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POSITIVE WEAK SOLUTIONS OF THE STEKLOV PROBLEM FOR A CLASS OF QUASI LINEAR ELLIPTIC OPERATORS CONTAINING THE $p(\cdot)$- LAPLACIAN AND THE MEAN CURVATURE OPERATOR

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Abstract

In this paper, we consider the existence of positive weak solutions for a class of quasilinear elliptic operators containing $p(\cdot)$-Laplacian and mean curvature operator with the Steklov boundary condition. The results for $p(\cdot)$-Laplacian are known, however, we attempt in this paper to extend the results for a class of quasilinear elliptic operators containing not only $p(\cdot)$-Laplacian but also the mean curvature operator.

1 Introduction

In this paper, we consider the following problem with the Steklov boundary value condition

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x,\boldsymbol{\nabla} u(x))] + \lambda |u(x)|^{p(x)-2}u(x) = f(x,u(x)) & \text{in } \Omega, \\ \boldsymbol{n}(x)\cdot \boldsymbol{a}(x,\boldsymbol{\nabla} u(x)) = g(x,u(x)) & \text{on } \partial\Omega. \end{cases} \qquad (Q_\lambda)$$

Here Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{1,\alpha}$-boundary $\partial\Omega$ for some

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$\alpha \in (0,1)$, the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to $\partial\Omega$. The function $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. Here we say that $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$, if for a.e. $x \in \Omega$, the map $\mathbb{R}^N \ni \boldsymbol{\xi} \mapsto \boldsymbol{a}(x,\boldsymbol{\xi})$ is continuous and for every $\boldsymbol{\xi} \in \mathbb{R}^N$, the map $\Omega \ni x \mapsto \boldsymbol{a}(x,\boldsymbol{\xi})$ is measurable on Ω.

The operator $u \mapsto \operatorname{div}[\boldsymbol{a}(x,\boldsymbol{\nabla} u(x))]$ is more general than the $p(\cdot)$-Laplacian $\Delta_{p(x)}u(x) = \operatorname{div}\left[|\boldsymbol{\nabla} u(x)|^{p(x)-2}\boldsymbol{\nabla} u(x)\right]$ and the mean curvature operator $\operatorname{div}\left[(1+|\boldsymbol{\nabla} u(x)|^2)^{(p(x)-2)/2}\boldsymbol{\nabla} u(x)\right]$. This generality brings about difficulties and requires some conditions.

$\lambda \in \mathbb{R}$ is a parameter, $p \in C^1_+(\overline{\Omega}) := \{p \in C^1(\Omega,\mathbb{R}); \min_{x\in\overline{\Omega}} p(x) > 1\}$ and functions $f = f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$ and $g = g(x,t) \in C(\partial\Omega \times \mathbb{R})$ satisfying some conditions.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [27]), in electrorheological fluids (Diening [11], Halsey [19], Mihăilescu and Rădulescu [22], Růžička [23]).

When $p(x) \equiv p =$ const., Abreu, Ó and Medeiros [1] considered the existence, nonexistence and multiplicity of positive solutions for inhomogeneous boundary value problem of the type

$$\begin{cases} -\Delta_p u(x) + \lambda u(x)^{p-1} = u(x)^q & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ |\boldsymbol{\nabla} u(x)|^{p-2}\frac{\partial u}{\partial \boldsymbol{n}}(x) = \varphi(x) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $1 < p < N$, $p-1 < q \le p^* - 1$, $p^* = Np/(N-p)$. The proofs rely on different methods: lower and upper solutions and variational approach. Deng [9] extended the result of [1] to the Robin boundary problem of the $p(\cdot)$-Laplacian:

$$\begin{cases} -\Delta_{p(x)}u(x) = \lambda f(x,u(x)) & \text{in } \Omega, \\ |\boldsymbol{\nabla} u|^{p(x)-2}\frac{\partial u}{\partial \boldsymbol{n}} + \beta(x)|u(x)|^{p(x)-2}u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Deng and Wang [10] considered the following problem

$$\begin{cases} -\Delta_{p(x)}u(x) + \lambda|u(x)|^{p(x)-2}u(x) = f(x,u(x)) & \text{in } \Omega, \\ |\boldsymbol{\nabla} u(x)|^{p(x)-2}\frac{\partial u}{\partial \boldsymbol{n}}(x) = g(x,u(x)) & \text{on } \partial\Omega. \end{cases}$$

The authors obtained the results on nonexistence, and existence and multiplicity of positive solutions, using the sub-supersolution method for the Steklov problem, which is similar to the given in Fan [13] for the Dirichlet problem for the $p(\cdot)$-Laplacian.

Our attempt is to extend the result of [10] to a class of quasilinear elliptic operators containing not only $p(\cdot)$-Laplacian but also the mean curvature operator. The underlying idea of proving main theorems is similar to that of [10]. To prove the main theorems we have to revise known results to fit a class of operators and the Steklov boundary condition. To overcome this, we succeeded to extend the strong maximum principle to our class of operators in Aramaki [8]. This paper has made it possible to treat the problem in a variety of ways.

The paper is organized as follows. In Subsection 2.1, we recall some notions on variable exponent Lebesgue-Sobolev spaces, and in Subsection 2.2, we give some assumptions and state some properties from it. In Section 3, we give regularity results and sub-supersolution principle. In Section 4, we give main theorems (Theorems 4.2, 4.6, 4.7 and 4.11) and their proofs.

2 Preliminaries

In the present paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \dots, a_N)$ and $\boldsymbol{b} = (b_1, \dots, b_N)$ in $\mathbb{R}^N$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{N} a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*, B}$.

2.1 Variable exponent Lebesgue and Sobolev spaces

Throughout this subsection, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz-boundary $\partial\Omega$. We recall some well-known results on variable exponent Lebesgue and Sobolev spaces. See Kováčik and Rácosník [21], Fan and Zhang [16], Diening et al. [12] and references therein for more detail. Define $C(\overline{\Omega}) = \{p; p \text{ is a continuous function on } \overline{\Omega}\}$, and for any $p \in C(\overline{\Omega})$, put

$$p^+ = p^+(\Omega) = \sup_{x \in \Omega} p(x) \text{ and } p^- = p^-(\Omega) = \inf_{x \in \Omega} p(x).$$

For any $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and for any measurable function u on Ω, a modular $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \rho_{p(\cdot),\Omega}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \tau > 0; \rho_{p(\cdot)} \left(\frac{u}{\tau} \right) \leq 1 \right\}.$$

We also define the Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \||\nabla u|\|_{L^{p(\cdot)}(\Omega)}. \tag{2.1}$$

Throughout this paper, we define a set by

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}); p^- > 1\}.$$

The following three propositions are well known (see Fan and Zhao [18]).

Proposition 2.1. *Let $p \in C_+(\overline{\Omega})$. Then the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive and separable Banach spaces.*

Proposition 2.2 (generalized Hölder inequality)**.** *Let $p \in C_+(\overline{\Omega})$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\int_\Omega |u(x)v(x)|dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, for any $p \in C_+(\overline{\Omega})$, $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$, that is, $p'(x) = p(x)/(p(x)-1)$ for $x \in \overline{\Omega}$.

For $p \in C_+(\overline{\Omega})$, define for $x \in \overline{\Omega}$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases} \tag{2.2}$$

Proposition 2.3. *If $q(\cdot) \in C(\overline{\Omega})$ satisfies $1 \le q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.*

Next we consider the trace (cf. Fan [15]). Let Ω be a bounded domain of $\mathbb{R}^N$ with a Lipschitz-boundary $\partial\Omega$ and $p \in C(\overline{\Omega})$ with $p^- \ge 1$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}(\Omega)$, the trace $\gamma(u) = u\big|_{\partial\Omega}$ to $\partial\Omega$ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1(\partial\Omega)$. We define

$$(\mathrm{Tr}W^{1,p(\cdot)})(\partial\Omega) = \{f; f \text{ is the trace to } \partial\Omega \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\partial\Omega)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_{\partial\Omega} = f\}$$

for $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\partial\Omega)$. Then we can see that $(\mathrm{Tr}W^{1,p(\cdot)})(\partial\Omega)$ is a Banach space. In the later we also write $F\big|_{\partial\Omega} = g$ by $F = g$ on $\partial\Omega$.

Let $r \in C_+(\partial\Omega) := \{q \in C(\partial\Omega); q^- := \inf_{x\in\partial\Omega} q(x) > 1\}$. We define

$$L^{r(\cdot)}(\partial\Omega) = \Big\{u; u : \partial\Omega \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma_x \\ \text{satisfying } \int_{\partial\Omega} |u(x)|^{r(x)} d\sigma_x < \infty\Big\},$$

where $d\sigma_x$ is the surface measure induced by the Lebesgue measure dx and the norm is defined by

$$\|u\|_{L^{r(\cdot)}(\partial\Omega)} = \inf\left\{\lambda > 0; \int_{\partial\Omega} \left|\frac{u(x)}{\lambda}\right|^{r(x)} d\sigma_x \le 1\right\}.$$

For $p \in C_+(\overline{\Omega})$, define for $x \in \overline{\Omega}$,

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases} \tag{2.3}$$

The following proposition follows from Yao [25, Proposition 2.6].

Proposition 2.4 ([25])**.** *Let $p \in C_+(\overline{\Omega})$. Then if $r \in C(\partial\Omega)$ satisfies $1 \le r(x) < p^\partial(x)$ for all $x \in \partial\Omega$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{r(\cdot)}(\partial\Omega)$ is well-defined and compact.*

Now we introduce a new norm on $W^{1,p(\cdot)}(\Omega)$ which is used later. For $u \in W^{1,p(\cdot)}(\Omega)$, define

$$\rho_\lambda(u) = \rho_{\lambda,\Omega}(u) = \int_\Omega |\boldsymbol{\nabla} u(x)|^{p(x)} dx + \lambda \int_\Omega |u(x)|^{p(x)} dx$$

and

$$\|u\|_\lambda = \inf\left\{\tau > 0; \rho_\lambda\left(\frac{u}{\tau}\right) \le 1\right\}.$$

Clearly we can see that if $\lambda_1 \ge \lambda_2 > 0$, then $\|u\|_{\lambda_1} \ge \|u\|_{\lambda_2}$.

Then we have the following proposition (cf. [9, Theorem 2.1 and Proposition 2.4]).

Proposition 2.5 ([9])**.** *When $\lambda > 0$, $\|\cdot\|_\lambda$ is a norm on $W^{1,p(\cdot)}(\Omega)$ which is equivalent to the norm defined by (2.1) of $W^{1,p(\cdot)}(\Omega)$. For $u, u_n \in W^{1,p(\cdot)}(\Omega)$ $(n = 1, 2, \dots)$, we have the following properties.*

(i) $\|u\|_\lambda \ge 1 \Longrightarrow \|u\|_\lambda^{p^-} \le \rho_\lambda(u) \le \|u\|_\lambda^{p^+}$.

(ii) $\|u\|_\lambda < 1 \Longrightarrow \|u\|_\lambda^{p^+} \le \rho_\lambda(u) \le \|u\|_\lambda^{p^-}$.

(iii) $\|u_n - u\|_\lambda \to 0 \Longleftrightarrow \rho_\lambda(u_n - u) \to 0$ *as* $n \to \infty$.

(iv) $\|u_n\|_\lambda \to \infty \Longleftrightarrow \rho_\lambda(u_n) \to \infty$ *as* $n \to \infty$.

2.2 Assumptions and some properties

In this subsection, we state some assumptions and some properties. Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{1,\alpha}$-boundary $\partial\Omega$ and let $p \in C^1_+(\overline{\Omega}) := \{q \in C^1(\overline{\Omega}); q^- > 1\}$ be fixed. We assume that the following (A.1)-(A.5) hold.

(A.1) Let $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying that for a.e. $x \in \Omega$ the function $A(x,\cdot) : \mathbb{R}^N \ni \boldsymbol{\xi} \mapsto A(x,\boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi} \in \mathbb{R}^N$ the function $A(\cdot,\boldsymbol{\xi}) : \Omega \ni x \mapsto A(x,\boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x,\mathbf{0}) = 0$, $A(x,-\boldsymbol{\xi}) = A(x,\boldsymbol{\xi})$ for a.e. $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^N$, and put $\boldsymbol{a}(x,\boldsymbol{\xi}) = \boldsymbol{\nabla}_{\boldsymbol{\xi}} A(x,\boldsymbol{\xi})$. Then $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$. Assume that $\boldsymbol{a}$ is of the form

$$\boldsymbol{a}(x,\boldsymbol{\xi}) = \chi(x,\boldsymbol{\xi})\boldsymbol{\xi}, \tag{2.4}$$

where $0 < \chi(x,\boldsymbol{\xi}) \in C(\overline{\Omega} \times (\mathbb{R}^N \setminus \{\mathbf{0}\}))$, and there exists a function $h \in C^1(\overline{\Omega})$ with $\min_{x\in\overline{\Omega}} h(x) > 0$ such that

$$\boldsymbol{a}(x,\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \leq p(x)A(x,\boldsymbol{\xi}) + h(x) \text{ for a.e. } x \in \Omega \text{ and all } \boldsymbol{\xi} \in \mathbb{R}^N. \tag{2.5}$$

(A.2) Put $\boldsymbol{a}(x,\boldsymbol{\xi}) = (a_1(x,\boldsymbol{\xi}), \dots, a_N(x,\boldsymbol{\xi}))$. Then $a_i \in C^1(\Omega \times (\mathbb{R}^N \setminus \{\mathbf{0}\})) \cap C(\Omega \times \mathbb{R}^N)$ for $i = 1, \dots, N$, and $a_i(x,\mathbf{0}) = 0$ for $i = 1, \dots, N$.

(A.3) There exists a constant $\mu \in [0,1]$ such that

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \xi_j}(x,\boldsymbol{\xi})\eta_i\eta_j \geq \Gamma_1(\mu + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}|\boldsymbol{\eta}|^2$$

for all $x \in \Omega$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N \setminus \{\mathbf{0}\}, \boldsymbol{\eta} = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$, where Γ_1 is a positive constant.

(A.4) $\displaystyle\sum_{i,j=1}^{N} \left|\frac{\partial a_i}{\partial \xi_j}(x,\boldsymbol{\xi})\right| \leq \Gamma_2(\mu + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}$ for all $x \in \Omega$ and $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, where Γ_2 is a positive constant.

