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SOME RESULTS ON THE FIFTH SINGER TRANSFER

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Abstract

We study the algebraic transfer constructed by Singer [16] using technique of the *hit problem*. In this paper, we show that Singer's conjecture for the algebraic transfer is true in the case of five variables and degree $r.2^s - 5$ with $r = 3, 4$ and s an arbitrary positive integer.

1 Introduction

Let V_k be an elementary abelian 2-group of rank k. Denote by BV_k the classifying space of V_k. It is well-known that

$$P_k := H^*(BV_k) \cong \mathbb{F}_2[x_1, x_2, \ldots, x_k],$$

a polynomial algebra in k variables $x_1, x_2, \ldots, x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements. Then, P_k is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is determined by the elementary properties of the Steenrod squares Sq^i and subject to the Cartan formula (see Steenrod and Epstein [18]).

Let GL_k be the general linear group over the field $\mathbb{F}_2$. This group acts naturally on P_k by matrix substitution. Since the two actions of $\mathcal{A}$ and GL_k upon P_k commute with each other, there is an inherited action of GL_k on $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

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Denote by $(P_k)_n$ the subspace of P_k consisting of all the homogeneous polynomials of degree n in P_k and by $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n$ the subspace of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ consisting of all the classes represented by the elements in $(P_k)_n$. In [16], Singer defined the algebraic transfer, which is a homomorphism

$$\varphi_k : \mathrm{Tor}^{\mathcal{A}}_{k,k+n}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)^{GL_k}_n$$

from the homology of the mod-2 Steenrod algebra to the subspace of $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n$ consisting of all the GL_k-invariant classes.

The Singer algebraic transfer was studied by many authors. (See Boardman [1], Bruner-Ha-Hung [2], Ha [7], Hung [8, 9], Chon-Ha [4, 5, 6], Minami [13], Nam [14], Hung-Quynh [10], Quynh [15], the first author [21] and others).

Singer showed in [16] that φ_k is an isomorphism for $k = 1, 2$. Boardman showed in [1] that φ_3 is also an isomorphism. However, for any $k \geqslant 4$, φ_k is not a monomorphism in infinitely many degrees (see Singer [16], Hung [9]). Singer made the following conjecture.

Conjecture 1.1 (Singer [16]). *The algebraic transfer φ_k is an epimorphism for any $k \geqslant 0$.*

The conjecture is true for $k \leqslant 3$. Based on the results in [19, 20], it can be verified for $k = 4$. We hope that it is also true in this case.

The purpose of the paper is to verify this conjecture for $k = 5$. The following is the main result of the paper.

Theorem 1.2. *Singer's conjecture is true for $k = 5$ and $n = r.2^s - 5$ with $r = 3, 4$ and s an arbitrary positive integer.*

We prove this theorem by studying the $\mathbb{F}_2$-vector space $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)^{GL_5}$. Based on the results in [23, 24], we have the following.

Theorem 1.3. *Let n be as in Theorem 1.2. Then, we have $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)^{GL_5}_n = 0$.*

Obviously, Theorem 1.3 implies Theorem 1.2. Note that for $r = 4$ and $s = 2$, the above results are due to Quynh [15].

Furthermore, from the results of Tangora [22], Lin [12] and Chen [3], for $r = 3$, $\mathrm{Ext}^{5,3.2^s}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) = 0$. By passing to the dual, one gets $\mathrm{Tor}^{\mathcal{A}}_{5,3.2^s}(\mathbb{F}_2, \mathbb{F}_2) = 0$. Hence, by Theorem 1.3, the homomorphism

$$\varphi_5 : \mathrm{Tor}^{\mathcal{A}}_{5,3.2^s}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)^{GL_5}_{3.2^s-5}$$

is an isomorphism. For $r = 4$,

$$\mathrm{Ext}^{5,4.2^s}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle P(h_2) \rangle, & \text{if} s = 2, \\ 0, & \text{otherwise.} \end{cases}$$

By passing to the dual, we obtain

$$\mathrm{Tor}^{\mathcal{A}}_{5,4.2^s}(\mathbb{F}_2,\mathbb{F}_2)=\begin{cases}\langle P(h_2)^*\rangle, & \text{if} s=2,\\ 0, & \text{otherwise.}\end{cases}$$

So, by Theorem 1.3, the homomorphism

$$\varphi_5:\mathrm{Tor}^{\mathcal{A}}_{5,4.2^s}(\mathbb{F}_2,\mathbb{F}_2)\to(\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)^{GL_5}_{4.2^s-5}$$

is an epimorphism. However, it is not a monomorphism for $s=2$.

In the remaining part of the paper we prove Theorem 1.3.

2 Preliminaries

In this section, we recall a result from Singer [17] which will be used in the next section.

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means

$$a=\alpha_0(a)2^0+\alpha_1(a)2^1+\alpha_2(a)2^2+\ldots,$$

for $\alpha_i(a)=0,1$ and $i\geqslant 0$.

Definition 2.1. For a monomial $x=x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}\in P_k$, we define two sequences associated with x by

$$\omega(x)=(\omega_1(x),\omega_2(x),\ldots,\omega_i(x),\ldots),\ \sigma(x)=(a_1,a_2,\ldots,a_k),$$

where $\omega_i(x)=\sum_{1\leqslant j\leqslant k}\alpha_{i-1}(a_j),\ i\geqslant 1$. The sequence $\omega(x)$ is called the weight vector of x.

Let $\omega=(\omega_1,\omega_2,\ldots,\omega_i,\ldots)$ be a sequence of non-negative integers. The sequence ω is called the weight vector if $\omega_i=0$ for $i\gg 0$.

The sets of all the weight vectors and the sigma vectors are given the left lexicographical order.

For a weight vector ω, we define $\deg\omega=\sum_{i>0}2^{i-1}\omega_i$. Denote by $P_k(\omega)$ the subspace of P_k spanned by monomials y such that $\deg y=\deg\omega$, $\omega(y)\leqslant\omega$, and by $P_k^-(\omega)$ the subspace of P_k spanned by monomials $y\in P_k(\omega)$ such that $\omega(y)<\omega$.

