) by its upper bound in (13) we obtain (24). The proof is completed.

Now, we are in the position to prove strong convergence of any sequence, generated by one of our iterative schemes.

Theorem 3.1 *Any sequence $\{x^k\}$, generated by one of our block-iterative schemes, as $k \to \infty$, converges strongly to a point $p_* \in \Gamma$, satisfying (5).*

Proof We need only to consider two cases.

Case 1. $\|x^{k+1}-p_*\| \leq \|x^k-p_*\|$ for all $k \geq \tilde{k}$, the integer in Lemma 3.3

Then, $\{x^k\}$ is bounded and $\lim_{k\to\infty}\|x^k-p_*\|$ exists. From (24) and the properties of $\theta_k, \alpha_k, \beta_k$ it follows that

$$d^k \leq \|x^k-p_*\|^2-\|x^{k+1}-p_*\|^2+d_k\alpha_k,$$

for all $k \geq \bar{k}$, a sufficiently large integer ($\bar{k} \geq \tilde{k}$), where $d_k = 2\|Fp_*\|(r'+\alpha_kM_1)$ and

$$d^k=\tilde{\rho}\sum_{t=1}^{s}\big[\|(V_t^k-I)x^{k,t-1}\|^2+\|(U_t^k-I)y^{k,t}\|^2\big].$$

We prove that $\lim_{k\to\infty} d^k = 0$. Indeed, if $d^k \leq d_k\alpha_k$ for all $k \geq \bar{k}$ then the limit exists and equal to zero. Otherwise, i.e. $d^k > d_k\alpha_k$, we have

$$0 \leq \sum_{k=\tilde{k}}^{N}(d^k-d_k\alpha_k) \leq \|x^{\tilde{k}}-p_*\|^2-\|x^{N+1}-p_*\|^2,$$

for any positive integer N. Thus, $\sum_{k=0}^{\infty}(d^k-d_k\alpha_k) \leq \|x^{\tilde{k}}-p_*\|^2$. Therefore, $\lim_{k\to\infty}(d^k-d_k\alpha_k)=0$, that yields $\lim_{k\to\infty} d^k=0$. This is equivalent to

$$\lim_{k\to\infty}\|(U_t^k-I)y^{k,t-1}\|=0,\ \lim_{k\to\infty}\|(V_t^k-I)x^{k,t-1}\|^2=0,$$

for $t = 1, ..., s$. Next, by the same argument, as in the proof of (21) and (22) in Lemma 3.2, we have

$$\lim_{k\to\infty}\|y^{k,t}-x^{k,t-1}\|=0,\ \lim_{k\to\infty}\|x^{k,t-1}-x^k\|=0.$$

Hence,

$$\lim_{k\to\infty}\|(U_t^k-I)x^k\|=0,\ \lim_{k\to\infty}\|(V_t^k-I)x^k\|=0, \tag{27}$$

for $t=1,...,s$.

Since $\{x^k\}$ is a bounded sequence in the Hilbert space $\mathcal{H}_0$, there exists a subsequence $\{x^{n_k}\}$ of $\{x^k\}$ such that $\{x^{n_k}\}$ converges weakly to a point $\tilde{p}\in\mathcal{H}_0$ as $k\to\infty$. First, we prove that $\tilde{p}\in\Gamma$. Let $t_1^k, t_2^k, \mathcal{N}_{t_1^k}^k$, and $\mathcal{M}_{t_2^k}^k$ be as the above. Replacing k and t, respectively, by n_k and t_1^k, t_2^k in (27), we obtain

$$\lim_{k\to\infty}\|(U_{t_1^k}^{n_k}-I)x^{n_k}\|=0,\ \lim_{k\to\infty}\|(V_{t_2^k}^{n_k}-I)x^{n_k}\|=0. \tag{28}$$

In the case that U_t^k and V_t^k are given by (14), using (8) with $T=U_{t_1^k}^{n_k}$ and $T=V_{t_2^k}^{n_k}$, we get

$$\begin{aligned}&\frac{\theta\rho_1}{2R}\|(U_i-I)x^{n_k}\|^2\le\|(U_{t_1^k}^{n_k}-I)x^{n_k}\|^2,\\&\frac{\theta\rho_2}{2R}\|(\mathcal{L}_{\sigma_j}(\{V_j\}-I)x^{n_k}\|^2\le\|(V_{t_1^k}^{n_k}-I)x^{n_k}\|^2,\end{aligned} \tag{29}$$

where R is some positive constant, because $\{x^{n_k}\}$ is bounded. Then, from (28) and (29) it follows that

$$\lim_{k\to\infty}\|(U_i-I)x^{n_k}\|=0,\ \lim_{k\to\infty}\|(\mathcal{L}_{\sigma_j}\{V_j\}-I)x^{n_k}\|=0. \tag{30}$$