(A.5) $\displaystyle\sum_{i,j=1}^{N} \left|\frac{\partial a_i}{\partial x_j}(x,\boldsymbol{\xi})\right| \leq \Gamma_3(\mu + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}|\boldsymbol{\xi}|(1 + |\log(\mu + |\boldsymbol{\xi}|^2)|)$ for all $x \in \Omega$ and $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, where Γ_3 is a positive constant.

Example 2.6. Let $h \in C^1(\overline{\Omega})$ with $\min_{x\in\overline{\Omega}} h(x) > 0$.

(i) $A(x,\boldsymbol{\xi}) = \frac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)}$, so $\boldsymbol{a}(x,\boldsymbol{\xi}) = h(x)|\boldsymbol{\xi}|^{p(x)-2}\boldsymbol{\xi}$.

(ii) $A(x,\boldsymbol{\xi}) = \frac{h(x)}{p(x)}((1 + |\boldsymbol{\xi}|^2)^{p(x)/2} - 1)$, so $\boldsymbol{a}(x,\boldsymbol{\xi}) = h(x)(1 + |\boldsymbol{\xi}|^2)^{p(x)-2}\boldsymbol{\xi}$.

Then (i) and (ii) satisfy (A.1)-(A.5) with $\mu = 0$ and $\mu = 1$, respectively.

Remark 2.7. In Example 2.6, when $h(x) \equiv 1$, the operator $u \mapsto \operatorname{div}[\boldsymbol{a}(\cdot, \boldsymbol{\nabla} u(\cdot))]$ of (i) corresponds to the $p(\cdot)$-Laplacian and that of (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

Lemma 2.8 ([8])**.** *Under (A.1), (A.2) and (A.4), there exists a constant* $C > 0$ *such that*

$$|\boldsymbol{a}(x, \boldsymbol{\xi})| \le C(1 + |\boldsymbol{\xi}|^{p(x)-1}), \ \textit{and}\ A(x, \boldsymbol{\xi}) \le 2C(1 + |\boldsymbol{\xi}|^{p(x)})$$

for a.e. $x \in \Omega$ *and all* $\boldsymbol{\xi} \in \mathbb{R}^N$.

Lemma 2.9 ([8])**.** *Under (A.1) and (A.3), the following properties hold.*

(i) There exists a constant $c > 0$ *such that*

$$(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \ge \begin{cases} c|\boldsymbol{\xi} - \boldsymbol{\eta}|^{p(x)} & \textit{if } p(x) \ge 2, \\ c(1 + |\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{(p(x)-2)/2} |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 & \textit{if } p(x) < 2, \end{cases}$$

for all $x \in \Omega$ *and* $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$.

(ii) There exist constants $c_1, c_2 > 0$ *and* $C_1, C_2 \ge 0$ *such that*

$$\boldsymbol{a}(x, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \ge c_1 |\boldsymbol{\xi}|^{p(x)} - C_1, \tag{2.6}$$

$$A(x, \boldsymbol{\xi}) \ge \begin{cases} \frac{c}{p^+} |\boldsymbol{\xi}|^{p(x)} & \textit{if } p(x) \ge 2, \\ \frac{c}{2}(1 + |\boldsymbol{\xi}|)^{p(x)-2} |\boldsymbol{\xi}|^2 & \textit{if } p(x) < 2 \end{cases} \tag{2.7}$$

and

$$A(x, \boldsymbol{\xi}) \ge c_2 |\boldsymbol{\xi}|^{p(x)} - C_2. \tag{2.8}$$

Throughout this paper the following assumption holds.

(fg) (i) $f = f(x, t) \in C(\overline{\Omega} \times \mathbb{R})$ and $f(x, t) \ge 0$ for $x \in \overline{\Omega}$ and $t \ge 0$.

(ii) $g = g(x, t) \in C(\partial\Omega) \times \mathbb{R})$ and $g(x, t) \ge 0$ for $x \in \partial\Omega$ and $t \ge 0$ and $g(x, 0) \not\equiv 0$.

(iii) For any $x_1, x_2 \in \partial\Omega, t_1, t_2 \in \mathbb{R}$,

$$|g(x_1, t_1) - g(x_2, t_2)| \le \Lambda(\max\{|t_1|, |t_2|\})(|x_1 - x_2|^{\beta_1} + |t_1 - t_2|^{\beta_2}),$$

where $\Lambda : [0, \infty) \to (0, \infty)$ is a nondecreasing continuous function and $\beta_1, \beta_2 \in (0, 1)$.

Definition 2.10. (i) We call $u \in W^{1,p(\cdot)}(\Omega)$ a weak solution of the problem (Q_λ), if u satisfies that

$$\int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) \cdot \boldsymbol{\nabla} v(x) dx + \lambda \int_\Omega |u(x)|^{p(x)-2} u(x) v(x) dx \\ = \int_\Omega f(x, u(x)) v(x) dx + \int_{\partial\Omega} g(x, u(x)) v(x) d\sigma_x$$

for all $v \in W^{1,p(\cdot)}(\Omega)$.

(ii) We call $u \in W^{1,p(\cdot)}(\Omega)$ a generalized solution of the equation

$$-\mathrm{div}\,[\boldsymbol{a}(x, \boldsymbol{\nabla} u(x))] + \lambda |u(x)|^{p(x)-2} u(x) = f(x, u(x)) \text{ in } \Omega, \tag{2.9}$$

if u satisfies that

$$\int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) \cdot \boldsymbol{\nabla} v(x) dx + \lambda \int_\Omega |u(x)|^{p(x)-2} u(x) v(x) dx = \int_\Omega f(x, u(x)) v(x) dx$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$.

(iii) We call $u \in W^{1,p(\cdot)}(\Omega)$ a subsolution (resp. supersolution) of the problem (Q_λ), if u satisfies that

$$\int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) \cdot \boldsymbol{\nabla} v(x) dx + \lambda \int_\Omega |u(x)|^{p(x)-2} u(x) v(x) d\sigma_x \\ \leq (\text{resp. } \geq) \int_\Omega f(x, u(x)) v(x) dx + \int_{\partial\Omega} g(x, u(x)) v(x) d\sigma_x$$

for all $v \in W^{1,p(\cdot)}(\Omega)$ with $v \geq 0$.

Here and from now on, for $u, v \in L^1_{\mathrm{loc}}(\Omega)$, we call $u \leq v$ (resp. $u < v$) in Ω, if $u(x) \leq v(x)$ (resp. $u(x) < v(x)$) a.e. $x \in \Omega$.

Let

$$\Lambda = \{\lambda \in \mathbb{R}; \text{there exists at least one positive weak solution of problem } (Q_\lambda)\} \tag{2.10}$$

and

$$\lambda_* = \inf \Lambda. \tag{2.11}$$

3 Regularity and sub-supersolution principle

In this section, we consider the regularity of weak solutions of (Q_λ) and sub-supersolution principle.

Define a functional by

$$\psi_\lambda(u) = \int_\Omega A(x, \boldsymbol{\nabla} u(x))dx + \lambda \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} dx \text{ for } u \in W^{1,p(\cdot)}(\Omega). \quad (3.1)$$

We have the following proposition which fulfills an important role in this paper. In particular, (v) in the following proposition is firstly derived by [8, Proposition 3.5].

Proposition 3.1. *Under the hypotheses (A.1)-(A.5), for $\lambda > 0$, the functional ψ_λ has the following properties.*

- *(i)* *$\psi_\lambda \in C^1(W^{1,p(\cdot)}(\Omega), \mathbb{R})$, ψ_λ is an even functional, that is, $\psi_\lambda(-u) = \psi_\lambda(u)$ for any $u \in W^{1,p(\cdot)}(\Omega)$, and its Fréchet derivative ψ'_λ satisfies that, for $u, v \in W^{1,p(\cdot)}(\Omega)$,*

  $$\langle \psi'_\lambda(u), v\rangle = \int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) \cdot \boldsymbol{\nabla} v(x) dx + \lambda \int_\Omega |u(x)|^{p(x)-2} u(x) v(x) dx.$$

  Here and hereafter, we denote the duality $\langle \cdot, \cdot \rangle_{(W^{1,p(\cdot)}(\Omega))^*, W^{1,p(\cdot)}(\Omega)}$ by simply $\langle \cdot, \cdot \rangle$.

- *(ii)* *ψ_λ is coercive, that is, $\psi_\lambda(u) \to \infty$ as $\|u\|_\lambda \to \infty$.*
- *(iii)* *ψ_λ is sequentially weakly lower-semicontinuous in $W^{1,p(\cdot)}(\Omega)$.*
- *(iv)* *ψ_λ is bounded on every bounded subset of $W^{1,p(\cdot)}(\Omega)$.*
- *(v)* *Let $\Omega_1 = \{x \in \Omega; p(x) \geq 2\}$ and $\Omega_2 = \{x \in \Omega; p(x) < 2\}$. Then ψ'_λ is uniformly monotone in the sense that there exist constants $c > 0$ and $C > 0$ such that*

  $$\begin{aligned} &\langle \psi'_\lambda(u) - \psi'_\lambda(v), u - v\rangle \geq c\rho_{\lambda,\Omega_1}(u - v) \\ &\quad + \left\{ c(C + \|u\|_\lambda + \|v\|_\lambda)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2} \rho_{\lambda,\Omega_2}(u - v) \right\}^{2/p^+(\Omega_2)} \\ &\qquad \wedge \left\{ c(C + \|u\|_\lambda + \|v\|_\lambda)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2} \rho_{\lambda,\Omega_2}(u - v) \right\}^{2/p^-(\Omega_2)} \end{aligned}$$

  for all $u, v \in W^{1,p(\cdot)}(\Omega)$. Here and from now on, we denote $a \wedge b = \min\{a, b\}$ for real numbers a and b.

  In particular, ψ'_λ is strictly monotone, that is, for $u, v \in W^{1,p(\cdot)}(\Omega)$,

  $$\langle \psi'_\lambda(u) - \psi'_\lambda(v), u - v\rangle \geq 0 \text{ and the equality holds only when } u = v.$$

- *(vi)* *ψ'_λ is bounded on every bounded subset of $W^{1,p(\cdot)}(\Omega)$.*

(vii) ψ'_λ *is coercive, that is,*

$$\lim_{\|u\|_\lambda\to\infty}\frac{\langle\psi'_\lambda(u),u\rangle}{\|u\|_\lambda}=\infty.$$

(viii) ψ'_λ *is of* (S_+)*-type, that is, if* $u_n\to u$ *weakly in* $W^{1,p(\cdot)}(\Omega)$ *as* $n\to\infty$ *and*

$$\limsup_{n\to\infty}\langle\psi'_\lambda(u_n),u_n-u\rangle\le 0,$$

then $u_n\to u$ *strongly in* $W^{1,p(\cdot)}(\Omega)$.

(ix) The mapping $\psi'_\lambda:W^{1,p(\cdot)}(\Omega)\to(W^{1,p(\cdot)}(\Omega))^*$ *is a homeomorphism.*

Proof. (i) follows from Aramaki [4, Proposition 4.2] and the argument of the proof.

(ii) By (2.8), for $u\in W^{1,p(\cdot)}(\Omega)$ with $\|u\|_\lambda>1$, we have

$$\begin{aligned}\psi_\lambda(u)&=\int_\Omega A(x,\nabla u(x))dx+\lambda\int_\Omega\frac{1}{p(x)}|u(x)|^{p(x)}dx\\&\ge c_2\int_\Omega|\nabla u(x)|^{p(x)}dx-C_2|\Omega|+\frac{\lambda}{p^+}\int_\Omega|u(x)|^{p(x)}dx\\&\ge(c_2\wedge 1/p^+)\rho_\lambda(u)-C_2|\Omega|\\&\ge(c_2\wedge 1/p^+)\|u\|_\lambda^{p^-}-C_2|\Omega|,\end{aligned}$$

where $|\Omega|$ denotes the volume of Ω and c_2,C_2 are constants in Lemma 2.9(ii). Thus we see that $\psi_\lambda(u)\to\infty$ as $\|u\|_\lambda\to\infty$.

(iii) From Aramaki [6, Proposition 3.3(iii)], the functional

$$W^{1,p(\cdot)}(\Omega)\ni u\mapsto\int_\Omega A(x,\nabla u(x))dx$$

is weakly lower semi-continuous on $W^{1,p(\cdot)}(\Omega)$. By Proposition 2.3, the embedding mapping $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{p(\cdot)}(\partial\Omega)$ is compact. Hence if $u_n\to u$ weakly in $W^{1,p(\cdot)}(\Omega)$, then $u_n\to u$ strongly in $L^{p(\cdot)}(\partial\Omega)$. If we use the argument of the Nemytskii operator (cf. [6, Proposition 2.14]), we can see that $L^{p(\cdot)}(\partial\Omega)\ni u\mapsto\lambda\int_\Omega|u(x)|^{p(x)}dx$ is continuous. Thus we see that

$$W^{1,p(\cdot)}(\Omega)\ni u\mapsto\lambda\int_\Omega|u(x)|^{p(x)}dx$$

is weakly continuous, so ψ_λ is weakly lower semi-continuous.

(iv) easily follows from Lemma 2.8.

(v) follows from Aramaki [7, Proposition 3.7] (cf. [8, Proposition 3.8]).

(vi) From Aramaki [5, Proposition 3.3(vi)], we can easily see that ψ'_λ is bounded on every bounded subset of $W^{1,p(\cdot)}(\Omega)$.

(vii) If $u \in W^{1,p(\cdot)}(\Omega)$ with $\|u\|_\lambda > 1$, then it follows from (2.6) that

$$\langle \psi'_\lambda(u), u\rangle \ge c_1\int_\Omega |\nabla u(x)|^{p(x)}dx - C_1|\Omega| + \lambda\int_\Omega |u(x)|^{p(x)}dx \ge (c_1\wedge 1)\|u\|_\lambda^{p^-} - C_1|\Omega|,$$

where c_1 and C_1 are constants in Lemma 2.9(ii). Since $p^- > 1$, we have

$$\lim_{\|u\|_\lambda\to\infty} \frac{\langle \psi'_\lambda(u), u\rangle}{\|u\|_\lambda} = \infty.$$

(viii) Assume that $u_n \to u$ weakly in $W^{1,p(\cdot)}(\Omega)$ and

$$\limsup_{n\to\infty} \langle \psi'_\lambda(u_n), u_n - u\rangle \le 0.$$

Since $u_n \to u$ weakly in $W^{1,p(\cdot)}(\Omega)$, we have $\lim_{n\to\infty}\langle \psi'_\lambda(u), u_n - u\rangle = 0$. From (v), since ψ'_λ is monotone, we see that

$$\langle \psi'_\lambda(u_n) - \psi'_\lambda(u), u_n - u\rangle \ge 0.$$

Hence

$$\lim_{n\to\infty} \langle \psi'_\lambda(u_n) - \psi'(u), u_n - u\rangle = 0.$$

Since $u_n \to u$ weakly in $W^{1,p(\cdot)}(\Omega)$, $\{u_n\}$ is bounded in $W^{1,p(\cdot)}(\Omega)$. Therefore, it follows from (v) and Proposition 2.3 that $u_n \to u$ strongly in $W^{1,p(\cdot)}(\Omega)$.