Definition 2.2. Let ω be a weight vector of degree n and $f,g\in(P_k)_n$.

i) $f\equiv g$ if and only if $f-g\in\mathcal{A}^+P_k$. If $f\equiv 0$, then f is called hit.
ii) $f\equiv_\omega g$ if and only if $f-g\in\mathcal{A}^+P_k+P_k^-(\omega)$.

Obviously, the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Note that if ω is a minimal sequence of degree n, then $f \equiv_\omega g$ if and only if $f \equiv g$ (see Theorem 2.4.) Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$. Then, we have

$$QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

It is easy to see that

$$QP_k(\omega) \cong QP_k^\omega := \langle\{[x] \in QP_k : x \text{isadmissibleand} \omega(x) = \omega\}\rangle.$$

So, we get

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n = \bigoplus_{\deg \omega = n} QP_k^\omega \cong \bigoplus_{\deg \omega = n} QP_k(\omega).$$

Hence, we can identify the vector space $QP_k(\omega)$ with $QP_k^\omega \subset QP_k$.

We note that the weight vector of a monomial is invariant under the permutation of the generators x_i, hence $QP_k(\omega)$ has an action of the symmetric group Σ_k. Furthermore, $QP_k(\omega)$ is also an GL_k-module.

For polynomials $f \in P_k$ and $g \in P_k(\omega)$, we denote by $[f]$ the class in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ represented by f, and by $[g]_\omega$ the class in $QP_k(\omega)$ represented by g. For $M \subset P_k$ and $S \subset P_k(\omega)$, denote

$$[M] = \{[f] : f \in M\} \text{ and } [S]_\omega = \{[g]_\omega : g \in S\}.$$

If ω is the minimal sequence, then $[S]_\omega = [S]$ and $[g]_\omega = [g]$.

Definition 2.3. A monomial $z = x_1^{b_1} x_2^{b_2} \ldots x_k^{b_k}$ is called a spike if $b_j = 2^{s_j} - 1$ for s_j a non-negative integer and $j = 1, 2, \ldots, k$. If z is a spike with $s_1 > s_2 > \ldots > s_{r-1} \geqslant s_r > 0$ and $s_j = 0$ for $j > r$, then it is called a minimal spike.

For a positive integer n, by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1 \leqslant i \leqslant r}(2^{d_i} - 1)$, where $d_i > 0$. In [17], Singer showed that if $\mu(n) \leqslant k$, then there exists uniquely a minimal spike of degree n in P_k.

The following is a criterion for the hit monomials in P_k.

Theorem 2.4 (Singer [17]). *Suppose $x \in P_k$ is a monomial of degree n, where $\mu(n) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$, then x is hit.*

Definition 2.5. Let x, y be monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds

i) $\omega(x) < \omega(y)$;

ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.6. A monomial x is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$ for $j = 1, 2, \ldots, t$ and $x \equiv y_1 + y_2 + \ldots + y_t$.

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n.

The proof of the following lemma is elementary.

Lemma 2.7.

i) *All the spikes in P_k are admissible and their weight vectors are weakly decreasing.*

ii) *If a weight vector ω is weakly decreasing and $\omega_1 \leqslant k$, then there is a spike z in P_k such that $\omega(z) = \omega$.*

One of the main tools in the study of the hit problem is Kameko's homomorphism $\widetilde{Sq}^0_* : \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \to \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. This homomorphism is an GL_k-homomorphism induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_* : P_k \to P_k$, given by

$$\widetilde{Sq}^0_*(x) = \begin{cases} y, & \text{if} x = x_1 x_2 \ldots x_k y^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However,

$$\widetilde{Sq}^0_* Sq^{2t} = Sq^t \widetilde{Sq}^0_*, \ \ \widetilde{Sq}^0_* Sq^{2t+1} = 0$$

for any non-negative integer t.

Observe obviously that $\widetilde{Sq}^0_*$ is surjective on P_k and therefore on $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. So, one gets

$$\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{2m+k} = \dim \operatorname{Ker}(\widetilde{Sq}^0_*)_{(k,m)} + \dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_m,$$

for any positive integer m. Here

$$(\widetilde{Sq}^0_*)_{(k,m)} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{2m+k} \to (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_m$$

denotes Kameko's homomorphism $\widetilde{Sq}^0_*$ in degree $2m + k$.

Theorem 2.8 (Kameko [11]). *Let m be a positive integer. If $\mu(2m+k) = k$, then*

$$(\widetilde{Sq}^0_*)_{(k,m)} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{2m+k} \to (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_m$$

is an isomorphism of GL_k-modules.

For $1 \leqslant i \leqslant k$, define the $\mathcal{A}$-homomorphism $g_i : P_k \to P_k$, which is determined by $g_i(x_i) = x_{i+1}, g_i(x_{i+1}) = x_i$, $g_i(x_j) = x_j$ for $j \neq i, i+1$, $1 \leqslant i < k$, and $g_k(x_1) = x_1 + x_2$, $g_k(x_j) = x_j$ for $j > 1$. Note that the general linear group GL_k is generated by the matrices associated with g_i, $1 \leqslant i \leqslant k$, and the symmetric group Σ_k is generated by g_i, $1 \leqslant i < k$.

So, a homogeneous polynomial $f \in P_k$ is an GL_k-invariant if and only if $g_i(f) \equiv f$ for $1 \leqslant i \leqslant k$. If $g_i(f) \equiv f$ for $1 \leqslant i < k$, then f is an Σ_k-invariant.

3 Proof of Theorem 1.3

From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k.