The latter limit and (12) yield $\lim_{k\to\infty}\|(V_j-I)A_jx^{n_k}\|=0$. Since A_j is a bounded linear operator, the property of $\{x^{n_k}\}$ and the demiclosedness property of U_i, V_j we have $\tilde{p}\in\mathrm{Fix}\,(U_i)$ and $A_j\tilde{p}\in\mathrm{Fix}\,(V_j)$ for each $i\in\mathcal{N}$ and $j\in\mathcal{M}$. It means that $\tilde{p}\in\Gamma$. Similarly, any weak cluster point of $\{x^k\}$ belongs to Γ. Therefore,

$$\limsup_{k\to\infty}\langle Fp_*, p_*-x^k\rangle\le 0.$$

By using this limit, the following one,

$$\|x^{k+1}-p_*\|^2\le(1-\theta_k\alpha_k\beta_k)\|x^k-p_*\|^2+2\alpha_k\big(\langle Fp_*,p_*-x^k\rangle+\alpha_k\|Fp_*\|M_1\big)$$

obtained by Lemma 3.3 with $p=p_*$, and Lemma **??** with properties of $\theta_k\alpha_k\beta_k$, we get $\lim_{k\to\infty}\|x^k-p_*\|=0$.

Now, we consider the case when U_t^k and V_t^k are given by (16). By the similar argument as in the proof of Lemma 3.2 we have

$$\begin{aligned}\lim_{k\to\infty}\big[&\|(U_iU_{i_{l-1}^k(t_1^k)}...U_{i_1^k(t_1^k)}-U_{i_{l-1}^k(t_1^k)}...U_{i_{1-2}^k(t_1^k)})x^k\|^2+\\&\|(U_{i_{l-2}^k(t_1^k)}...U_{i_1^k(t_1^k)}-U_{i_{l-3}^k(t_1^k)}...U_{i_1^k(t_1^k)})x^k\|^2+...+\\&\|(U_{i_2^k(t_1^k)}U_{i_1^k(t_1^k)}-U_{i_1^k(t_1^k)})x^k\|^2+\|(U_{i_1^k(t_1^k)}-I)x^k\|^2\big]=0\end{aligned}$$

and

$$\lim_{k\to\infty}\big[\|(\mathcal{L}_{\sigma_j}-\mathcal{L}^{\sigma_{j^k_{l'-1}}(t^k_2)})x^k\|^2+\|(\mathcal{L}^{\sigma_{j^k_{l'-1}}(t^k_2)}-\mathcal{L}^{\sigma_{j^k_{l'-2}}(t^k_2)})x^k\|^2+\dots$$
$$+\|(\mathcal{L}^{\sigma_{j^k_2}(t^k_2)}-\mathcal{L}^{\sigma_{j^k_1}(t^k_2)})x^k\|^2+\|(\mathcal{L}^{\sigma_{j^k_1}(t^k_2)}-I)x^k\|^2\big]=0.$$

Therefore, (30) is satisfied. Next, by the same argument as above we obtain the same result as for U^k_t, V^k_t defined by (14). Clearly, when U^k_t and V^k_t are given by (19) or (20), we also obtain the same result.

Case 2. There exists a subsequence $\{l_k\}\subset\{k\}$ such that

$$\|x^{l_k}-p_*\|\le\|x^{l_k+1}-p_*\|\ \forall k\ge\tilde{k}.$$

Then, by Lemma **??**, there exists a non-decreasing sequence $\{n_k\}\subseteq\{k\}$ such that $n_k\to\infty$,

$$\|x^{n_k}-p_*\|\le\|x^{n_k+1}-p_*\|,\qquad \|x^k-p_*\|\le\|x^{n_k+1}-p_*\| \tag{31}$$

for each $k\ge\tilde{k}$. So, from (24) with k replaced by n_k and the first inequality in (31), it follows that

$$\|x^{n_k}-p_*\|^2\le\frac{4}{\theta\eta}\big[\langle Fp_*,p_*-x^{n_k}\rangle+\alpha_{n_k}\|Fp\|M_1\big] \tag{32}$$