(ix) can be proved by replacing Φ_λ in [5, Proposition 3.3(ix)] with ψ_λ and similar reasoning, using the Minty-Browder theorem (cf. Zeidler [26, Theorem 26.A]) and (viii). □

We need the global regularity for the weak solution of (Q_λ). In order to do so, we assume the following properties.

(f.1) There exist positive constants c_1 and c_2 such that

$$|f(x,t)| \le c_1 + c_2|t|^{q(x)-1} \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

where $q \in C(\overline{\Omega})$ with $1 \le q(x) < p^*(x)$ for $x \in \overline{\Omega}$.

(g.1) There exist positive constants $\bar{c}_1$ and $\bar{c}_2$ such that

$$|g(x,t)| \le \bar{c}_1 + \bar{c}_2|t|^{r(x)-1} \text{ for all } (x,t) \in \partial\Omega \times \mathbb{R},$$

where $r \in C(\partial\Omega)$ with $1 \le r(x) < p^\partial(x)$ for $x \in \partial\Omega$.

Proposition 3.2 (Fan and Zhao [17], Fan [14], [10]). *Assume that (A.1)-(A.5) hold.*

(i) *Let (f.1) and (g.1) hold. If* $u \in W^{1,p(\cdot)}(\Omega)$ *is a weak solution of* (Q_λ), *then* $u \in L^\infty(\Omega)$ *and* $\|u\|_{L^\infty(\Omega)}$ *depends only on* $\|u\|_\lambda, p^-, p^+, N, \lambda, q^+, c_1, c_2, r^+, \overline{c}_1$ *and* $\overline{c}_2$.

(ii) *If* $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ *is a weak solution of* (Q_λ), *then* $u \in C^{0,\alpha_1}(\overline{\Omega})$ *for some* $\alpha_1 \in (0,1)$.

(iii) *Suppose that (iii) of (fg) holds. If* $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ *is a weak solution of* (Q_λ), *then* $u \in C^{1,\alpha_1}(\overline{\Omega})$ *for some* $\alpha_1 \in (0,1)$ *and* $\|u\|_{C^{1,\alpha_1}(\overline{\Omega})}$ *depends only on* $\|u\|_{L^\infty(\Omega)}$, $\|p\|_{C^1(\overline{\Omega})}$, $p^-, p^+, N, \lambda, q^+, c_1, c_2, r^+, \overline{c}_1, \overline{c}_2$ *and* Ω.

Proposition 3.3 ([8] A strong maximum principle). *Let* Ω *be an open subset of* $\mathbb{R}^N$ $(N \geq 2)$. *Suppose that (A.1)-(A.5) hold and* $f = f(x,t)$ *is a Carathéodory function satisfying* $f(x,t) \geq 0$ *for a.e.* $x \in \Omega$ *and* $t \geq 0$. *Let* $u \in W^{1,p(\cdot)}(\Omega)$ *be a generalized solution of* (Q_λ) *with* $\lambda \geq 0$, $u \geq 0$ *in* Ω *and* $u \not\equiv 0$. *Then* $u > 0$ *in* Ω. *If* Ω *satisfies the interior ball condition at* $x_0 \in \partial\Omega$ *and* $u \in C^1(\Omega \cup \{x_0\})$ *with* $u(x_0) = 0$, *then we have* $\frac{\partial u}{\partial \boldsymbol{n}}\big|_{x=x_0} < 0$.

Proof. Let $u \in W^{1,p(\cdot)}(\Omega)$ be a generalized solution of (Q_λ) with $\lambda \geq 0$, $u \geq 0$ and $u \not\equiv 0$. Then we have

$$\langle \psi_\lambda'(u), w \rangle = \int_\Omega f(x, u(x)) w(x) dx \geq 0$$

for all $w \in W_0^{1,p(\cdot)}(\Omega)$ with $w \geq 0$. Thus u is a weak supersolution of the equation

$$-\text{div}\,[\boldsymbol{a}(x, \boldsymbol{\nabla} u(x))] + \lambda |u(x)|^{p(x)-2} u(x) = 0.$$

Hence we can apply [8, Theorem 1.1 and 1.2] with $d(x) \equiv \lambda$. □

Proposition 3.4 (A comparison principle). *Assume that (A.1)-(A.5) hold and let* $\lambda > 0$. *If* $u, v \in W^{1,p(\cdot)}(\Omega)$ *satisfy that*

$$\langle \psi_\lambda'(v), w \rangle \leq \langle \psi_\lambda'(u), w \rangle \text{ for all } w \in W^{1,p(\cdot)}(\Omega) \text{ with } w \geq 0 \text{ in } \Omega,$$

then we can see that $v \leq u$ *in* Ω.

Proof. By the hypothesis, we have

$$\int_\Omega (\boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) - \boldsymbol{a}(x, \boldsymbol{\nabla} v(x))) \cdot \boldsymbol{\nabla} w(x) dx \\ + \lambda \int_\Omega \left[|u(x)|^{p(x)-2} u(x) - |v(x)|^{p(x)-2} v(x) \right] w(x) dx \geq 0 \quad (3.2)$$

for all $w \in W^{1,p(\cdot)}(\Omega)$ with $w \geq 0$ in Ω. Let $\Omega_0 = \{x \in \Omega; u(x) < v(x)\}$ and take $w(x) = \max\{-(u(x) - v(x)), 0\}$ as a test function of (3.2). We note that $w \in W^{1,p(\cdot)}(\Omega)$ with $w \geq 0$ in Ω and

$$\boldsymbol{\nabla} w(x) = \begin{cases} -(\boldsymbol{\nabla} u(x) - \boldsymbol{\nabla} v(x)) & \text{if } x \in \Omega_0, \\ \mathbf{0} & \text{if } x \in \Omega \setminus \Omega_0. \end{cases}$$

We use the following well-known inequality (cf. Kichenassamy and Veron [20] and Thelin [24]).

$$\begin{aligned} &(|\boldsymbol{\xi}|^{p(x)-2}\boldsymbol{\xi} - |\boldsymbol{\eta}|^{p(x)-2}\boldsymbol{\eta}) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \\ &\qquad \geq \left\{ \begin{array}{ll} c|\boldsymbol{\xi} - \boldsymbol{\eta}|^{p(x)} & \text{if } p(x) \geq 2, \\ c(1 + |\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi} - \boldsymbol{\eta}|^2 & \text{if } 1 < p(x) < 2 \end{array} \right\} \geq 0 \end{aligned} \tag{3.3}$$

for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$ $(N \geq 1)$ with a positive constant c, and in the last inequality, the equality is only valid when $\boldsymbol{\xi} = \boldsymbol{\eta}$. Hence, from (3.2) and Lemma 2.9(i) we have

$$\begin{aligned} 0 &\leq \int_\Omega (\boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) - \boldsymbol{a}(x, \boldsymbol{\nabla} v(x))) \cdot \boldsymbol{\nabla} w(x) dx \\ &\quad + \lambda \int_\Omega \big[|u(x)|^{p(x)-2}u(x) - |v(x)|^{p(x)-2}v(x)\big] w(x) dx \\ &= -\int_{\Omega_0} (\boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) - \boldsymbol{a}(x, \boldsymbol{\nabla} v(x))) \cdot \boldsymbol{\nabla}(u(x) - v(x)) dx \\ &\quad - \lambda \int_{\Omega_0} \big[|u(x)|^{p(x)-2}u(x) - |v(x)|^{p(x)-2}v(x)\big](u(x) - v(x)) dx \leq 0. \end{aligned} \tag{3.4}$$

If $|\Omega_0| > 0$, then it follows from Lemma 2.9(i) and (3.4) that $\boldsymbol{\nabla} u(x) - \boldsymbol{\nabla} v(x) = 0$ and $u(x) - v(x) = 0$ in Ω_0. This implies that $\boldsymbol{\nabla} w = \mathbf{0}$ in Ω, so $w(x) = \text{const.}$ in Ω. Since $w = 0$ in Ω_0 and $|\Omega_0| > 0$, $w \equiv 0$. This contradicts $|\Omega_0| > 0$. Hence we see that $|\Omega_0| = 0$, so $v \leq u$ in Ω. □

Proposition 3.5. *Suppose that (A.1)-(A.5) hold and let $\lambda > 0$. Let $q \in C_+(\overline{\Omega})$ and $r \in C_+(\partial\Omega)$ satisfy that $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $r(x) < p^\partial(x)$ for all $x \in \partial\Omega$. Then for each $h \in L^{q'(\cdot)}(\Omega)$ and $k \in L^{r'(\cdot)}(\partial\Omega)$, the following problem*

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x, \boldsymbol{\nabla} u(x))] + \lambda|u(x)|^{p(x)-2}u(x) = h(x) & \text{in } \Omega, \\ \boldsymbol{n}(x) \cdot \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) = k(x) & \text{on } \partial\Omega \end{cases} \tag{N_λ}$$

has a unique weak solution $u \in W^{1,p(\cdot)}(\Omega)$ which is a global minimizer of a functional

$$\widehat{\varphi}_\lambda(v) = \psi_\lambda(v) - \int_\Omega h(x)v(x)dx - \int_{\partial\Omega} k(x)v(x)d\sigma_x \text{ for } v \in W^{1,p(\cdot)}(\Omega). \tag{3.5}$$

Proof. Recall that the functional ψ_λ on $W^{1,p(\cdot)}(\Omega)$ is defined by (3.1). We note that it follows from Proposition 2.2 and 2.3 that $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ are compact embeddings. By Proposition 3.1(i), it is known that $\psi_\lambda \in C^1(W^{1,p(\cdot)}(\Omega), \mathbb{R})$ and

$$\langle \psi'_\lambda(u), v\rangle = \int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) \cdot \boldsymbol{\nabla} v(x)dx + \lambda \int_\Omega |u(x)|^{p(x)-2}u(x)v(x)dx$$

for $u, v \in W^{1,p(\cdot)}(\Omega)$, and $\psi'_\lambda : W^{1,p(\cdot)}(\Omega) \to (W^{1,p(\cdot)}(\Omega))^*$ is strictly monotone, bounded on every bounded subset of $W^{1,p(\cdot)}(\Omega)$ and a homeomorphism from Proposition 3.1(ix). Moreover, we note that since $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, we have $L^{q'(\cdot)}(\Omega) \hookrightarrow (W^{1,p(\cdot)}(\Omega))^*$, and since $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$, we have $L^{r'(\cdot)}(\partial\Omega) \hookrightarrow (W^{1,p(\cdot)}(\Omega))^*$. For each $h \in L^{q'(\cdot)}(\Omega)$ and $k \in L^{r'(\cdot)}(\partial\Omega)$, if we define a functional on $W^{1,p(\cdot)}(\Omega)$ by

$$\langle S, v\rangle = \int_\Omega h(x)v(x)dx + \int_{\partial\Omega} k(x)v(x)d\sigma_x \text{ for } v \in W^{1,p(\cdot)}(\Omega),$$

then $S \in (W^{1,p(\cdot)}(\Omega))^*$, so there exists a unique $u \in W^{1,p(\cdot)}(\Omega)$ such that $\psi'_\lambda(u) = S$, that is,

$$\langle \widehat{\varphi}'_\lambda(u), v\rangle = \langle \psi'_\lambda(u), v\rangle - \int_\Omega h(x)v(x)dx - \int_{\partial\Omega} k(x)v(x)d\sigma_x = 0 \tag{3.6}$$

for all $v \in W^{1,p(\cdot)}(\Omega)$.

On the other hand, the functional $\widehat{\varphi}_\lambda$ defined by (3.5) is clearly sequentially weakly lower semi-continuous, and coercive on $W^{1,p(\cdot)}(\Omega)$. Indeed, it follows from the Hölder inequality (Proposition 2.2) and $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ that we have

$$\left|\int_\Omega h(x)v(x)dx\right| \le 2\|h\|_{L^{q'(\cdot)}(\Omega)}\|v\|_{L^{q(\cdot)}(\Omega)} \le C'\|h\|_{L^{q'(\cdot)}(\Omega)}\|v\|_\lambda \le C'_1\|v\|_\lambda$$

and similarly,

$$\left|\int_{\partial\Omega} k(x)v(x)dx\right| \le C'_1\|v\|_\lambda$$

for some positive constants C' and C'_1. So we obtain that

$$\widehat{\varphi}_\lambda(v) \ge c_2\|v\|_\lambda^{p^-} - C_2 - C'_1\|v\|_\lambda$$

for $v \in W^{1,p(\cdot)}(\Omega)$ with $\|v\|_\lambda > 1$. Since $p^- > 1$, we see that $\widehat{\varphi}_\lambda(v) \to \infty$ as $\|v\|_\lambda \to \infty$.

Hence it follows from [26, Theorem 25.D] that $\widehat{\varphi}_\lambda$ has a global minimizer u_0 in $W^{1,p(\cdot)}(\Omega)$. Thus $\widehat{\varphi}'_\lambda(u_0) = 0$, that is,

$$\langle \widehat{\varphi}'_\lambda(u_0), v\rangle = \langle \psi'_\lambda(u_0), v\rangle - \int_\Omega h(x)v(x)dx - \int_{\partial\Omega} k(x)v(x)d\sigma_x = 0 \tag{3.7}$$

for all $v \in W^{1,p(\cdot)}(\Omega)$. From (3.6) and (3.7), we have

$$0 = \langle \widehat{\varphi}'_\lambda(u) - \widehat{\varphi}'_\lambda(u_0), u - u_0 \rangle = \langle \psi'_\lambda(u) - \psi'_\lambda(u_0), u - u_0 \rangle.$$

Since ψ'_λ is strictly monotone from Proposition 3.1(v), we see that $u = u_0$ is a unique weak solution of (N_λ). □

Define an operator $K = K_\lambda : L^{q'(\cdot)}(\Omega) \times L^{r'(\cdot)}(\partial\Omega) \to W^{1,p(\cdot)}(\Omega)$ by $K(h,k) = u$ for $h \in L^{q'(\cdot)}(\Omega)$, $k \in L^{r'(\cdot)}(\partial\Omega)$ and u is a unique weak solution of (N_λ). K is called the solution operator.