For any monomials $z, z_1, z_2, \ldots, z_m$ in $(P_k)_n$ with $m \geqslant 1$, we denote

$$\begin{aligned}\Sigma_k(z_1, z_2, \ldots, z_m) &= \{\sigma z_t : \sigma \in \Sigma_k, 1 \leqslant t \leqslant m\} \subset (P_k)_n,\\ [B(z_1, z_2, \ldots, z_m)]_\omega &= [B_k(n)]_\omega \cap \langle [\Sigma_k(z_1, z_2, \ldots, z_m)]_\omega \rangle,\\ p(z) &= \sum_{y \in B_k(n) \cap \Sigma_k(z)} y.\end{aligned}$$

If ω is the minimal sequence of degree n, then we write

$$[B(z_1, z_2, \ldots, z_m)]_\omega = [B(z_1, z_2, \ldots, z_m)].$$

3.1 The case $r = 3$

For $r = 3$, we have $n = 2^{s+1} + 2^s - 5$. If $s > 3$, then $\mu(n) = 5$. Hence, using Theorem 2.8, we see that the iterated Kameko's homomorphism

$$(\widetilde{Sq}^0_*)^{s-3}_{(5,3.2^{s-1}-5)} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{2^{s+1}+2^s-5} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{19}$$

is an isomorphism of the GL_5-modules. So, we need only to prove the theorem for $s = 1, 2, 3$. For $s = 1$, we have $n = 1$. By a simple computation, one gets the following.

Proposition 3.1.1. $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_1 = 5$ *and* $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_1^{GL_5} = 0$.

For $s = 2$, we have $n = 7$.

Proposition 3.1.2. $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7^{GL_5} = 0$.

Since Kameko's homomorphism

$$(\widetilde{Sq}^0_*)_{(5,1)} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7 \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_1$$

is a homomorphism of GL_5-modules and $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_1^{GL_5} = 0$, we have

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7^{GL_5} \subset \mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,1)}.$$

From a result in [24], we see that $\dim(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,1)}) = 105$ with the basis $\bigcup_{i=1}^7 [B_5(u_i)]$, where

$$\begin{aligned}&u_1 = x_1^7,\ u_2 = x_1x_2^6,\ u_3 = x_1x_2^2x_3^4,\ u_4 = x_1x_2^3x_3^3,\\ &u_5 = x_1x_2^2x_3^2x_4^2,\ u_6 = x_1x_2x_3^2x_5^3,\ u_7 = x_1x_2x_3x_4^2x_5^2.\end{aligned}$$

By a routine computation we obtained the following.

Lemma 3.1.3.

i) *The subspaces* $\langle[\Sigma_5(u_i)]\rangle$, $1 \leqslant i \leqslant 4$, $\langle[\Sigma_5(u_5, u_6)]\rangle$ *and* $\langle[\Sigma_5(u_7)]\rangle$ *are* Σ_5*-submodules of* $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7$.

ii) *We have the direct summand decompositions of the* Σ_5*-modules:*

$$(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,1)} = \bigoplus_{i=1}^{4}\langle[\Sigma_5(u_i)]\rangle \bigoplus \langle[\Sigma_5(u_5, u_6)]\rangle \bigoplus \langle \Sigma_5[(u_7)]\rangle.$$

Lemma 3.1.4. $\langle[\Sigma_5(u_i)]\rangle^{\Sigma_5} = \langle[p(u_i)]\rangle$, $i = 1, 2, 3, 4$, $\langle[\Sigma_5(u_7)]\rangle^{\Sigma_5} = 0$ *and* $\langle[\Sigma_5(u_5, u_6)]\rangle^{\Sigma_5} = \langle[p(u_5]\rangle$.

Proof. We compute $\langle[\Sigma_5(u_i)]\rangle^{\Sigma_5}$ for $i = 3, 7$. The others can be proved by a similar computation.

Note that $\dim\langle[\Sigma_5(u_3)]\rangle = 10$ with a basis consisting of all the classes represented by the following admissible monomials:

$$a_1 = x_3x_4^2x_5^4,\ a_2 = x_2x_4^2x_5^4,\ a_3 = x_2x_3^2x_5^4,\ a_4 = x_2x_3^2x_4^4,\ a_5 = x_1x_4^2x_5^4,$$
$$a_6 = x_1x_3^2x_5^4,\ a_7 = x_1x_3^2x_4^4,\ a_8 = x_1x_2^2x_5^4,\ a_9 = x_1x_2^2x_4^4,\ a_{10} = x_1x_2^2x_3^4.$$

Suppose $p = \sum_{j=1}^{10}\gamma_ja_j$ and $[p] \in \langle[\Sigma_5(u_3)]\rangle^{\Sigma_5}$ with $\gamma_j \in \mathbb{F}_2$. By a direct computation, one gets

$$g_1(p) + p \equiv (\gamma_2 + \gamma_5)(a_2 + a_5) + (\gamma_3 + \gamma_6)(a_3 + a_6) + (\gamma_4 + \gamma_7)(a_4 + a_7) \equiv 0,$$
$$g_2(p) + p \equiv (\gamma_1 + \gamma_2)(a_1 + a_2) + (\gamma_6 + \gamma_8)(a_6 + a_8) + (\gamma_7 + \gamma_9)(a_7 + a_9) \equiv 0,$$
$$g_3(p) + p \equiv (\gamma_2 + \gamma_3)(a_2 + a_3) + (\gamma_5 + \gamma_6)(a_5 + a_6) + (\gamma_9 + \gamma_{10})(a_9 + a_{10}) \equiv 0,$$
$$g_4(p) + p \equiv (\gamma_3 + \gamma_4)(a_3 + a_4) + (\gamma_6 + \gamma_7)(a_6 + a_7) + (\gamma_8 + \gamma_9)(a_8 + a_9) \equiv 0.$$

These relations imply $\gamma_j = \gamma_1$, for $j = 2, 3, \ldots, 10$.