and $d^{n_k}\le d_{n_k}\alpha_{n_k}$ for sufficiently $n_k\ge\tilde{k}$. Hence, $\lim_{k\to\infty}d^{n_k}=0$ by the properties of d_{n_k} and α_{n_k}. By the same argument as in the proof of the *Case 1*, we obtain (30). Evidently, there exists a subsequence $\{x^{m_k}\}$ of $\{x^{n_k}\}$ such that $\{x^{m_k}\}$ converges weakly to a point $\underline{p}\in\mathcal{H}_0$ as $k\to\infty$. With each $i\in\mathcal{N}$ and $j\in\mathcal{M}$, for each $k\ge0$ let $t^k_1, t^k_2, \mathcal{N}^k_{t^k_1}$, and $\mathcal{M}^k_{t^k_2}$ be defined as the above. Next, again, by using the same argument as in the proof for the *Case 1* above, $\underline{p}\in\Gamma$ and any weak cluster point of $\{x^{m_k}\}$ belongs to Γ. Thus,

$$\limsup_{k\to\infty}\langle Fp_*,p_*-x^{n_k}\rangle\le0.$$

This, (32), and the property of α_k deduce $\lim_{k\to\infty}\|x^{n_k}-p_*\|=0$. Further, from Lemma **??** with k and p replaced, respectively, by n_k and p_*, we get

$$\|x^{n_k+1}-p_*\|^2\le\|x^{n_k}-p_*\|^2+2\alpha_{n_k}\|Fp_*\|\big(r'+\alpha_{n_k}M_1\big).$$

This, the last limit, and the property of α_{n_k} yield $\lim_{k\to\infty}\|x^{n_k+1}-p_*\|^2=0$, from which and the second inequality in (31) we have $\lim_{k\to\infty}\|x^k-p_*\|=0$. The proof is completed.

Remarks

1. If we take $F=I$, then F is η-strongly monotone and l-Lipschitz continuous mapping on $\mathcal{H}_0$ with $\eta\le1$ and $l\ge1$. Then, we obtain

$$x^{k+1}=(1-\xi_k)(1-t_k)x^k+\xi_k x^{k,s}, \tag{33}$$

since $\mu_k = 1$ for all $k \geq 0$. Method (33) is an improved modification of Krasnoselski-Mann [19, 21] iterative method for the considered problem.
2. Let α be any positive real number in $(0, 1)$. Take $F = (1-\alpha)I - (1-\alpha)u$, where u is a fixed point in $\mathcal{H}_0$. Clearly, F is η-strongly monotone and l-Lipschitz continuous operator on $\mathcal{H}_0$ with $l \geq \eta > 0$. Then, from (11) we deduce that

$$x^{k+1} = (1-\theta_k)[(1-t'_k)x^k + t'_k u] + \theta_k x^{k,s},$$

where $t'_k = (1-\alpha)t_k$. This method is a combination of the Krasnoselski-Mann iterative method with the Halpern one [18].
3. The results above have still value, if (12) is replaced by either

$$x^{k+1} = (1-\theta_k)x^k + \theta_k(I - t_k\mu_k F)x^{k,s} \tag{34}$$

with $\theta_k \in [\theta, 1]$ or

$$\begin{aligned} x^{k+1} &= (1-\theta_k)(I - t_k\mu_k F)x^{k,s} + \theta_k x^{k,s} \\ &= (I - \tilde{\theta}_k t_k \mu_k F)x^{k,s}, \ \tilde{\theta}_k = 1 - \theta_k, \end{aligned}$$

with $\theta_k \in [\theta, 1-\theta]$, where

$$\mu_k = \frac{\langle Fx^{k,s} - Fx^{k-1,s}, x^{k,s} - x^{k-1,s}\rangle}{\|Fx^{k.s} - Fx^{k-1,s}\|^2},$$

if $x^{k,s} \neq x^{k-1,s}$, μ_{k-1}, otherwise, and $x^{-1,s} = x^{-1}$ and $x^{0,s} = x^0$. Taking $\theta_k = 1$ and $F = I$, (34) becomes a simple iterative method $x^{k+1} = (1-t_k)x^{k,s}$.
4. The approach in this paper can be extended to a generation of (1), that is to find a point

$$p \in C := \cap_{i\in\mathcal{N}}\mathrm{Fix}(U_i), \ A_j p \in \cap_{l\in\mathcal{M}_{l,j}}\mathrm{Fix}(V_{l,j}), \forall j \in \mathcal{M},$$

where $V_{l,j}$ is $\rho_{l,j}$-strongly quasi-nonexpansive operator on $\mathcal{H}_j$ with $\rho_{l,j} > 0$ for every $l \in \mathcal{M}_{l,j} = \{1, ..., q_{l,j}\}$ with integers $q_{l,j} \geq 1$.
5. Some similar split problems with iterative methods have been recently considered in [7, 6, 5].