Proposition 3.6. *Assume that (A.1)-(A.5) hold and let $\lambda > 0$. Then the following properties hold.*

- *(i) If $q \in C_+(\overline{\Omega})$ and $r \in C_+(\partial\Omega)$ satisfy that $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $r(x) < p^\partial(x)$ for all $x \in \partial\Omega$, respectively, then the operator $K : L^{q'(\cdot)}(\Omega) \times L^{r'(\cdot)}(\partial\Omega) \to W^{1,p(\cdot)}(\Omega)$ is continuous and bounded on every bounded subset of $L^{q'(\cdot)}(\Omega) \times L^{r'(\cdot)}(\partial\Omega)$.*

  Moreover, if $q_1 \in C_+(\overline{\Omega})$ satisfies that $q_1(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then $K : L^{q'(\cdot)}(\Omega) \times L^{r'(\cdot)}(\partial\Omega) \to L^{q_1(\cdot)}(\Omega)$ is weakly-strongly continuous, that is, if $(h_n, k_n) \to (h, k)$ weakly in $L^{q'(\cdot)}(\Omega) \times L^{r'(\cdot)}(\partial\Omega)$, then $K(h_n, k_n) \to K(h, k)$ strongly in $L^{q_1(\cdot)}(\Omega)$ as $n \to \infty$.
- *(ii) The operator $K : L^\infty(\Omega) \times C^{0,\alpha}(\partial\Omega) \to C^{1,\alpha_1}(\overline{\Omega})$ for some $\alpha_1 \in (0,1)$ is continuous, bounded on every bounded subset of $L^\infty(\Omega) \times C^{0,\alpha}(\partial\Omega)$, and hence $K : L^\infty(\Omega) \times C^{0,\alpha}(\partial\Omega) \to C^1(\overline{\Omega})$ is a compact operator.*
- *(iii) K is an increasing operator, that is, if $h_1, h_2 \in L^{q'(\cdot)}(\Omega)$ with $h_1 \le h_2$ in Ω and $k_1, k_2 \in L^{r'(\cdot)}(\partial\Omega)$ with $k_1 \le k_2$ on $\partial\Omega$, then $K(h_1, k_1) \le K(h_2, k_2)$ in Ω.*

Proof. (i) Let $(h_n, k_n) \to (h_0, k_0)$ in $L^{q'(\cdot)}(\Omega) \times L^{r'(\cdot)}(\partial\Omega)$ as $n \to \infty$. Define $S_n \in (W^{1,p(\cdot)}(\Omega))^*$ by

$$\langle S_n, v \rangle = \int_\Omega h_n(x) v(x) dx + \int_{\partial\Omega} k_n(x) v(x) d\sigma_x \quad (n = 0, 1, 2, \ldots)$$

for $v \in W^{1,p(\cdot)}(\Omega)$. By the Hölder inequality (Proposition 2.2), we have

$$\begin{aligned} |\langle S_n - S_0, v \rangle| &\le 2\|h_n - h_0\|_{L^{q'(\cdot)}(\Omega)} \|v\|_{L^{q(\cdot)}(\Omega)} + 2\|k_n - k_0\|_{L^{r'(\cdot)}(\partial\Omega)} \|v\|_{L^{r(\cdot)}(\partial\Omega)} \\ &\le C(\|h_n - h_0\|_{L^{q'(\cdot)}(\Omega)} + \|k_n - k_0\|_{L^{r'(\cdot)}(\partial\Omega)}) \|v\|_\lambda \end{aligned}$$

for some constant $C > 0$. Thus $S_n \to S_0$ in $(W^{1,p(\cdot)}(\Omega))^*$ as $n \to \infty$. If we put $u_n = K(h_n, k_n)$ and $u_0 = K(h_0, k_0)$, then $\psi'_\lambda(u_n) = S_n$ and $\psi'_\lambda(u_0) = S_0$. Since $(\psi'_\lambda)^{-1}$ is continuous from Proposition 3.1(ix), $u_n \to u_0$ in $W^{1,p(\cdot)}(\Omega)$ as $n \to \infty$.

Let $\{(h_n, k_n)\}_{n=1}^{\infty} \subset L^{q'(\cdot)}(\Omega) \times L^{r'(\cdot)}(\partial\Omega)$ be a bounded set and $\{K(h_n, k_n)\}$ be not bounded. If we put $\{S_n\} \subset (W^{1,p(\cdot)}(\Omega))^*$ as above, then there exists $M > 0$ such that $\|S_n\|_{(W^{1,p(\cdot)}(\Omega))^*} \leq M$ and$\{u_n = (\psi'_\lambda)^{-1} S_n\}$ is not bounded. Passing to a subsequence, we can assume that $\|u_n\|_\lambda \to \infty$. Hence we have

$$\frac{\langle \psi'_\lambda(u_n), u_n \rangle}{\|u_n\|_\lambda} = \frac{\langle S_n, u_n \rangle}{\|u_n\|_\lambda} \leq C_1 M$$

for some constant C_1. This contradicts the coerciveness of ψ'_λ.

(ii) When $h \in L^\infty(\Omega)$ and $k \in C^{0,\alpha}(\partial\Omega)$, if we put $f(x,t) = h(x)$ and $g(x,t) = k(x)$, (ii) follows from Proposition 3.2.

(iii) Let $h_1, h_2 \in L^{q'(\cdot)}(\Omega)$ satisfy $h_1 \leq h_2$ in Ω and $k_1, k_2 \in L^{r'(\cdot)}(\partial\Omega)$ satisfy $k_1 \leq k_2$ on $\partial\Omega$, and put $u_1 = K(h_1, k_1), u_2 = K(h_2, k_2)$. For $w \in W^{1,p(\cdot)}(\Omega)$ with $w \geq 0$ in Ω, it follows from (3.6) that

$$\begin{aligned} \langle \psi'_\lambda(u_1), w \rangle &= \int_\Omega h_1(x) w(x) dx + \int_{\partial\Omega} k_1(x) w(x) d\sigma_x \\ &\leq \int_\Omega h_2(x) w(x) dx + \int_{\partial\Omega} k_2(x) w(x) d\sigma_x = \langle \psi'_\lambda(u_2), w \rangle. \end{aligned}$$

Hence it follows from a comparison principle (Proposition 3.4) that we have $u_1 \leq u_2$ in Ω. □

Proposition 3.7. *Assume that (A.1)-(A.5) hold and let* $\lambda > 0$. *Let* $q \in C_+(\overline{\Omega})$ *such that* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *and* $r \in C_+(\partial\Omega)$ *such that* $r(x) < p^\partial(x)$ *for all* $x \in \partial\Omega$. *If* $h \in L^{q'(\cdot)}(\Omega)$ *with* $h \geq 0$ *and* $k \in L^{r'(\cdot)}(\partial\Omega)$ *with* $k \geq 0$, *then* $K(h,k) \geq 0$.

In addition, if $h \in L^\infty(\Omega)$ *and* $k \in C^{0,\alpha}(\partial\Omega)$ *with* $k \not\equiv 0$, *then* $K(h,k) > 0$ *on* $\overline{\Omega}$.

Proof. Let $h \in L^{q'(\cdot)}(\Omega)$ with $h \geq 0$ and $k \in L^{r'(\cdot)}(\partial\Omega)$ with $k \geq 0$, and put $u = K_\lambda(h,k)$. By Proposition 3.5, u is the unique global minimizer of the energy functional $\widehat{\varphi}_\lambda$ defined by (3.5). Since $A(x, -\boldsymbol{\xi}) = A(x, \boldsymbol{\xi})$ from (A.1) and

$$\boldsymbol{\nabla}|u(x)| = \begin{cases} \boldsymbol{\nabla} u(x) & \text{if } u(x) > 0, \\ \mathbf{0} & \text{if } u(x) = 0, \\ -\boldsymbol{\nabla} u(x) & \text{if } u(x) < 0, \end{cases}$$

we can see that $|u| \in W^{1,p(\cdot)}(\Omega)$ and $A(x, \boldsymbol{\nabla}|u(x)|) = A(x, \boldsymbol{\nabla} u(x))$, so $\psi_\lambda(|u|) = \psi_\lambda(u)$. Since $h(x)u(x) \leq h(x)|u(x)|$ and $k(x)u(x) \leq k(x)|u(x)|$, we have $\widehat{\varphi}_\lambda(u) \geq \widehat{\varphi}_\lambda(|u|)$, so $|u|$ is also a global minimizer of $\widehat{\varphi}_\lambda$. Hence $K_\lambda(h,k) = u = |u| \geq 0$.

In particular, if $h \in L^\infty(\Omega)$ with $h \geq 0$, and $k \in C^{0,\alpha}(\partial\Omega)$ with $k \geq 0$ and $k \not\equiv 0$, then it follows from Proposition 3.2(iii) that $u = K_\lambda(h,k) \in C^{1,\alpha_1}(\overline{\Omega})$ with $u \geq 0$. We claim that $u \not\equiv 0$. Indeed, for $1 > \varepsilon > 0$, we have

$$\widehat{\varphi}_\lambda(\varepsilon) \leq \lambda \varepsilon^{p^-} \int_\Omega \frac{1}{p(x)} dx - \varepsilon \int_{\partial\Omega} k(x) d\sigma_x.$$

Since $k \geq 0$ and $k \not\equiv 0$, we see that $\int_{\partial\Omega} k(x)d\sigma_x > 0$. Since $p^- > 1$, if we choose a small $\varepsilon > 0$, then we can see that $\widehat{\varphi}_\lambda(u) \leq \widehat{\varphi}_\lambda(\varepsilon) < 0 = \widehat{\varphi}_\lambda(0)$. Thus $u \not\equiv 0$. By a strong maximum principle (Proposition 3.3), $u > 0$ in Ω. If there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0$, it follows from the boundary condition that $\boldsymbol{n}(x_0) \cdot \boldsymbol{a}(x_0, \boldsymbol{\nabla} u(x_0)) = k(x_0) \geq 0$. This implies that $\frac{\partial u}{\partial \boldsymbol{n}}(x_0) \geq 0$ from (2.4) in (A.1). This contradicts the result of a strong maximum principle (Proposition 3.3). Hence $u > 0$ on $\overline{\Omega}$. □

Proposition 3.8. *Assume that (A.1)-(A.5) hold. Let* $h \in L^\infty(\Omega)$ *with* $h \geq 0$, $k \in C^{0,\alpha}(\partial\Omega)$ *with* $k \geq 0$, *and for* $\lambda > 0$, *put* $u_\lambda = K_\lambda(h, k)$. *Then we have the following properties.*

- *(i)* $\|u_\lambda\|_\lambda$ *is bounded uniformly for* $\lambda \in [1, \infty)$ *and* $u_\lambda \to 0$ *in* $L^{p(\cdot)}(\Omega)$ *as* $\lambda \to \infty$.
- *(ii) If* $\lambda_1 \geq \lambda_2 > 0$, *then* $K_{\lambda_1}(h, k) \leq K_{\lambda_2}(h, k)$.
- *(iii) We obtain that* $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ *as* $\lambda \to \infty$.

Proof. First we note that it follows from Propositions 3.2 and 3.7 that $u_\lambda \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and $u_\lambda \geq 0$. Since u_λ is a weak solution of the problem (N_λ), we have

$$\langle \psi'_\lambda(u_\lambda), v\rangle = \int_\Omega h(x)v(x)dx + \int_{\partial\Omega} k(x)v(x)d\sigma_x \tag{3.8}$$

for all $v \in W^{1,p(\cdot)}(\Omega)$.

(i) Let $\lambda \geq \lambda_0 = 1$. Taking $v = u_\lambda$ as a test function of (3.8) and noting that if $\|u_\lambda\|_\lambda \geq 1$, then

$$\langle \psi'_\lambda(u_\lambda), u_\lambda\rangle \geq c_1 \left(\int_\Omega |\boldsymbol{\nabla} u_\lambda(x)|^{p(x)}dx + \lambda \int_\Omega u_\lambda(x)^{p(x)}dx\right) - C_1 \geq c_1\|u_\lambda\|_\lambda^{p^-} - C_1$$

for some positive constants $c_1 \leq 1$ and C_1 from (2.6). Hence it follows from the Hölder inequality (Proposition 2.2) that

$$\begin{aligned} c_1\|u_\lambda\|_\lambda^{p^-} &\leq C_1 + \langle \psi'_\lambda(u_\lambda), u_\lambda\rangle \\ &= C_1 + \int_\Omega h(x)u_\lambda(x)dx + \int_{\partial\Omega} k(x)u_\lambda(x)d\sigma_x \\ &\leq C_1 + 2\|h\|_{L^{p'(\cdot)}(\Omega)}\|u_\lambda\|_{L^{p(\cdot)}(\Omega)} + 2\|k\|_{L^{p'(\cdot)}(\partial\Omega)}\|u_\lambda\|_{L^{p(\cdot)}(\partial\Omega)} \\ &\leq C_1 + C_2\|u_\lambda\|_{\lambda_0} \leq C_1 + C_2\|u_\lambda\|_\lambda \end{aligned}$$

for some constant C_2 independent of λ. Since $p^- > 1$, we see that $\|u_\lambda\|_\lambda$ is bounded uniformly for $\lambda \in [1, \infty)$ and we have

$$\lambda \int_\Omega u_\lambda(x)^{p(x)}dx \leq \int_\Omega |\boldsymbol{\nabla} u_\lambda(x)|^{p(x)}dx + \lambda \int_\Omega u_\lambda(x)^{p(x)}dx \leq C_3$$

for some constant $C_3 > 0$ independent of λ. This implies that $u_\lambda \to 0$ in $L^{p(\cdot)}(\Omega)$ as $\lambda \to \infty$.

(ii) Let $\lambda_1 \geq \lambda_2 > 0$ and put $u_1 = K_{\lambda_1}(h,k)$ and $u_2 = K_{\lambda_2}(h,k)$. Then it follows from Proposition 3.7 that $u_1, u_2 \geq 0$. Then for any $w \in W^{1,p(\cdot)}(\Omega)$ with $w \geq 0$,

$$\langle \psi'_{\lambda_1}(u_1), w\rangle = \int_\Omega h(x)w(x)dx + \int_{\partial\Omega} k(x)w(x)d\sigma_x = \langle \psi'_{\lambda_2}(u_2), w\rangle \leq \langle \psi'_{\lambda_1}(u_2), w\rangle.$$

By a comparison principle (Proposition 3.4), we have $u_1 \leq u_2$.