For $i = 7$, $\dim\langle[\Sigma_5(u_7)]\rangle = 5$, with a basis consisting of the classes represented by the following admissible monomials:

$$b_1 = x_1x_2x_3x_4^2x_5^2,\ b_2 = x_1x_2x_3^2x_4x_5^2,\ b_3 = x_1x_2x_3^2x_4^2x_5,$$
$$b_4 = x_1x_2^2x_3x_4x_5^2,\ b_5 = x_1x_2^2x_3x_4^2x_5.$$

If $q = \sum_{j=1}^{5}\gamma_j[b_j] \in \langle[\Sigma_5(u_7)]\rangle^{\Sigma_5}$ with $\gamma_j \in \mathbb{F}_2$, then

$$g_1(q) + q \equiv (\gamma_4 + \gamma_5)b_1 + \gamma_4b_2 + \gamma_5b_3 \equiv 0.$$

This implies $\gamma_4 = \gamma_5 = 0$. So, $q = \gamma_1b_1 + \gamma_2b_2 + \gamma_3b_3$. A simple computation shows

$$g_2(q) + q \equiv \gamma_2(b_2 + b_4) + \gamma_3)(b_3 + b_5) \equiv 0,$$
$$g_3(q) + q \equiv (\gamma_1 + \gamma_2)(b_1 + b_2) \equiv 0.$$

From the last equalities, we get $\gamma_1 = \gamma_2 = \gamma_3 = 0$. $\square$

Proof of Proposition 3.1.2. Let $f \in (P_5)_7$ such that $[f] \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7^{GL_5}$. Since $[f] \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7^{\Sigma_5}$, using Proposition 3.1.1, Lemmas 3.1.3 and 3.1.4, we have $f \equiv \sum_{j=1}^{5} \gamma_j p(u_j)$ with $\gamma_j \in \mathbb{F}_2$. By computing $g_5(f) + f$ in terms of the admissible monomials, we obtain

$$\begin{aligned} g_5(f) + f \equiv (\gamma_1 + \gamma_2)x_2^7 + (\gamma_2 + \gamma_3 + \gamma_5)x_2x_3^6 + (\gamma_3 + \gamma_4)x_2x_3^2x_4^4 \\ + \gamma_4 x_2x_3^2x_4^2x_5^2 + \gamma_5 x_1x_3^3x_3^3 + \text{otherterms} \equiv 0. \end{aligned}$$

This relation implies $\gamma_j = 0$ for $1 \leqslant j \leqslant 5$. The proposition is proved. $\square$

We now prove Theorem 1.3 for $r = 3$ and $s = 3$. Then, we have $n = 19$.

Since Kameko's homomorphism $(\widetilde{Sq}_*^0)_{(5,7)} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{19} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7$ is a homomorphism of GL_5-module and $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7^{GL_5} = 0$, we have

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_7^{GL_5} \subset \mathrm{Ker}(\widetilde{Sq}_*^0)_{(5,7)}.$$

From a result in [24], we see that $\dim(\mathrm{Ker}(\widetilde{Sq}_*^0)_{(5,7)}) = 802$ and

$$\mathrm{Ker}(\widetilde{Sq}_*^0)_{(5,7)} \cong QP_5(\omega) \bigoplus QP_5(\bar{\omega}) \bigoplus QP_5(\widetilde{\omega}).$$

Here $\omega = (3, 2, 1, 1)$, $\bar{\omega} = (3, 2, 3)$ and $\widetilde{\omega} = (3, 4, 2)$.

Proposition 3.1.5. $QP_5(\widetilde{\omega})^{GL_5} = 0$ *and* $QP_5(\bar{\omega})^{GL_5} = 0$.

According to a result in [24], $\dim(QP_5(\widetilde{\omega})) = 55$ with the basis $\bigcup_{j=1}^{3}[B_5(v_j)]_{\widetilde{\omega}}$, where

$$v_1 = x_1x_2^2x_3^2x_4^7x_5^7,\ v_2 = x_1x_2^2x_3^3x_4^6x_5^7,\ v_3 = x_1x_2^3x_3^3x_4^6x_5^6;$$

$\dim(QP_5(\bar{\omega})) = 47$ with the basis $\bigcup_{j=4}^{6}[B_5(v_j)]_{\bar{\omega}}$, where

$$v_4 = x_1x_2^2x_3^4x_4^5x_5^7,\ v_5 = x_1x_2^2x_3^3x_4^6x_5^7,\ v_6 = x_1^2x_2^3x_3^4x_4^5x_5^5.$$

By a simple computation using technique as given in the proof of Lemma 3.1.4, we obtain the following.

Lemma 3.1.6.

i) *The subspaces* $\langle[\Sigma_5(v_i)]_{\widetilde{\omega}}\rangle$, $i = 1, 2, 3$, *are* Σ_5*-submodules of* $QP_5(\widetilde{\omega})$; $\langle[\Sigma_5(v_4)]_{\bar{\omega}}\rangle$ *and* $\langle[\Sigma_5(v_5, v_6)]_{\bar{\omega}}\rangle$ *are* Σ_5*-submodules of* $QP_5(\bar{\omega})$.

ii) *We have the direct summand decompositions of the* Σ_5*-modules:*

$$\begin{aligned} QP_5(\widetilde{\omega}) &= \langle[\Sigma_5(v_1)]_{\widetilde{\omega}}\rangle \bigoplus \langle[\Sigma_5(v_2)]_{\widetilde{\omega}}\rangle \bigoplus \langle[\Sigma_5(v_3)]_{\widetilde{\omega}}\rangle, \\ QP_5(\bar{\omega}) &= \langle[\Sigma_5(v_4)]_{\bar{\omega}}\rangle \bigoplus \langle[\Sigma_5(v_5, v_6)]_{\bar{\omega}}\rangle. \end{aligned}$$

Lemma 3.1.7. *We have*

$$\langle[\Sigma_5(v_i)]_{\widetilde{\omega}}\rangle^{\Sigma_5} = \langle[p(v_i)]_{\widetilde{\omega}}\rangle,\ i = 1, 2, 3,$$
$$\langle[\Sigma_5(v_4)]_{\bar{\omega}}\rangle^{\Sigma_5} = \langle[p(v_4)]_{\bar{\omega}}\rangle,\ \langle[\Sigma_5(v_5, v_6)]_{\bar{\omega}}\rangle^{\Sigma_5} = 0.$$