4 An application to the MSSFP given by level sets and computational experiments

Let φ_i and ψ_j be boundedly Lipschitz continuous and convex functions on real Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_j$, respectively, for $i \in \mathcal{N}$ and $j \in \mathcal{M}$. We consider the MSSFP: Find a point

$$\begin{aligned} &p \in C = \{p \in \mathcal{H}_0 : \varphi_i(p) \leq 0\} \ \forall i \in \mathcal{N}, \\ &A_j p \in \{v \in \mathcal{H}_j : \psi_j(v) \leq 0\} \ \forall j \in \mathcal{M} \end{aligned} \tag{35}$$

where A_j is a bounded linear operator from $\mathcal{H}_0$ into $\mathcal{H}_j$, for all $j \in \mathcal{M}$. The solution set of (35) is denoted by Γ.

Let $\partial f(x) = \{g \in \mathcal{H} : f(y) - f(x) \geq \langle g, y - x\rangle, \text{ for all } y \in \mathcal{H}\}$, the subdifferential of a continuous convex function f on a Hilbert space $\mathcal{H}$. Since f is continuous, the set $\partial f(x) \neq \emptyset$ (see, [3]). For an $\eta(x) \in \partial f(x)$, the operator P_f, so-called subgradient projection relative to f, is defined by

$$P_f x := x - \frac{[f(x)]_+}{\|\eta(x)\|^2}\eta(x)$$

if $\eta(x) \neq 0$; x, otherwise. where $[t]_+ = \max\{0, t\}$, has the properties: $\mathrm{Fix}\,(P_f) = \{p \in \mathcal{H} : f(p) \leq 0\}$ and P_f is 1-strongly quasi-nonexpansive, which satisfies the demiclosedness principle (see, [10] Theorem 4.2.7 and Corollary 4.2.6). Thus, p_* is a solution of (35) if and only if it belongs to Γ and solves (1) with $U_i = P_{\varphi_i}$ and $V_j = P_{\psi_j}$.

For computations, we take $\mathcal{H}_0 = \mathbf{E}^4, \mathcal{H}_j = \mathbf{E}^{j+1}$, Euclidean spaces, with $j \in \mathcal{M} = \{1, 2, 3, 4\}$.

$$\begin{aligned}
&\varphi_i(x) = \|(I - P_{\tilde{C}_i})x\|^2/2, \tilde{C}_i = \{x \in \mathbf{E}^4 : \|x - a^i\|^2 \leq 1\}, a^i \in \mathbf{E}^4,\\
&\psi_j(y) = \|(I - P_{\tilde{Q}_j})y\|^2/2, \tilde{Q}_j = \{y \in \mathbf{E}^j : \|y - b^j\|^2 \leq 1\}, b^j \in \mathbf{E}^j,\\
&a^i = (1 - i/10, 0, 0, 0),\ b^j = (1, 0, ..., 0),
\end{aligned}$$

for all $i \in \mathcal{N} = \{1, ..., 6\}$ and for all $j \in \mathcal{M}$ with a matrix A_1, having two rows $r_1 = (1, 2, 1, 2)$ and $r_2 = (2, 4, 2, 4)$, a matrix A_2, having three rows $r_1 = (1, 2, 0, 0)$, $r_2 = (2, 4, 0, 0)$ and $r_3 = (0, 0, 1, 0)$, a matrix A_3, having four rows $r_1 = (2, 1, 0, 0), r_2 = (0, 2, 1, 0)$, $r_3 = (0, 0, 2, 1), r_4 = (0, 0, 4, 2)$, and a matrix A_4, having fine rows $r_1 = (2, 1, 0, 0), r_2 = (0, 2, 1, 0)$, $r_3 = (0, 0, 2, 1), r_4 = (0, 0, 4, 2)$, $r_5 = (0, 0, 0, 1)$. It is easy to see that $\cap_{i \in L}\tilde{C}_i \neq \emptyset$ and the considered example has the unique minimum norm solution $p_* = (0, 0, 0, 0)$.