(iii) It follows from (ii) that $\{u_\lambda\}$ is decreasing with respect to $\lambda > 0$, so is $\|u_\lambda\|_{L^\infty(\Omega)}$. This implies that $\|u_\lambda\|_{L^\infty(\Omega)}$ is convergent as $\lambda \to \infty$. By Proposition 3.2(iii), the boundedness of $\{\|u_\lambda\|_{L^\infty(\Omega)}\}$ implies the boundedness of $\{\|u_\lambda\|_{C^{1,\alpha_1}(\overline{\Omega})}\}$, where $\alpha_1 \in (0,1)$ is a constant. Hence $\{|\nabla u_\lambda|\}$ is bounded.

Suppose that $\lim_{\lambda\to\infty} \|u_\lambda\|_{L^\infty(\Omega)} = 3a > 0$. Then there exists $x_0 \in \overline{\Omega}$ such that $|u_\lambda(x_0)| \geq 2a$ for large $\lambda > 0$. Since $u_\lambda \to 0$ in $L^{p(\cdot)}(\Omega)$ as $\lambda \to \infty$ from (i), for large $\lambda > 0$, there exist $r_\lambda > 0$ and $x_1 \in \overline{\Omega}$ with $r_\lambda < |x_1 - x_0| < 2r_\lambda$ such that $|u_\lambda(x)| \geq a$ for all $x \in B_{\overline{\Omega}}(x_0, r_\lambda) := \{x \in \overline{\Omega}; |x - x_0| < r_\lambda\}$ and $|u_\lambda(x_1)| < a$. It is easy to see that we can choose $r_\lambda \to 0$ as $\lambda \to \infty$. Then $|\nabla u_\lambda|$ is unbounded. This is a contradiction. □

Proposition 3.9 (A sub-supersolution principle)**.** *Assume that (A.1)-(A.5), (fg), (f.1) and (g.1) hold. Let* $\lambda > 0$. *Suppose that* $u_0, v^0 \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, u_0 *and* v^0 *are a subsolution and a supersolution of* (Q_λ), *respectively, with* $u_0 \leq v^0$. *If* f *and* g *satisfy the conditions:*

$f(x,t)$ *is nondecreasing in* $[\inf u_0(x), \sup v^0(x)]$ *for all* $x \in \Omega$,

$g(x,t)$ *is nondecreasing in* $[\inf u_0(x), \sup v^0(x)]$ *for all* $x \in \partial\Omega$,

then (Q_λ) *has a minimal weak solution* u_* *and a maximal weak solution* v^* *in the order interval* $[u_0, v^0]$ *in the sense that* $u_0 \leq u_* \leq v^* \leq v^0$ *in* Ω *and if* u *is any weak solution of* (Q_λ) *such that* $u_0 \leq u \leq v^0$ *in* Ω, *then* $u_* \leq u \leq v^*$ *in* Ω.

Proof. For $u \in L^\infty(\Omega)$, it follows from (f.1) and (g.1) that $f(\cdot, u(\cdot)) \in L^\infty(\Omega)$ and $g(\cdot, u(\cdot)) \in L^\infty(\partial\Omega)$. So we can define an operator T_λ by

$$T_\lambda(u) = K_\lambda(f(\cdot, u(\cdot)), g(\cdot, u(\cdot))) \text{ for } u \in L^\infty(\Omega).$$

Then it follows from (fg) and Proposition 3.2(iii) that $T_\lambda : L^\infty(\Omega) \to C^1(\overline{\Omega})$ is a compact operator, and by Proposition 3.6(iii), T_λ is an increasing operator on the order interval $[u_0, v^0]$. We show that $u_0 \leq T_\lambda(u_0)$ and $T_\lambda(v^0) \leq v^0$. Since $u_0 \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ is a subsolution of (Q_λ), we have

$$\langle \psi'_\lambda(u_0), v\rangle \leq \int_\Omega f(x, u_0(x))v(x)dx + \int_{\partial\Omega} g(x, u_0(x))v(x)d\sigma_x \tag{3.9}$$

for any $v \in W^{1,p(\cdot)}(\Omega)$ with $v \geq 0$. If we put

$$\overline{u}_0 = T_\lambda(u_0) = K_\lambda(f(\cdot, u_0(\cdot)), g(\cdot, u_0(\cdot))),$$

then $\overline{u}_0$ is a weak solution of the problem

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x, \overline{u}_0(x))] + \lambda|\overline{u}_0(x)|^{p(x)-2}\overline{u}_0(x) = f(x, u_0(x)) & \text{in } \Omega, \\ \boldsymbol{n}(x) \cdot \boldsymbol{a}(x, \boldsymbol{\nabla}\overline{u}_0(x)) = g(x, u_0(x)) & \text{on } \partial\Omega. \end{cases}$$

Hence for $v \in W^{1,p(\cdot)}(\Omega)$ with $v \geq 0$ in Ω, it follows from (3.9) that

$$\langle \psi'_\lambda(\overline{u}_0), v\rangle = \int_\Omega f(x, u_0(x))v(x)dx + \int_{\partial\Omega} g(x, u_0(x))v(x)d\sigma_x \geq \langle \psi'_\lambda(u_0), v\rangle.$$

By a comparison principle (Proposition 3.4), we see that $u_0 \leq \overline{u}_0 = T_\lambda(u_0)$ in Ω. Similarly we can see that $T_\lambda(v^0) \leq v^0$ in Ω. Thus we have $T_\lambda : [u_0, v^0] \to [u_0, v^0]$. It is clear that the cone of all non-negative functions in $L^\infty(\Omega)$ is normal, that is, the norm is semimonotone (that is, if $0 \leq u \leq v$, then $\|u\|_{L^\infty(\Omega)} \leq \delta\|v\|_{L^\infty(\Omega)}$ for some $\delta > 0$). Hence this proposition follows from applying the fixed point theorem for the increasing operator on the order interval. See Amann [2, Corollary 6.2]. □

Proposition 3.10. *Assume that (A.1)-(A.5) hold. Let* $\lambda > 0$. *If* $u \in C^1(\overline{\Omega})$ *is a local minimizer of a functional* φ_λ *in the* $C^1(\overline{\Omega})$*-topology, where the functional* φ_λ *is defined by*

$$\varphi_\lambda(v) = \psi_\lambda(v) - \int_\Omega F(x, v(x))dx - \int_{\partial\Omega} G(x, v(x))d\sigma_x \text{ for } v \in W^{1,p(\cdot)}(\Omega), \tag{3.10}$$

$F(x,t) = \int_0^t f(x,s)ds$ *and* $G(x,t) = \int_0^t g(x,s)ds$, *then* u *is also a local minimizer of* φ_λ *in the* $W^{1,p(\cdot)}(\Omega)$*-topology.*

For the proof, see Fan [13, Theorem 3.1 and its proof].

Proposition 3.11 (A strong comparison principle)**.** *Assume that (A.1)-(A.5), (fg), (f.1) and (g.1) hold. Moreover, assume the following properties.*

(f.2) For eacn $x \in \Omega$, $f(x,t)$ *is nondecreasing with respect to* $t \geq 0$.

(g.2) For eacn $x \in \partial\Omega$, $g(x,t)$ *is nondecreasing with respect to* $t \geq 0$.

Let $\lambda_1, \lambda_2 \in \Lambda$ *with* $0 < \lambda_2 < \lambda_1$ *and* u_{λ_1} *and* u_{λ_2} *be the positive weak solutions of* (Q_{λ_1}) *and* (Q_{λ_2}), *respectively, with* $u_{\lambda_1} \leq u_{\lambda_2}$. *Then* $u_{\lambda_1} < u_{\lambda_2}$ *on* $\overline{\Omega}$.

Proof. By (fg), (f.1) and (g.1), it follows from Proposition 3.2 that $u_{\lambda_2} \in C^1(\overline{\Omega})$ and $u_{\lambda_2} > 0$ on $\overline{\Omega}$. Hence there exists $b > 0$ such that $b < u_{\lambda_2}(x)$ for all $x \in \overline{\Omega}$.

Let $0 < \varepsilon < \left(1-(\lambda_2/\lambda_1)^{1/(p^+-1)}\right) b$ and put $v_\varepsilon = u_{\lambda_2} - \varepsilon$. We note that $\lambda_1 v_\varepsilon(x)^{p(x)-1} > \lambda_2 u_{\lambda_2}(x)^{p(x)-1}$. If we put

$$h_1(x) = f(x, u_{\lambda_2}(x)) + \lambda_1 v_\varepsilon(x)^{p(x)-1} - \lambda_2 u_{\lambda_2}(x)^{p(x)-1},$$

then it easily follows from (f.2) that $h_1(x) \geq f(x, u_{\lambda_2}(x)) \geq f(x, u_{\lambda_1}(x))$. For any $w \in W^{1,p(\cdot)}(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} v_\varepsilon(x)) \cdot \boldsymbol{\nabla} w(x) dx + \lambda_1 \int_\Omega v_\varepsilon(x)^{p(x)-1} w(x) dx \\
&\quad = \int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u_{\lambda_2}(x)) \cdot \boldsymbol{\nabla} w(x) dx + \lambda_2 \int_\Omega u_{\lambda_2}(x)^{p(x)-1} w(x) dx \\
&\qquad + \int_\Omega (\lambda_1 v_\varepsilon(x)^{p(x)-1} - \lambda_2 u_{\lambda_2}(x)^{p(x)-1}) w(x) dx \\
&\quad = \int_\Omega \left(f(x, u_{\lambda_2}(x)) + \lambda_1 v_\varepsilon(x)^{p(x)-1} - \lambda_2 u_{\lambda_2}(x)^{p(x)-1}\right) w(x) dx \\
&\qquad + \int_{\partial\Omega} g(x, u_{\lambda_2}(x)) w(x) d\sigma_x \\
&\quad = \int_\Omega h_1(x) w(x) dx + \int_{\partial\Omega} g(x, u_{\lambda_2}(x)) w(x) d\sigma_x.
\end{aligned}$$

Thus v_ε is a weak solution of the problem

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x, \boldsymbol{\nabla} u(x))] + \lambda_1 |u(x)|^{p(x)-2} u(x) = h_1(x) & \text{in } \Omega, \\ \boldsymbol{n}(x) \cdot \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) = g(x, u_{\lambda_2}(x)) & \text{on } \partial\Omega. \end{cases}$$

Therefore, it follows from Proposition 3.6(iii) that

$$v_\varepsilon = K_{\lambda_1}(h_1(\cdot), g(\cdot, u_{\lambda_2}(\cdot))) \geq K_{\lambda_1}(f(x, u_{\lambda_1}(\cdot)), g(\cdot, u_{\lambda_1}(\cdot))) = u_{\lambda_1}.$$

So we see that $u_{\lambda_2} \geq u_{\lambda_1} + \varepsilon > u_{\lambda_1}$ on $\overline{\Omega}$. □

4 Main theorems and their proofs

In this section, we give main theorems and their proofs. Since we consider only the positive solutions, without loss of its essence, we can assume that

(f_+) $f(x,t) = f(x,0)$ for all $\overline{\Omega}$ and $t < 0$,

(g_+) $g(x,t) = g(x,0)$ for all $\partial\Omega$ and $t < 0$.

Lemma 4.1. *Assume that (A.1)-(A.5), (fg), (f.1) and (g.1) hold and let* $\lambda > 0$. *If* u_λ *is a weak solution of* (Q_λ) *with* $u_\lambda \geq 0$ *and* $g(\cdot, u_\lambda(\cdot)) \not\equiv 0$, *then* $u_\lambda > 0$ *on* $\overline{\Omega}$.

Proof. We note that if u_λ is a weak solution of (Q_λ), then it follows from Proposition 3.2 that $u_\lambda \in C^{1,\alpha_1}(\overline{\Omega})$ for some $\alpha_1 \in (0,1)$. If u_λ satisfies that $u_\lambda \geq 0$ and $g(\cdot, u_\lambda(\cdot)) \not\equiv 0$, then it follows from (f.1) and (fg) that $0 \leq f(\cdot, u_\lambda(\cdot)) \in L^\infty(\Omega)$ and $0 \leq g(\cdot, u_\lambda(\cdot)) \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. Since we can write $u_\lambda = K_\lambda(f(\cdot, u_\lambda(\cdot)), g(\cdot, u_\lambda(\cdot)))$, it follows from Proposition 3.7 that $u_\lambda > 0$ on $\overline{\Omega}$. □

Theorem 4.2. *Suppose that (A.1)-(A.5), (fg), (f.1), (g.1), (f.2) and (g.2) hold. Then $\Lambda \neq \emptyset, \lambda_* \geq 0$ and $(\lambda_*, \infty) \subset \Lambda$. Moreover, for every $\lambda > \lambda_*$, there exists a minimal positive weak solution u_λ of (Q_λ) such that $u_{\lambda_1} \leq u_{\lambda_2}$ if $\lambda_* < \lambda_2 < \lambda_1$.*

Proof. Step 1. $\Lambda \neq \emptyset$ and $\lambda_* \geq 0$.