Proof of Proposition 3.1.5. Let $p \in (P_5)_{19}$ such that $[p]_{\widetilde{\omega}} \in QP_5(\widetilde{\omega})^{GL_5}$. Since $[p]_{\widetilde{\omega}} \in QP_5(\widetilde{\omega})^{\Sigma_5}$, using Lemma 3.1.6, one gets $p \equiv_{\widetilde{\omega}} \sum_{j=1}^{3} \gamma_j p(v_j)$ with $\gamma_j \in \mathbb{F}_2$. By computing $g_5(p) + p$ in terms of the admissible monomials, we obtain

$$\begin{aligned} g_5(p) + p \equiv_{\widetilde{\omega}} (\gamma_1 + \gamma_2)x_1x_2^7x_3^2x_4^2x_5^7 + \gamma_2 x_1x_2^3x_3^2x_4^6x_5^7 \\ + \gamma_3 x_1x_3^3x_3^3x_4^6x_5^6 + \text{otherterms} \equiv_{\widetilde{\omega}} 0. \end{aligned}$$

The last equality implies $\gamma_1 = \gamma_2 = \gamma_3 = 0$.

Now, let $q \in (P_5)_{19}$ such that $[p]_{\bar{\omega}} \in QP_5(\bar{\omega})^{GL_5}$. Since $[p]_{\bar{\omega}} \in QP_5(\bar{\omega})^{\Sigma_5}$, using Lemma 3.1.6, we have $q \equiv_{\bar{\omega}} \gamma p(v_4)$ with $\gamma \in \mathbb{F}_2$. By a direct computation, we get

$$g_5(q) + q \equiv_{\bar{\omega}} \gamma x_1x_3^3x_3^4x_4^4x_5^7 + \text{otherterms} \equiv_{\bar{\omega}} 0.$$

From this relation it implies $\gamma = 0$. The proposition follows. $\square$

Using Propositions 3.1.2 and 3.1.5, we obtain $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{19}^{GL_5} = QP_5(\omega)^{GL_5}$. In the remain part of this subsection, we prove the following.

Proposition 3.1.8. $QP_5(\omega)^{GL_5} = 0$.

Based on the results in [24], we see that $\dim QP_5(\omega) = 700$ with the basis $\bigcup_{j=1}^{10}[B_5(w_j)]_{\omega}$, where

$$\begin{aligned} &w_1 = x_1x_2^3x_3^{15},\ w_2 = x_1x_2^7x_3^{11},\ w_3 = x_1^3x_2^7x_3^9,\ w_4 = x_1x_2x_3^2x_4^{15},\\ &w_5 = x_1x_2^3x_3^6x_4^9,\ w_6 = x_1x_2x_3^2x_4^4x_5^{11},\ w_7 = x_1x_2^2x_3^3x_4^{13},\\ &w_8 = x_1x_2x_3^2x_4^6x_5^9,\ w_9 = x_1x_2^3x_3^4x_4^{11},\ w_{10} = x_1x_2^2x_3^3x_4^5x_5^8. \end{aligned}$$

By a direct computation, using technique as given in the proof of Lemma 3.1.4, we obtain the following lemmas.

Lemma 3.1.9.

i) *The subspaces* $\langle[\Sigma_5(w_i)]\rangle, 1 \leqslant i \leqslant 6$, $\langle[\Sigma_5(w_7, w_9)]\rangle$ *and* $\langle[\Sigma_5(w_8, w_{10})]\rangle$ *are* Σ_5*-submodules of* $QP_5(\omega)$.

ii) *We have a direct summand decomposition of the* Σ_5*-modules:*

$$QP_5(\omega) = \bigoplus_{i=1}^{6}\langle[\Sigma_5(w_i)]\rangle \bigoplus \langle[\Sigma_5(w_7, w_9)]\rangle \bigoplus \langle[\Sigma_5(w_8, w_{10})]\rangle.$$

Lemma 3.1.10. *We have*

i) $\langle[\Sigma_5(w_i)]\rangle^{\Sigma_5} = \langle[p(u_i)]\rangle$, *for* $i = 1, 2$ *and* $\langle[\Sigma_5(w_4)]\rangle^{\Sigma_5} = \langle[\Sigma_5(w_6)]\rangle^{\Sigma_5} = 0$.

ii) $\langle[\Sigma_5(w_3)]\rangle^{\Sigma_5} = \langle[p_{(1,\omega)}]\rangle$, *where*

$$p_{(1,\omega)} = \sum_{1\leqslant i<j<t\leqslant 5}\left(x_i^3x_j^3x_t^{13} + x_i^3x_j^{13}x_t^3 + x_i^7x_j^3x_t^9 + x_i^7x_j^9x_t^3\right).$$

iii) $\langle[\Sigma_5(w_5)]\rangle^{\Sigma_5} = \langle[p_{(2,\omega)}]\rangle$, *where*

$$\begin{aligned}p_{(2,\omega)} = &\sum_{1\leqslant i<j<t<u\leqslant 5}\left(x_i^3x_jx_t^5x_u^{10} + x_i^3x_jx_t^6x_u^9 + x_i^3x_j^3x_t^4x_u^9 + x_i^3x_j^3x_t^5x_u^8\right)\\ &+ \sum_{1\leqslant i<j<t,u\leqslant 5}\left(x_ix_j^3x_t^3x_u^{12} + x_ix_j^6x_t^3x_u^9 + x_i^3x_j^4x_t^3x_u^9 + x_i^3x_j^5x_t^2x_u^9 + x_i^3x_j^5x_t^3x_u^8\right).\end{aligned}$$

iv) $\langle[\Sigma_5(w_7, w_9)]\rangle^{\Sigma_5} = \langle[p_{(3,\omega)} + p_{(4,\omega)}], [p_{(4,\omega)} + p_{(5,\omega)}]\rangle$, *where*