Clearly, $\partial\varphi_i(x) = \varphi_i'(x) = x - P_{\tilde{C}_i}x$, that has the form (see, [10]): $P_{\tilde{C}_i}x = a^i + (1/\|x - a^i\|)(x - a^i)$ if $\|x - a^i\| > 1$ and x otherwise. Then, $P_{\varphi_i}x = (x + P_{\tilde{C}_i}x)/2$ if $x \notin \tilde{C}_i$; otherwise, $P_{\varphi_i}x = x$ and P_{ψ_j} is similarly defined in $\mathbf{E}^{j+1}$. For computations, by taking $s = 2$, $\mathcal{N} = \mathcal{N}_1^k \cup \mathcal{N}_2^k$ and $\mathcal{M} = \mathcal{M}_1^k \cup \mathcal{M}_2^k$ with $\mathcal{N}_1^k = \{1, 2, 3\}$, $\mathcal{N}_2^k = \{4, 5, 6\}$, $\mathcal{M}_1^k = \{1, 2\}$, $\mathcal{M}_2^k = \{3, 4\}$, $\sigma_{j^k(t)}(x) = \tau_{j^k(t)}(x)$, and the starting point $x^0 = (3, 2, 5, 4)$, and using method (33), we have the following scheme,

$$\begin{aligned}
&x^{k,t} = U_t^k y^{k,t} y^{k,t} = V_t^k x^{k,t-1},\ t = 1, 2,\\
&x^{k+1} = (1 - \xi_k)(1 - t_k)x^k + \xi_k x^{k,2}.
\end{aligned} \tag{36}$$

where $\tau_{j^k(t)}(x)$ is defined in the block-iterative scheme. Computational results by (36) with (14), (15), (19), and (20), $t_k = 1/(k + 1)$, $\lambda_{i^k(t)}^k = 1/3$, and $\xi_k = \theta_{j^k(t)}^k = 1/2$ are given in the following tables.

Table 1: Result by (36) and (14).

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}	x_4^{k+1}
10	0.0088308449	0.0152433682	0.0454595158	0.0246655582
20	0.0004772087	0.0008681698	0.0027052801	0.0014691992
30	0.0000327203	0.0000595935	0.0001860741	0.0001010072
40	0.0000024101	0.0000043897	0.0000137075	0.0000074410
50	0.0000001846	0.0000003361	0.0000010497	0.0000005698

Table 2: Result by (36) and (16).

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}	x_4^{k+1}
10	0.0100743460	0.0151863428	0.0230477441	0.0110144905
20	0.0004430946	0.0006780287	0.0010239829	0.0004835695
30	0.0000231565	0.0000354934	0.0000535205	0.0000252499
40	0.0000012387	0.0000019907	0.0000030017	0.0000014155
50	0.0000000758	0.0000001161	0.0000001751	0.0000000826

Table 3: Result by (36) and (19).

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}	x_4^{k+1}
10	0.0044369738	0.0087323594	0.0387153882	0.0248808418
20	0.0002209196	0.0004826718	0.0022353282	0.0014344862
30	0.0000147412	0.0000328535	0.0001495178	0.0009595348
40	0.0000010566	0.0000023118	0.0000107169	0.0000068776
50	0.0000000787	0.0000001722	0.0000007986	0.0000005125

Table 4: Result by (36) and (20).

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}	x_4^{k+1}
10	0.0091396752	0.0147419320	0.0304096230	0.0155142279
20	0.0003689121	0.0006096060	0.0012723778	0.0064546818
30	0.0000180284	0.0000298010	0.0000622267	0.0000315668
40	0.0000009468	0.0000015352	0.0000032682	0.0000016579
50	0.0000000518	0.0000000856	0.0000001787	0.0000000906

The tables of numerical results show the effectiveness of the introduced iterative schemes.

5 Conclusion

In this paper, using the extrapolated Landweber-type operators, we proposed new strongly convergent iterative schemes for solving the the split feasibility problem with multiple output sets, governed by the fixed point sets of demi-closed strongly quasi-nonexpansive operators on infinite-dimensional Hilbert spaces. The strong convergence has been proved without the closedness property the range of A_j as well as the bounded regular one of the family of fixed point sets of the operators, assumed recently in the literature for the similar problems. An application to the MSSFP with multiple output level sets was given with numerical experiments.

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