Indeed, consider the following problem

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x, \boldsymbol{\nabla} u(x))] + \lambda |u(x)|^{p(x)-2} u(x) = 0 & \text{in } \Omega, \\ \boldsymbol{n}(x) \cdot \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) = g(x,1) & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

From (fg)(iii), for any $x_1, x_2 \in \partial\Omega$, we have $|g(x_1,1) - g(x_2,1)| \leq \Lambda(1)|x_1 - x_2|^{\beta_1}$, so $g(x,1) \in C^{0,\beta_1}(\partial\Omega)$. Since $g(x,1) \geq g(x,0) \not\equiv 0$, if $\lambda > 0$, then it follows from Proposition 3.5, 3.6 and 3.7 that (4.1) has a unique weak solution $w_\lambda \in C^1(\overline{\Omega})$ with $w_\lambda > 0$ on $\overline{\Omega}$. Moreover, from Proposition 3.8(iii), $\|w_\lambda\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to \infty$. Hence there exist $\lambda_0 > 0$ large enough and $\varepsilon = \varepsilon(\lambda_0) \in (0,1)$ such that $K_{\lambda_0}(0, g(\cdot,1)) = w_{\lambda_0}(x) \leq 1$ with $w_{\lambda_0}(x) \geq \varepsilon$ for all $x \in \overline{\Omega}$. Put $d = \sup_{x \in \overline{\Omega}} f(x, w_{\lambda_0}(x))$, $M = d/\varepsilon^{p^+-1}$ and $\lambda_1 = \lambda_0 + M$. For any $v \in W^{1,p(\cdot)}(\Omega)$ with $v \geq 0$, it follows from (g.2) that

$$\begin{aligned} \langle \psi'_{\lambda_1}(w_{\lambda_0}), v \rangle &= \langle \psi'_{\lambda_0}(w_{\lambda_0}), v \rangle + M \int_\Omega w_{\lambda_0}(x)^{p(x)-1} v(x) dx \\ &\geq \int_{\partial\Omega} g(x,1) v(x) d\sigma_x + d \int_\Omega v(x) dx \\ &\geq \int_{\partial\Omega} g(x, w_{\lambda_0}(x)) v(x) d\sigma_x + \int_\Omega f(x, w_{\lambda_0}(x)) v(x) dx. \end{aligned}$$

Thus w_{λ_0} is a supersolution of (Q_{λ_1}). By (fg), 0 is clearly a subsolution of (Q_{λ_1}). By a sub-supersolution principle (Proposition 3.9), (Q_{λ_1}) has a weak solution u_{λ_1} such that $0 \leq u_{\lambda_1} \leq w_{\lambda_0}$. Since $g(x, u_{\lambda_1}(x)) \geq g(x,0) \not\equiv 0$ from (fg) and (g.2), it follows from Lemma 4.1 that we see that $u_{\lambda_1} > 0$ on $\overline{\Omega}$. This implies that $\lambda_1 \in \Lambda$, so $\Lambda \neq \emptyset$.

Let $\lambda \in \Lambda$ and u_λ be a positive weak solution of (Q_λ). Taking $v = 1$ as a test function of Definition 2.10(i), it follows from (fg) and (g.2) that

$$\lambda \int_\Omega u_\lambda^{p(x)-1} dx = \int_\Omega f(x, u_\lambda(x)) dx + \int_{\partial\Omega} g(x, u_\lambda(x)) d\sigma_x \geq \int_{\partial\Omega} g(x,0) d\sigma_x > 0.$$

Then $\lambda > 0$, so $\lambda_* \geq 0$.

Step 2. If $\lambda_0 \in \Lambda$, then for any $\lambda > \lambda_0$, we have $\lambda \in \Lambda$.

Indeed, let $\lambda_0 \in \Lambda$, $\lambda > \lambda_0$ and u_{λ_0} be a positive weak solution of (Q_{λ_0}). For any $v \in W^{1,p(\cdot)}(\Omega)$ with $v \geq 0$,

$$\langle \psi'_\lambda(u_{\lambda_0}), v\rangle \geq \langle \psi'_{\lambda_0}(u_{\lambda_0}), v\rangle = \int_\Omega f(x, u_{\lambda_0}(x))v(x)dx + \int_{\partial\Omega} g(x, u_{\lambda_0}(x))v(x)d\sigma_x.$$

Thus u_{λ_0} is a supersolution of (Q_λ). By (fg), 0 is a subsolution of (Q_λ). Using again sub-supersolution principle (Proposition 3.9), (Q_λ) has a minimal weak solution u_λ such that $0 \leq u_\lambda \leq u_{\lambda_0}$. By (g.2) and Lemma 4.1, $u_\lambda > 0$ on $\overline{\Omega}$. This implies that $\lambda \in \Lambda$.

Step 3. For every $\lambda > \lambda_*$, there exists a minimal positive weak solution u_λ of (Q_λ) such that $u_{\lambda_1} \leq u_{\lambda_2}$ if $\lambda_* < \lambda_2 < \lambda_1$.

Indeed, for every $\lambda > \lambda_*$, by Step 2, we have $\lambda \in \Lambda$ and (Q_λ) has a minimal positive weak solution u_λ of (Q_λ). Let $\lambda_* < \lambda_2 < \lambda_1$, u_{λ_1} and u_{λ_2} be the minimal positive weak solutions of (Q_{λ_1}) and (Q_{λ_2}), respectively. It is easy to see that

$$\langle \psi'_{\lambda_1}(u_{\lambda_2}), v\rangle \geq \langle \psi'_{\lambda_2}(u_{\lambda_2}), v\rangle = \int_\Omega f(x, u_{\lambda_2}(x))v(x)dx + \int_{\partial\Omega} g(x, u_{\lambda_2}(x))v(x)d\sigma_x$$

for all $v \in W^{1,p(x)}(\Omega)$ with $v \geq 0$. Thus u_{λ_2} is a supersolution of (Q_{λ_1}), and from (fg), clearly 0 is a subsolution of (Q_{λ_1}). Hence it follows from a sub-supersolution principle (Proposition 3.9), we have $0 \leq u_{\lambda_1} \leq u_{\lambda_2}$. □

Corollary 4.3. *Suppose that (A.1)-(A.5), (fg), (f.2) and (g.2) hold. Moreover, we assume (f_+) and (g_+) and the following conditions hold.*

$(\widetilde{f}, 1)$ *There exist constants $M_1, c'_1, c'_2 > 0$ such that*

$$f(x,t) \leq c'_1 + c'_2 t^{q_0(x)-1} \text{ for all } x \in \Omega \text{ and all } t \geq M_1,$$

where $q_0 \in C(\overline{\Omega})$ with $1 \leq q_0(x) < p(x)$ for all $x \in \overline{\Omega}$.

$(\widetilde{g}, 1)$ *There exist constants $\overline{M}_1, \overline{c}_1, \overline{c}_2 > 0$ such that*

$$g(x,t) \leq \overline{c}_1 + \overline{c}_2 t^{r_0(x)-1} \text{ for all } x \in \partial\Omega \text{ and all } t \geq \overline{M}_1,$$

where $r_0 \in C(\partial\Omega)$ with $1 \leq r_0(x) \leq r_0^+ < p^-$ for all $x \in \partial\Omega$.

Then we have $\lambda_* = 0$.

Proof. Let $\lambda > 0$ be arbitrary. By $(\widetilde{f}, 1)$ and (f_+), since $q_0(x) < p(x)$ for $x \in \overline{\Omega}$, there exists $M'_1 > 0$ such that

$$|F(x,t)| \leq \frac{\lambda}{2p^+}|t|^{p(x)} \text{ for all } x \in \overline{\Omega} \text{ and } |t| \geq M'_1,$$

and by $(\widetilde{g}, 1)$ and (g_+), there exist constants $\overline{c}_1'$ and $\overline{c}_2'$ such that

$$|G(x,t)| \leq \overline{c}_1' + \overline{c}_2' |t|^{r_0(x)} \text{ for all } (x,t) \in \partial\Omega \times \mathbb{R}.$$

We consider the energy functional φ_λ defined by (3.10). Let $u \in W^{1,p(\cdot)}(\Omega)$ with $\|u\|_\lambda \geq 1$. Then using (2.8), we have

$$\begin{aligned}\varphi_\lambda(u) &\geq c_2 \int_\Omega |\boldsymbol{\nabla} u(x)|^{p(x)} dx - C_2|\Omega| + \frac{\lambda}{p^+} \int_\Omega |u(x)|^{p(x)} dx \\ &\quad - \frac{\lambda}{2p^+} \int_\Omega |u(x)|^{p(x)} dx - c_3 \|u\|_\lambda^{r_0^+} - c_4 \\ &\geq (c_2 \wedge 1/(2p^+)) \|u\|_\lambda^{p^-} - c_3 \|u\|_\lambda^{r_0^+} - c_5\end{aligned}$$

for some constants c_3, c_4 and c_5. Since $r_0^+ < p^-$, we see that φ_λ is coercive on $W^{1,p(\cdot)}(\Omega)$. Since φ_λ is sequentially weakly lower semi-continuous on $W^{1,p(\cdot)}(\Omega)$, φ_λ has a global minimizer $u_0 \in W^{1,p(\cdot)}(\Omega)$, so u_0 is a weak solution of (Q_λ). Under the hypotheses, it follows from Proposition 3.2 that $u_0 \in C^1(\overline{\Omega})$. By (f_+), (g_+), (f.2) and (g.2), $F(x,t)$ and $G(x,t)$ are nondecreasing with respect to $t \in \mathbb{R}$. Hence

$$\varphi_\lambda(u_0) \geq \psi_\lambda(|u_0|) - \int_\Omega F(x, |u_0(x)|) dx - \int_{\partial\Omega} G(x, |u_0(x)|) d\sigma_x = \varphi_\lambda(|u_0|).$$

Since $|u_0| \in W^{1,p(\cdot)}(\Omega)$, $|u_0|$ is also a global minimizer of φ_λ. Thus we can assume that the minimizer u_0 satisfies that $u_0 \geq 0$ in $\overline{\Omega}$. Since $g(x, u_0(x)) \geq g(x, 0) \not\equiv 0$, it follows from Lemma 4.1 that $u_0 > 0$ on $\overline{\Omega}$, so $\lambda \in \Lambda$. Since $\lambda > 0$ is arbitrary, we see that $\lambda_* = 0$. □

Proposition 4.4. *Let (A.1)-(A.5), (fg), (f.1), (g.1), (f.2) and (g.2) hold, and let* $\lambda_1, \lambda_2 \in \Lambda$ *with* $\lambda_2 < \lambda < \lambda_1$. *Suppose that* u_{λ_1} *and* u_{λ_2} *are positive weak solutions of* (Q_{λ_1}) *and* (Q_{λ_2}), *respectively, with* $u_{\lambda_1} \leq u_{\lambda_2}$. *Then there exists a positive weak solution* v_λ *of* (Q_λ) *such that* v_λ *is a global minimizer of the restriction of the functional* φ_λ *defined by* (3.10) *to the order interval* $[u_{\lambda_1}, u_{\lambda_2}] \cap W^{1,p(\cdot)}(\Omega)$.

Proof. Define $\widetilde{f} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(x,t) = \begin{cases} f(x, u_{\lambda_1}(x)) & \text{if } t < u_{\lambda_1}(x), \\ f(x,t) & \text{if } u_{\lambda_1}(x) \leq t \leq u_{\lambda_2}(x), \\ f(x, u_{\lambda_2}(x)) & \text{if } t > u_{\lambda_2}(x) \end{cases}$$

and define $\widetilde{g} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{g}(x,t) = \begin{cases} g(x, u_{\lambda_1}(x)) & \text{if } t < u_{\lambda_1}(x), \\ g(x,t) & \text{if } u_{\lambda_1}(x) \leq t \leq u_{\lambda_2}(x), \\ g(x, u_{\lambda_2}(x)) & \text{if } t > u_{\lambda_2}(x). \end{cases}$$

Let $\widetilde{F}(x,t)=\int_0^t \widetilde{f}(x,s)ds$ and $\widetilde{G}(x,t)=\int_0^t \widetilde{g}(x,s)ds$. Moreover, define a functional

$$\widetilde{\varphi}_\lambda(v)=\psi_\lambda(v)-\int_\Omega \widetilde{F}(x,u(x))dx-\int_{\partial\Omega}\widetilde{G}(x,u(x))d\sigma_x$$

for $v\in W^{1,p(\cdot)}(\Omega)$. Then it is easy to see that $\widetilde{\varphi}_\lambda$ is sequentially weakly lower semi-continuous and coercive on $W^{1,p(\cdot)}(\Omega)$. Thus the global minimum of $\widetilde{\varphi}_\lambda$ is achieved at some $v_\lambda\in W^{1,p(\cdot)}(\Omega)$. Hence v_λ is a weak solution of the following problem.

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x,\boldsymbol{\nabla} u(x))]+\lambda|u(x)|^{p(x)-2}u(x)=\widetilde{f}(x,u(x)) & \text{in } \Omega,\\ \boldsymbol{n}(x)\cdot\boldsymbol{a}(x,\boldsymbol{\nabla} u(x))=\widetilde{g}(x,u(x)) & \text{on } \partial\Omega \end{cases} \qquad (\widetilde{Q}_\lambda)$$

and $v_\lambda\in C^1(\overline{\Omega})$. For $\lambda_2<\lambda<\lambda_1$, using the definitions of $\widetilde{f}$ and $\widetilde{g}$ and that $f(x,t)$ and $g(x,t)$ are nondecreasing with respect to $t\geq 0$, it is easy to see that

$$\begin{aligned} f(x,u_{\lambda_1}(x))&=\widetilde{f}(x,u_{\lambda_1}(x))\leq\widetilde{f}(x,v_\lambda(x))\leq\widetilde{f}(x,u_{\lambda_2}(x))=f(x,u_{\lambda_2}(x)),\\ g(x,u_{\lambda_1}(x))&=\widetilde{g}(x,u_{\lambda_1}(x))\leq\widetilde{g}(x,v_\lambda(x))\leq\widetilde{g}(x,u_{\lambda_2}(x))=g(x,u_{\lambda_2}(x)). \end{aligned}$$

By Proposition 3.6(iii), we obtain that $u_{\lambda_1}\leq v_\lambda\leq u_{\lambda_2}$. Hence $\widetilde{f}(x,v_\lambda(x))=f(x,v_\lambda(x))$ and $\widetilde{g}(x,v_\lambda(x))=g(x,v_\lambda(x))$ and so v_λ is also a positive weak solution of (Q_λ).

For $u\in[u_{\lambda_1},u_{\lambda_2}]\cap W^{1,p(\cdot)}(\Omega)$, we have

$$\begin{aligned}\int_\Omega \widetilde{F}(x,u(x))dx&=\int_\Omega\left(\int_0^{u(x)}\widetilde{f}(x,s)ds\right)dx=\int_\Omega\left(\int_0^{u(x)}f(x,s)ds\right)dx+C_1\\ &=\int_\Omega F(x,u(x))dx+C_1\end{aligned}$$

for some constant C_1. Similarly we have

$$\int_{\partial\Omega}\widetilde{G}(x,u(x))d\sigma_x=\int_{\partial\Omega}G(x,u(x))d\sigma_x+C_2$$

for some constant C_2. Thus we have $\widetilde{\varphi}_\lambda(u)=\varphi_\lambda(u)+C_1+C_2$ for all $u\in[u_{\lambda_1},u_{\lambda_2}]\cap W^{1,p(\cdot)}(\Omega)$. Hence v_λ is a global minimizer of $\varphi_\lambda\big|_{[u_{\lambda_1},u_{\lambda_2}]\cap W^{1,p(\cdot)}(\Omega)}$. □

Remark 4.5. If we replace u_{λ_1} with $u_0=0$ in Proposition 4.4, then we get the same conclusion.