$$\begin{aligned}p_{(3,\omega)} &= \sum_{1\leqslant i<j,t,u\leqslant 5}\left(x_ix_jx_t^3x_u^{14} + x_i^7x_jx_t^3x_u^8\right),\\ p_{(4,\omega)} &= \sum_{1\leqslant i<j<t,u\leqslant 5}\left(x_i^3x_jx_tx_u^{14} + x_i^3x_j^{13}x_tx_u^2 + x_i^7x_jx_tx_u^{10} + x_i^7x_j^9x_tx_u^2\right),\\ p_{(5,\omega)} &= \sum_{1\leqslant i<j,t,u\leqslant 5;\ t<u}\left(x_ix_jx_t^6x_u^{11} + x_ix_jx_t^7x_u^{10} + x_i^3x_jx_t^4x_u^{11} + x_i^3x_jx_t^7x_u^8\right).\end{aligned}$$

v) $\langle[\Sigma_5(w_8, w_{10})]\rangle^{\Sigma_5} = \langle[p_{(6,\omega)}], [p_{(7,\omega)}]\rangle$, *where*

$$\begin{aligned}p_{(6,\omega)} = &\ x_1x_2x_3^6x_4x_5^{10} + x_1x_2x_3^6x_4^{10}x_5 + x_1x_2x_3^3x_4^{12}x_5^2 + x_1x_2x_3^2x_4^5x_5^{10}\\ &+ x_1x_2^2x_3x_4^5x_5^{10} + x_1x_2^2x_3x_4^6x_5^9 + x_1x_2x_3^6x_4^3x_5^8 + x_1x_2x_3^6x_4^8x_5^3\\ &+ x_1x_2^2x_3^5x_4^2x_5^9 + x_1x_2^2x_3^5x_4^9x_5^2 + x_1x_2x_3^2x_4^3x_5^{12} + x_1x_2x_3^2x_4^{12}x_5^3\\ &+ x_1x_2x_3^3x_4^2x_5^{12} + x_1x_2x_3^2x_4^6x_5^9 + x_1x_2x_3^6x_4^2x_5^9 + x_1x_2x_3^6x_4^9x_5^2\\ &+ x_1x_2^3x_3x_4^4x_5^{10} + x_1x_2^3x_3^4x_4x_5^{10} + x_1x_2^3x_3^4x_4^{10}x_5 + x_1x_2^3x_3x_4^6x_5^8\\ &+ x_1x_2^3x_3^6x_4x_5^8 + x_1x_2^3x_3^6x_4^8x_5 + x_1x_2^3x_3^3x_4^4x_5^8 + x_1x_2^3x_3^4x_4^3x_5^8\\ &+ x_1x_2^3x_3^4x_4^8x_5^3 + x_1^3x_2x_3^3x_4^4x_5^8 + x_1^3x_2x_3^4x_4^3x_5^8 + x_1^3x_2x_3^4x_4^8x_5^3\\ &+ x_1^3x_2^3x_3x_4^4x_5^8 + x_1^3x_2^3x_3^4x_4x_5^8 + x_1^3x_2^3x_3^4x_4^8x_5 + x_1x_2^2x_3x_4^3x_5^{12}\\ &+ x_1x_2^2x_3x_4^{12}x_5^3 + x_1x_2^2x_3^3x_4x_5^{12} + x_1x_2^2x_3^3x_4^{12}x_5 + x_1x_2^2x_3^{12}x_4x_5^3\\ &+ x_1x_2^2x_3^{12}x_4^3x_5 + x_1x_2^6x_3x_4^3x_5^8 + x_1x_2^6x_3x_4^8x_5^3 + x_1x_2^6x_3^3x_4x_5^8\\ &+ x_1x_2^6x_3^3x_4^8x_5 + x_1x_2^6x_3^8x_4x_5^3 + x_1x_2^6x_3^8x_4^3x_5 + x_1x_2^2x_3^3x_4^4x_5^9\\ &+ x_1x_2^2x_3^4x_4^3x_5^9 + x_1x_2^2x_3^4x_4^9x_5^3 + x_1x_2^2x_3^3x_4^5x_5^8 + x_1x_2^2x_3^5x_4^3x_5^8\\ &+ x_1x_2^2x_3^5x_4^8x_5^3 + x_1^3x_2^4x_3x_4^3x_5^8 + x_1^3x_2^4x_3x_4^8x_5^3 + x_1^3x_2^4x_3^3x_4x_5^8\\ &+ x_1^3x_2^4x_3^3x_4^8x_5 + x_1^3x_2^4x_3^8x_4x_5^3 + x_1^3x_2^4x_3^8x_4^3x_5.\end{aligned}$$

$$\begin{aligned}p_{(7,\omega)} &= x_1x_2x_3x_4^6x_5^{10} + x_1^3x_2x_3x_4^4x_5^{10} + x_1^3x_2x_3x_4^6x_5^8 + x_1x_2x_3^6x_4x_5^{10}\\ &+ x_1x_2x_3^6x_4^{10}x_5 + x_1x_2^6x_3x_4x_5^{10} + x_1x_2^6x_3x_4^{10}x_5 + x_1x_2^3x_3^{12}x_4x_5^2\\ &+ x_1x_2^3x_3^{12}x_4^2x_5 + x_1x_2^6x_3^9x_4x_5^2 + x_1x_2^6x_3^9x_4^2x_5 + x_1^3x_2x_3^4x_4x_5^{10}\\ &+ x_1^3x_2x_3^4x_4^{10}x_5 + x_1^3x_2^4x_3x_4x_5^{10} + x_1^3x_2^4x_3x_4^{10}x_5 + x_1x_2^3x_3^6x_4x_5^8\\ &+ x_1x_2^3x_3^6x_4^8x_5 + x_1^3x_2x_3^6x_4x_5^8 + x_1^3x_2x_3^6x_4^8x_5 + x_1^3x_2^4x_3^9x_4x_5^2\\ &+ x_1^3x_2^4x_3^9x_4^2x_5 + x_1x_2^3x_3^5x_4^2x_5^8 + x_1x_2^3x_3^5x_4^8x_5^2 + x_1^3x_2^5x_3x_4^2x_5^8\\ &+ x_1^3x_2^5x_3x_4^8x_5^2 + x_1^3x_2^5x_3^2x_4x_5^8 + x_1^3x_2^5x_3^2x_4^8x_5 + x_1^3x_2^5x_3^8x_4x_5^2\\ &+ x_1^3x_2^5x_3^8x_4^2x_5.\end{aligned}$$

Proof of Proposition 3.1.8. Let $f \in (P_5)_{19}$ such that $[f] \in QP_5(\omega)^{GL_5}$. Since $[f] \in QP_5(\omega)^{\Sigma_5}$, using Lemmas 3.1.9 and 3.1.10, we have

$$\begin{aligned}f &\equiv \gamma_1 p(u_1) + \gamma_2 p(u_2) + \gamma_3 p_{(1,\omega)} + \gamma_4 p_{(2,\omega)}\\ &\quad + \gamma_5(p_{(3,\omega)} + p_{(4,\omega)}) + \gamma_6(p_{(4,\omega)} + p_{(5,\omega)}) + \gamma_7 p_{(6,\omega)} + \gamma_8 p_{(7,\omega)},\end{aligned}$$

with $\gamma_j \in \mathbb{F}_2$. By computing $g_5(f) + f$ in terms of the admissible monomials, we obtain

$$\begin{aligned}g_5(f) + f &\equiv \gamma_1 x_1x_2^3x_3^{15} + \gamma_2 x_1x_2^7x_3^{11} + \gamma_3 x_1x_2x_3^3x_4^{14}\\ &+ \gamma_4 x_1x_2^3x_3^{12}x_4^3 + \gamma_5 x_1x_2^{14}x_3x_4^3 + \gamma_6 x_1x_2^7x_3x_4^{10}\\ &+ \gamma_7 x_1x_2^3x_3^3x_4^4x_5^8 + \gamma_8 x_1x_2^7x_3x_4^2x_5^8 + \text{otherterms} \equiv 0.\end{aligned}$$

This relation implies $\gamma_j = 0$ for $1 \leqslant j \leqslant 8$. The proposition is proved. □

Combining the above results, we get $(\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)^{GL_5}_{3.2^s-5} = 0$. So, Theorem 1.3 is proved for the case $r = 3$.

3.2 The case $r = 4$

For $r = 4$, we have $n = 2^{s+2} - 5$. If $s > 2$, then $\mu(2^{s+2} - 5) = 5$. Using Theorem 2.8, we see that the iterated Kameko's homomorphism

$$(\widetilde{Sq}^0_*)^{s-2}_{(5,2^{s+1}-5)} : (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_{2^{s+2}-5} \longrightarrow (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_{11}$$

is an isomorphism. So, we need only to prove the theorem for $s = 1, 2$. For $s = 1$, we have $n = 3$. By a simple computation, we obtain

Proposition 3.2.1. $\dim(\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_3 = 25$ *and* $(\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)^{GL_5}_3 = 0$.

For $s = 2$, we have $n = 11$. Since Kameko's homomorphism

$$(\widetilde{Sq}^0_*)_{(5,3)} : (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_{11} \longrightarrow (\mathbb{F}_2{\otimes}_{\mathcal{A}}P_5)_3$$

is a homomorphism of GL_5-module and $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_3^{GL_5} = 0$, we have

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}^{GL_5} \subset \mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,3)}.$$

From the results in [23], we see that

$$\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,3)} = QP_5(3,2,1) \bigoplus QP_5(3,4)$$

and $\dim QP_5(3,4) = 10$. By a direct computation, using the admissible monomial basis of $QP_5(3,4)$, we easily obtain the following.

Proposition 3.2.2. $QP_5(3,4)^{GL_5} = 0$.

Now, we compute $QP_5(3,2,1)^{GL_5}$. From the results in [23], we can see that $\dim QP_5(3,2,1) = 280$ with the basis $\bigcup_{i=1}^5 [B(\bar{u}_i)]$, where

$$\bar{u}_1 = x_1x_2^3x_3^7,\ \bar{u}_2 = x_1^3x_2^3x_3^5,\ \bar{u}_3 = x_1x_2x_3^2x_4^7,$$
$$\bar{u}_4 = x_1x_2^2x_3^3x_4^5,\ \bar{u}_5 = x_1x_2x_3^2x_4^3x_5^4.$$

A simple computation, using the results in [23], one gets the following.

Lemma 3.2.3.
i) *The subspaces* $\langle[\Sigma_5(\bar{u}_i)]\rangle,\ 1 \leqslant i \leqslant 5$, *are* Σ_5*-submodules of* $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$.
ii) *We have a direct summand decomposition of the* Σ_5*-modules:*

$$QP_5(3,2,1) = \bigoplus_{i=1}^{5} \langle[\Sigma_5(\bar{u}_i)]\rangle.$$

Lemma 3.2.4. *We have*

i) $\langle[\Sigma_5(\bar{u}_1)]\rangle^{\Sigma_5} = \langle[p(\bar{u}_1)]\rangle$, $\langle[\Sigma_5(\bar{u}_i)]\rangle^{\Sigma_5} = 0$ *for* $i = 2,3,5$.
ii) $\langle[\Sigma_5(\bar{u}_4)]\rangle^{\Sigma_5} = \langle[\bar{p}]\rangle$, *where*

$$\bar{p} = \sum_{1 \leqslant i < j,t,u \leqslant 5} \left(x_ix_jx_t^3x_u^6 + x_i^3x_jx_t^3x_u^4 \right).$$

Proof. We prove that $\langle[\Sigma_5(\bar{u}_2)]\rangle^{\Sigma_5} = 0$. The others can be proved by a similar computation.