Theorem 4.6. *Assume that (A.1)-(A.5), (fg), (f.1), (g.1), (f.2) and (g.2) hold. Then for each* $\lambda\in(\lambda_*,\infty)$, *the problem* (Q_λ) *has a positive weak solution* u_λ *which is a local minimizer of the energy functional* φ_λ *defined by* (3.10). *Moreover, for* $\lambda_*<\lambda_2<\lambda_1$, *there exist the local minimizers* u_{λ_1} *and* u_{λ_2} *of the corresponding energy functionals* φ_{λ_1} *and* φ_{λ_2}, *respectively, such that* $u_{\lambda_1}<u_{\lambda_2}$ *on* $\overline{\Omega}$.

Proof. For $\lambda > \lambda_*$, choose $\lambda_1, \lambda_2 \in \Lambda$ so that $\lambda_2 < \lambda < \lambda_1$. By Theorem 4.2, (Q_{λ_1}) and (Q_{λ_2}) have positive weak solutions u_{λ_1} and u_{λ_2}, respectively, such that $u_{\lambda_1} \le u_{\lambda_2}$. By Proposition 4.4, there exists a positive weak solution u_λ of (Q_λ) which is a global minimizer of $\varphi_\lambda\big|_{[u_{\lambda_1}, u_{\lambda_2}] \cap W^{1,p(\cdot)}(\Omega)}$. By a strong comparison principle (Proposition 3.11), $u_{\lambda_1} < u_\lambda < u_{\lambda_2}$ on $\overline{\Omega}$. Thus there exists a $C^0(\overline{\Omega})$-neighborhood U of u_λ in the $C^0(\overline{\Omega})$-topology such that $U \subset [u_{\lambda_1}, u_{\lambda_2}]$. Hence u_λ is a local minimizer of φ_λ in the $C^0(\overline{\Omega})$-topology, so in the $C^1(\overline{\Omega})$-topology. By Proposition 3.10, u_λ is also a local minimizer of φ_λ in the $W^{1,p(\cdot)}(\Omega)$-topology.

Moreover, for $\lambda_* < \lambda_2 < \lambda_1$, let u_{λ_2} be a positive weak solution of (Q_{λ_2}) and a local minimizer of φ_{λ_2} constructed as above, then $u_{\lambda_2} > 0$ on $\overline{\Omega}$. By Remark 4.5, there exists a positive weak solution u_{λ_1} of (Q_{λ_1}) which is a global minimizer of $\varphi_{\lambda_1}\big|_{[0, u_{\lambda_2}] \cap W^{1,p(\cdot)}(\Omega)}$ and so $0 < u_{\lambda_1} \le u_{\lambda_2}$. By using again a strong comparison principle (Proposition 3.11), we see that $0 < u_{\lambda_1} < u_{\lambda_2}$ on $\overline{\Omega}$. So u_{λ_1} is a local minimizer of φ_{λ_1}. □

Theorem 4.7. *Assume that (A.1)-(A.5), (fg), (f.1), ($\widetilde{g}$,1) in Corollary 4.3, (f.2) and (g.2) hold. Moreover, suppose that* (f_+), (g_+) *and*

(f.3) There exist $M_2 > 0$ *and* $\theta > p^+$ *such that*

$$0 < \theta F(x,t) \le t f(x,t) \text{ for all } x \in \Omega \text{ and all } t \ge M_2.$$

Then for each $\lambda \in (\lambda_*, \infty)$*, the problem* (Q_λ) *has at least two positive weak solutions* u_λ *and* v_λ *with* $u_\lambda \le v_\lambda$*, where* u_λ *is a local minimizer of the energy functional* φ_λ*.*

Proof. Step 1. Existence of the first solution.

For $\lambda > \lambda_*$, choose $\lambda_1, \lambda_2 \in \Lambda$ so that $\lambda_2 < \lambda < \lambda_1$ and let $u_{\lambda_1} \le u_\lambda \le u_{\lambda_2}$ be, as in Theorem 4.6, the positive weak solutions of (Q_{λ_1}), (Q_λ), (Q_{λ_2}), respectively with u_λ being a local minimizer of φ_λ in the $W^{1,p(\cdot)}(\Omega)$-topology. Define

$$\overline{f}_\lambda(x,t) = \begin{cases} f(x,t) & \text{if } t > u_\lambda(x), \\ f(x, u_\lambda(x)) & \text{if } t \le u_\lambda(x), \end{cases} \quad \overline{g}_\lambda(x,t) = \begin{cases} g(x,t) & \text{if } t > u_\lambda(x), \\ g(x, u_\lambda(x)) & \text{if } t \le u_\lambda(x), \end{cases}$$

and $\overline{F}_\lambda(x,t) = \int_0^t \overline{f}_\lambda(x,s)ds$, $\overline{G}_\lambda(x,t) = \int_0^t \overline{g}_\lambda(x,s)ds$. Consider the problem

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x, \boldsymbol{\nabla} u(x))] + \lambda |u(x)|^{p(x)-2} u(x) = \overline{f}_\lambda(x, u(x)) & \text{in } \Omega, \\ \boldsymbol{n}(x) \cdot \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) = \overline{g}_\lambda(x, u(x)) & \text{on } \partial\Omega \end{cases} \qquad (\overline{Q}_\lambda)$$

and denote by $\overline{\varphi}_\lambda$ the energy functional corresponding to $(\overline{Q}_\lambda)$. By the definitions of $\overline{f}_\lambda$, $\overline{g}_\lambda$, (f.2) and (g.2), for every $u \in W^{1,p(\cdot)}(\Omega)$, we have

$$\overline{f}_\lambda(x, u(x)) \ge f(x, u_\lambda(x)) \text{ and } \overline{g}_\lambda(x, u(x)) \ge g(x, u_\lambda(x)).$$

For each weak solution u of $(\overline{Q}_\lambda)$, we have

$$\begin{aligned}\langle \psi_\lambda'(u), v\rangle &= \int_\Omega \overline{f}_\lambda(x,u(x))v(x)dx + \int_{\partial\Omega} \overline{g}_\lambda(x,u(x))v(x)d\sigma_x \\ &\geq \int_\Omega f(x,u_\lambda(x))v(x)dx + \int_{\partial\Omega} g(x,u_\lambda(x))v(x)d\sigma_x \\ &= \langle \psi_\lambda'(u_\lambda), v\rangle\end{aligned}$$

for all $v \in W^{1,p(\cdot)}(\Omega)$ with $v \geq 0$. Hence it follows from a comparison principle (Proposition 3.4) that $u \geq u_\lambda$. So, $\overline{f}_\lambda(x,u(x)) = f(x,u(x))$ and $\overline{g}_\lambda(x,u(x)) = g(x,u(x))$. Therefore, u is also a weak solution of (Q_λ). It is easy to see that u_{λ_1} and u_{λ_2} are a subsolution and supersolution of $(\overline{Q}_\lambda)$, respectively. By a sub-supersolution principle (Proposition 3.9) and the proof of Theorem 4.6, there exists $u_\lambda^* \in [u_{\lambda_1}, u_{\lambda_2}] \cap C^1(\overline{\Omega})$ such that u_λ^* is a weak solution of $(\overline{Q}_\lambda)$ and is also a local minimizer of $\overline{\varphi}_\lambda$ in the $C^1(\overline{\Omega})$-topology, so in the $W^{1,p(\cdot)}(\Omega)$-topology by Proposition 3.10. Hence $u_\lambda^* \geq u_\lambda$ and u_λ^* is also a positive weak solution of (Q_λ).

Step 2. Existence of the second solution.

If $u_\lambda^* \neq u_\lambda$, then the assertion of Theorem 4.7 already holds with $v_\lambda = u_\lambda^*$, hence we can assume that $u_\lambda^* = u_\lambda$. Thus u_λ is a local minimizer of $\overline{\varphi}_\lambda$ in the $C^1(\overline{\Omega})$-topology, so in the $W^{1,p(\cdot)}(\Omega)$-topology from Proposition 3.10. We can assume that u_λ is a strictly local minimizer of $\overline{\varphi}_\lambda$ in the $W^{1,p(\cdot)}(\Omega)$-topology. Indeed, if u_λ is not a strictly local minimizer of $\overline{\varphi}_\lambda$ in the $W^{1,p(\cdot)}(\Omega)$-topology, then for any $\varepsilon > 0$, there exists $\overline{u}_\lambda \in [u_{\lambda_1}, u_{\lambda_2}] \cap W^{1,p(\cdot)}(\Omega)$ such that $0 < \|\overline{u}_\lambda - u_\lambda\|_\lambda < \varepsilon$ and $\overline{u}_\lambda$ is a local minimizer of $\overline{\varphi}_\lambda$ in the $W^{1,p(\cdot)}(\Omega)$-topology. Hence $\overline{\varphi}_\lambda(\overline{u}_\lambda) = \overline{\varphi}_\lambda(u_\lambda)$, $\overline{u}_\lambda \neq u_\lambda$ and $\overline{u}_\lambda \geq u_\lambda$, so the assertion of Theorem 4.7 holds.

Thus we can assume that u_λ is a strictly local mimimizer of $\overline{\varphi}_\lambda$. Then there exists $\delta > 0$ such that

$$\overline{\varphi}_\lambda(u_\lambda) < c_\lambda := \inf\{\overline{\varphi}_\lambda(u); \|u - u_\lambda\|_\lambda = \delta, u \in W^{1,p(\cdot)}(\Omega)\}.$$

Under the hypotheses, we can see that $\overline{\varphi}_\lambda$ satisfies the (PS)-condition. Indeed, let $\{u_n\}_{n=1}^\infty \subset W^{1,p(\cdot}(\Omega)$ satisfy that $c := \sup_{n\in\mathbb{N}} |\overline{\varphi}_\lambda(u_n)| < \infty$ and $\|\overline{\varphi}_\lambda'(u_n)\|_{(W^{1,p(\cdot)}(\Omega))^*} \to 0$ as $n \to \infty$. Then by using (2.5), we have

$$\begin{aligned} c \geq \overline{\varphi}_\lambda(u_n) &= \psi_\lambda(u_n) - \int_\Omega \overline{F}_\lambda(x,u_n(x))dx - \int_{\partial\Omega} \overline{G}_\lambda(x,u_n(x))d\sigma_x \\ &\geq \frac{1}{p^+}\langle \psi_\lambda'(u_n), u_n\rangle - \int_\Omega \frac{h(x)}{p(x)}dx - \int_\Omega \overline{F}_\lambda(x,u_n(x))dx - \int_{\partial\Omega} \overline{G}_\lambda(x,u_n(x))d\sigma_x. \end{aligned} \tag{4.2}$$

Here we note that if $t \leq 0$, then it follows from (f_+) that

$$\theta F(x,t) = \theta \int_0^t f(x,s)ds = \theta t f(x,0) \leq t f(x,t).$$

Moreover, for $0 \le t \le M_2$, it follows from (f.2) that

$$\theta F(x,t) \le \theta t f(x,t) = tf(x,t) + (\theta-1)tf(x,t) \le tf(x,t) + (\theta-1)M_2 f(x,M_2).$$

Thus we have

$$\theta F(x,t) \le tf(x,t) + c_1 \text{ for all } x \in \Omega \text{ and all } t \in \mathbb{R}, \tag{4.3}$$

where c_1 is a positive constant. By simple calculation we can see that

$$\int_\Omega \overline{F}_\lambda(x,u_n(x))dx \le \int_\Omega F(x,u_n(x))dx + c_2 \le \frac{1}{\theta}\int_\Omega u_n(x)f(x,u_n(x))dx + c_1 + c_2, \tag{4.4}$$

where c_2 is a positive constant. Moreover, we have

$$\begin{aligned}\frac{1}{\theta}\langle \overline{\varphi}'_\lambda(u_n), u_n\rangle &= \frac{1}{\theta}\langle \psi'_\lambda(u_n), u_n\rangle \\ &\quad - \frac{1}{\theta}\int_\Omega \overline{f}_\lambda(x,u_n(x))u_n(x)dx - \frac{1}{\theta}\int_{\partial\Omega} \overline{g}_\lambda(x,u_n(x))u_n(x)d\sigma_x. \end{aligned} \tag{4.5}$$

Here we note that

$$\begin{aligned}&\int_\Omega \overline{f}_\lambda(x,u_n(x))u_n(x)dx \\ &\quad = \int_\Omega f(x,u_n(x))u_n(x)dx \qquad (4.6)\\ &\qquad + \int_\Omega (\overline{f}_\lambda(x,u_n(x)) - f(x,u_n(x)))u_n(x)dx \\ &\quad = \int_\Omega f(x,u_n(x))u_n(x)dx + \int_{\Omega\cap\{x\in\Omega; u_n(x)\le u_\lambda(x)\}} (f(x,u_\lambda(x)) - f(x,u_n(x)))u_n(x)dx \\ &\quad \ge \int_\Omega f(x,u_n(x))u_n(x)dx + \int_{\Omega\cap\{x\in\Omega; u_n(x)<0\}} f(x,u_\lambda(x))u_n(x)dx \\ &\quad \ge \int_\Omega f(x,u_n(x))u_n(x)dx - c_3\|u_n\|_\lambda, \qquad (4.7)\end{aligned}$$

for some constant c_3. Thus from (4.2)-(4.7), we obtain that

$$\begin{aligned}
c \geq \overline{\varphi}_\lambda(u_n) &\geq \frac{1}{p^+}\langle \psi'_\lambda(u_n), u_n\rangle - \int_\Omega F(x, u_n(x))dx - c_2 - \int_{\partial\Omega} \overline{G}_\lambda(x, u_n(x))d\sigma_x \\
&\geq \frac{1}{p^+}\langle \psi'_\lambda(u_n), u_n\rangle - \frac{1}{\theta}\int_\Omega u_n(x) f(x, u_n(x))dx \\
&- c_1 - c_2 - \int_{\partial\Omega} \overline{G}_\lambda(x, u_n(x))d\sigma_x \\
&\geq \frac{1}{p^+}\langle \psi'_\lambda(u_n), u_n\rangle - \frac{1}{\theta}\int_\Omega \overline{f}_\lambda(x, u_n(x))u_n(x)dx \\
&- c_1 - c_2 - \int_{\partial\Omega} \overline{G}_\lambda(x, u_n(x))d\sigma_x \\
&= \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\langle \psi'_\lambda(u_n), u_n\rangle + \frac{1}{\theta}\int_{\partial\Omega} \overline{g}_\lambda(x, u_n(x))u_n(x)d\sigma_x \\
&\quad + \frac{1}{\theta}\langle \overline{\varphi}'_\lambda(u_n), u_n\rangle - \int_{\partial\Omega} \overline{G}_\lambda(x, u_n(x))d\sigma_x - c_1 - c_2 \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) c_4\|u_n\|_\lambda^{p^-} - c_5\|u_n\|_\lambda^{r_0^+} - c_6\|u_n\|_\lambda - c_7,
\end{aligned}$$

where c_4, c_5, c_6 and c_7 are positive constants. Since $p^- > r_0^+ > 1$, $\{u_n\}$ is bounded in $W^{1,p(\cdot)}(\Omega)$. Since $\{u_n\}$ is bounded in a reflexive Banach space $W^{1,p(\cdot)}(\Omega)$, there exist a subsequene $\{u_{n'}\}$ of $\{u_n\}$ and $u_0 \in W^{1,p(\cdot)}(\Omega)$ such that $u_{n'} \to u_0$ weakly in $W^{1,p(\cdot)}(\Omega)$, so $u_{n'} \to u_0$ strongly in $L^{q(\cdot)}(\Omega)$ as $n' \to \infty$ from Proposition 2.3. Thus we see that

$$\lim_{n'\to\infty}\langle \psi'_\lambda(u_{n'}), u_{n'} - u_0\rangle = \lim_{n'\to\infty}\langle \overline{\varphi}'_\lambda(u_{n'}), u_{n'} - u_0\rangle = 0.$$

Since ψ'_λ is of (S_+)-type, it follows from Proposition 3.1(viii) that $u_{n'} \to u_0$ strongly in $W^{1,p(\cdot)}(\Omega)$. Thus $\overline{\varphi}_\lambda$ satisfies the (PS)-condition.