From the result in [23], $\langle[\Sigma_5(\bar{u}_2)]\rangle$ is an $\mathbb{F}_2$-vector space of dimension 20 with a basis consisting of all the classes represented by the following admissible monomials:

$$\begin{array}{llll}
a_1 = x_3^3x_4^3x_5^5 & a_2 = x_3^3x_4^5x_5^3 & a_3 = x_2^3x_4^3x_5^5 & a_4 = x_2^3x_4^5x_5^3 \\
a_5 = x_2^3x_3^3x_5^5 & a_6 = x_2^3x_3^3x_4^5 & a_7 = x_2^3x_3^5x_5^3 & a_8 = x_2^3x_3^5x_4^3 \\
a_9 = x_1^3x_4^3x_5^5 & a_{10} = x_1^3x_4^5x_5^3 & a_{11} = x_1^3x_3^3x_5^5 & a_{12} = x_1^3x_3^3x_4^5 \\
a_{13} = x_1^3x_3^5x_5^3 & a_{14} = x_1^3x_3^5x_4^3 & a_{15} = x_1^3x_2^3x_5^5 & a_{16} = x_1^3x_2^3x_4^5 \\
a_{17} = x_1^3x_2^3x_3^5 & a_{18} = x_1^3x_2^5x_5^3 & a_{19} = x_1^3x_2^5x_4^3 & a_{20} = x_1^3x_2^5x_3^3.
\end{array}$$

Suppose that p is a polynomial such that $[p] \in \langle [\Sigma_5(\bar{u}_2)] \rangle^{\Sigma_5}$ and

$$p \equiv \sum_{1 \leqslant i \leqslant 20} \gamma_i a_i,$$

where $\gamma_i \in \mathbb{F}_2$, $1 \leqslant i \leqslant 20$. By a direct computation, we obtain

$$\begin{aligned}
g_1(p) + p \equiv\ & (\gamma_3 + \gamma_9)a_3 + (\gamma_4 + \gamma_{10})a_4 + (\gamma_5 + \gamma_{11})a_5 + (\gamma_6 + \gamma_{12})a_6 \\
& + (\gamma_7 + \gamma_{13})a_7 + (\gamma_8 + \gamma_{14})a_8 + (\gamma_3 + \gamma_9)a_9 + (\gamma_4 + \gamma_{10})a_{10} \\
& + (\gamma_5 + \gamma_{11})a_{11} + (\gamma_6 + \gamma_{12})a_{12} + (\gamma_7 + \gamma_{13})a_{13} \\
& + (\gamma_8 + \gamma_{14})a_{14} + \gamma_{18}a_{15} + \gamma_{19}a_{16} + \gamma_{20}a_{17} \equiv 0, \\
g_2(p) + p \equiv\ & (\gamma_1 + \gamma_3)a_1 + (\gamma_2 + \gamma_4)a_2 + (\gamma_1 + \gamma_3)a_3 + (\gamma_2 + \gamma_4)a_4 \\
& + \gamma_7 a_5 + \gamma_8 a_6 + (\gamma_{11} + \gamma_{15})a_{11} + (\gamma_{12} + \gamma_{16})a_{12} \\
& + (\gamma_{13} + \gamma_{18})a_{13} + (\gamma_{14} + \gamma_{19})a_{14} + (\gamma_{11} + \gamma_{15})a_{15} \\
& + (\gamma_{12} + \gamma_{16})a_{16} + (\gamma_{17} + \gamma_{20})a_{17} + (\gamma_{13} + \gamma_{18})a_{18} \\
& + (\gamma_{14} + \gamma_{19})a_{19} + (\gamma_{17} + \gamma_{20})a_{20} \equiv 0.
\end{aligned}$$

These relations imply $\gamma_i = 0$ for $i = 7, 8, 13, 14, 17, 18, 19, 20$. From this we get

$$\begin{aligned}
g_3(p) + p \equiv\ & \gamma_2 a_1 + (\gamma_3 + \gamma_5)a_3 + \gamma_4 a_4 + (\gamma_3 + \gamma_5)a_5 + \gamma_6 a_6 \\
& + \gamma_4 a_7 + \gamma_6 a_8 + (\gamma_9 + \gamma_{11})a_9 + \gamma_{10}a_{10} + (\gamma_9 + \gamma_{11})a_{11} \\
& + \gamma_{12}a_{12} + \gamma_{10}a_{13} + \gamma_{12}a_{14} + \gamma_{16}a_{16} + \gamma_{16}a_{17} \equiv 0, \\
g_4(p) + p \equiv\ & (\gamma_1 + \gamma_2)a_1 + (\gamma_1 + \gamma_2)a_2 + (\gamma_3 + \gamma_4)a_3 + (\gamma_3 + \gamma_4)a_4 \\
& + (\gamma_5 + \gamma_6)a_5 + (\gamma_5 + \gamma_6)a_6 + (\gamma_9 + \gamma_{10})a_9 \\
& + (\gamma_9 + \gamma_{10})a_{10} + (\gamma_{11} + \gamma_{12})a_{11} + (\gamma_{11} + \gamma_{12})a_{12} \\
& + (\gamma_{15} + \gamma_{16})a_{15} + (\gamma_{15} + \gamma_{16})a_{16} \equiv 0.
\end{aligned}$$

Combining the above equalities gives $\gamma_i = 0$ for $i = 1, 2, \ldots, 20$. $\square$

Proposition 3.2.5. $QP_5(3,2,1)^{GL_5} = 0$.

Proof. Let $h \in (P_5)_{11}$ such that $[h] \in QP_5(3,2,1)^{GL_5}$. Since $[h] \in QP_5(3,2,1)^{\Sigma_5}$, using Lemmas 3.2.3 and 3.2.4, we have

$$h \equiv \gamma_1 p(\bar{u}_1) + \gamma_2 \bar{p},$$

with $\gamma_1, \gamma_2 \in \mathbb{F}_2$. Computing $g_5(h) + h$ in terms of the admissible monomials, we obtain

$$g_5(h) + h \equiv \gamma_1 x_1 x_2^3 x_3^7 + \gamma_2 x_1 x_2 x_3^2 x_4^2 x_5^5 + \text{otherterms} \equiv 0.$$

This relation implies $\gamma_1 = \gamma_2 = 0$, hence $h = 0$. The proposition is proved. $\square$

From Propositions 3.2.1, 3.2.2 and 3.2.5, we get $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)^{GL_5}_{2^{s+2}-5} = 0$ for all $s \geqslant 1$. Theorem 1.3 is completely proved.

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