(f.3) implies that there exist positive constants c_8 and c_9 such that $F(x,t) \geq c_8 t^\theta - c_9$ for $x \in \overline{\Omega}$ and $t \geq 0$. Let $t \in (1, \infty)$ and $t > \max_{x\in\overline{\Omega}} u_\lambda(x)$. Then

$$\begin{aligned}
\overline{\varphi}_\lambda(t) &\leq \lambda\int_\Omega \frac{1}{p(x)}t^{p(x)}dx - \int_\Omega F(x,t)dx - \int_{\partial\Omega} G(x,t)d\sigma_x + c_{10} \\
&\leq \lambda t^{p^+}\int_\Omega \frac{1}{p(x)}dx - c_8 t^\theta|\Omega| + c_9 + c_{10}
\end{aligned}$$

for some positive constant c_{10}. Since $\theta > p^+$, we see that $\overline{\varphi}_\lambda(t) \to -\infty$ as $t \to \infty$. Hence $\inf\{\overline{\varphi}_\lambda(u); u \in W^{1,p(\cdot)}(\Omega)\} = -\infty$. Using the Mountain Pass Lemma (Ambrosetti and Rabinowitz [3]), $(\overline{Q}_\lambda)$ has a weak solution v_λ such that $\overline{\varphi}_\lambda(v_\lambda) \geq c_\lambda > \overline{\varphi}_\lambda(u_\lambda)$. Hence $v_\lambda \neq u_\lambda$ and $v_\lambda \geq u_\lambda$. Thus v_λ is also a positive weak solution of (Q_λ). □

Now we discuss the relation between λ_* and Λ.

Proposition 4.8. *Let (A.1)-(A.5), (fg), (f.2) and (g.2) hold. Suppose that there exists* $M_3 \geq 1$ *such that*

$$f(x,t) \geq t^{p(x)-1} \text{ for all } x \in \Omega \text{ and } t \geq M_3. \tag{4.8}$$

Then

$$\lambda_* \geq \min\left\{ \frac{b}{M_3^{p^+-1}|\Omega|}, 1 \right\},$$

where $b = \int_{\partial\Omega} g(x,0)d\sigma_x > 0$.

Proof. Let $\lambda \in \Lambda$ and u be a positive weak solution of (Q_λ). Taking $v \equiv 1$ as a test function of Definition 2.10(i), it follows from (g.2) that

$$\lambda\int_\Omega u(x)^{p(x)-1}dx - \int_\Omega f(x,u(x))dx = \int_{\partial\Omega} g(x,u(x))d\sigma_x \geq \int_{\partial\Omega} g(x,0)d\sigma_x = b > 0. \tag{4.9}$$

Arguing by contradiction, let $\lambda_* < \min\{b/(M_3^{p^+-1}|\Omega|), 1\}$. Then it follows from Theorem 4.2 that there exists $\lambda \in \Lambda$ such that $\lambda < 1$ and $\lambda < b/(M_3^{p^+-1}|\Omega|)$. Setting $\Omega_1 = \{x \in \Omega; u(x) < M_3\}$ and $\Omega_2 = \{x \in \Omega; u(x) \geq M_3\}$, then

$$\begin{aligned}
&\lambda \int_\Omega u(x)^{p(x)-1}dx - \int_\Omega f(x,u(x))dx \\
&\qquad \leq \lambda \int_{\Omega_1} u(x)^{p(x)-1}dx + \int_{\Omega_2} u(x)^{p(x)-1}dx - \int_{\Omega_2} f(x,u(x))dx \\
&\qquad\qquad \leq \lambda \int_{\Omega_1} u(x)^{p(x)-1}dx \leq \lambda M_3^{p^+-1}|\Omega| < b.
\end{aligned}$$

This contradicts (4.9). ☐

Remark 4.9. (i) (f.3) implies that $F(x,t) \geq c_3 t^\theta - c_4$ for $t > 0$ with some positive constants c_3 and c_4, and (4.8) holds.

(ii) Let $u \geq 0$. Then it follows from (f.2) that

$$\begin{aligned}
\int_\Omega F(x,u(x))dx &= \int_\Omega \left(\int_0^{u(x)} f(x,s)ds \right) dx \\
&\leq \int_\Omega \left(\int_0^{u(x)} f(x,u(x))ds \right) dx = \int_\Omega f(x,u(x))u(x)dx.
\end{aligned}$$

Similarly, it follows from (g.2) that

$$\int_{\partial\Omega} G(x,u(x))d\sigma_x \leq \int_{\partial\Omega} g(x,u(x))u(x)d\sigma_x.$$

Proposition 4.10. *Let (A.1)-(A.5), (fg), (f.1), (g.1), (f.2) and (g.2) hold. Then for each* $\lambda > \lambda_*$*, the problem* (Q_λ) *has a positive weak solution* u_λ *which is a local minimizer of* φ_λ *such that* $\varphi_\lambda(u_\lambda) < 0$*.*

Proof. Let $\lambda > \lambda_*$. Choose $\lambda_2 \in (\lambda_*, \lambda)$ and let u_{λ_2} be a positive weak solution of (Q_{λ_2}). By Remark 4.5, the problem (Q_λ) has a positive weak solution $u_\lambda \in [0, u_{\lambda_2}]$ such that

$$\varphi_\lambda(u_\lambda) = \inf\{\varphi_\lambda(u); u \in [0, u_{\lambda_2}] \cap W^{1,p(\cdot)}(\Omega)\}.$$

Hence $\varphi_\lambda(u_\lambda) \le \varphi_\lambda(0) = 0$.

On the other hand, putting $v_t \equiv t$ on $\overline{\Omega}$, where t is a positive constant, then for sufficiently small $t > 0$, we have $v_t \in [0, u_{\lambda_2}] \cap W^{1,p(\cdot)}(\Omega)$ and

$$\begin{aligned}\varphi_\lambda(v_t) &= \lambda \int_\Omega \frac{1}{p(x)} t^{p(x)} dx - \int_\Omega F(x,t) dx - \int_{\partial\Omega} \left(\int_0^t g(x,s) ds \right) d\sigma_x \\ &\le \lambda t^{p^-} \int_\Omega \frac{1}{p(x)} dx - t \int_{\partial\Omega} g(x,0) d\sigma_x < 0.\end{aligned}$$

Hence $\varphi_\lambda(u_\lambda) \le \varphi_\lambda(v_t) < 0$. □

Theorem 4.11. *Under the assumptions of Theorem 4.7, if there exists a sequence* $\{\lambda_n\} \subset \Lambda$ *such that* $\lambda_n \to \lambda_0$ *as* $n \to \infty$*, then* $\lambda_0 > 0$*,* $\lambda_0 \in \Lambda$ *and* $u_{\lambda_n} \to u_{\lambda_0}$ *in* $C^1(\overline{\Omega})$*, where* u_{λ_n} *is, as in Theorem 4.6, a local minimizer of* φ_{λ_n} *and* u_{λ_0} *is a positive weak solution of* (Q_{λ_0})*. In particular,* $\lambda_* > 0$ *and* $\lambda_* \in \Lambda$*.*

Proof. Let $\{\lambda_n\} \subset \Lambda$ such that $\lambda_n \to \lambda_0$ as $n \to \infty$. By Proposition 4.8 and Remark 4.9(i), $\lambda_0 \ge \lambda_* > 0$. By Proposition 4.10, for each n, φ_{λ_n} has a local minimizer u_{λ_n} such that $\varphi_{\lambda_n}(u_{\lambda_n}) < 0$, that is,

$$\psi_{\lambda_n}(u_{\lambda_n}) < \int_\Omega F(x, u_{\lambda_n}(x)) dx + \int_{\partial\Omega} G(x, u_{\lambda_n}(x)) d\sigma_x.$$

On the other hand, since u_{λ_n} is a positive weak solution of (Q_{λ_n}), we have

$$\langle \psi'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} \rangle = \int_\Omega f(x, u_{\lambda_n}(x)) u_{\lambda_n}(x) dx + \int_{\partial\Omega} g(x, u_{\lambda_n}(x)) u_{\lambda_n}(x) d\sigma_x.$$

By (f.3) and Remark 4.9(ii),

$$\begin{aligned}&\int_\Omega F(x, u_{\lambda_n}(x)) dx + \int_{\partial\Omega} G(x, u_{\lambda_n}(x)) d\sigma_x \\ &\qquad \le \frac{1}{\theta} \int_\Omega f(x, u_{\lambda_n}(x)) u_{\lambda_n}(x) dx + \int_{\partial\Omega} g(x, u_{\lambda_n}(x)) u_{\lambda_n}(x) d\sigma_x + c_3\end{aligned}$$

for some constant $c_3 > 0$. Thus we have

$$\frac{1}{p^+}\langle \psi'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n}\rangle \le \psi_{\lambda_n}(u_{\lambda_n}) + c_4$$
$$< \frac{1}{\theta}\langle \psi'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n}\rangle + \left(1-\frac{1}{\theta}\right)\int_{\partial\Omega} g(x, u_{\lambda_n}(x))u_{\lambda_n}(x))d\sigma_x + c_4$$

for some constant $c_4 > 0$. Hence, for large n,

$$\left(\frac{1}{p^+}-\frac{1}{\theta}\right)\langle \psi'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n}\rangle \le \left(1-\frac{1}{\theta}\right)\int_{\partial\Omega} g(x, u_{\lambda_n}(x))u_{\lambda_n}(x)d\sigma_x + c_4$$
$$. \le \left(1-\frac{1}{\theta}\right)c_5\int_{\partial\Omega} |u_{\lambda_n}(x)|^{r_0^+}d\sigma_x + c_6$$

for some positive constants c_5 and c_6. Hence if $\|u_{\lambda_n}\|_{\lambda_n} \ge 1$, then

$$\left(\frac{1}{p^+}-\frac{1}{\theta}\right)c_7\|u_{\lambda_n}\|_{\lambda_n}^{p^-} \le c_8\|u_{\lambda_n}\|_{\lambda_0/2}^{r_0^+} + c_9 \le c_8\|u_{\lambda_n}\|_{\lambda_n}^{r_0^+} + c_9$$

for some positive constants c_7, c_8 and c_9. Since $\theta > p^+$ and $p^- > r_0^+$, we see that $\|u_{\lambda_n}\|_{\lambda_n}$ is bounded, so $\|u_{\lambda_n}\|_{W^{1,p(\cdot)}(\Omega)}$ is bounded uniformly in n. Since $W^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space, passing to a subsequence, we can assume that $u_{\lambda_n} \to u_0$ weakly in $W^{1,p(\cdot)}(\Omega)$, and so we may assume that $u_{\lambda_n}(x) \to u_0(x) \in L^{p(\cdot)}(\Omega)$ for a.e. $x \in \Omega$. By Proposition 3.2(i), the boundedness of $\{\|u_{\lambda_n}\|_{\lambda_n}\}$ implies the boundedness of $\{\|u_{\lambda_n}\|_{L^\infty(\Omega)}\}$. By Proposition 3.2(iii), the boundedness of $\{\|u_{\lambda_n}\|_{L^\infty(\Omega)}\}$ implies the boundedness of $\{\|u_{\lambda_n}\|_{C^{1,\alpha}(\overline{\Omega})}\}$, where $\alpha \in (0,1)$ is a constant. Passing again to a subsequence, we can assume that $u_{\lambda_n} \to u_0$ in $C^1(\overline{\Omega})$. For every $v \in W^{1,p(\cdot)}(\Omega)$, since u_{λ_n} is a weak solution of (Q_{λ_n}), we have

$$\int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u_{\lambda_n}(x))\cdot\boldsymbol{\nabla} v(x)dx + \lambda_n\int_\Omega |u_{\lambda_n}(x)|^{p(x)-2}u_{\lambda_n}(x)v(x)dx$$
$$= \int_\Omega f(x, u_{\lambda_n}(x))v(x)dx + \int_{\partial\Omega} g(x, u_{\lambda_n}(x))v(x)d\sigma_x. \quad (4.10)$$

Letting $n \to \infty$ in (4.10), we have

$$\int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u_0(x))\cdot\boldsymbol{\nabla} v(x)dx + \lambda_0\int_\Omega |u_0(x)|^{p(x)-2}u_0(x)v(x)dx$$
$$= \int_\Omega f(x, u_0(x))v(x)dx + \int_{\partial\Omega} g(x, u_0(x))v(x)d\sigma_x$$

which shows that u_0 is a weak solution of (Q_{λ_0}). Since $u_{\lambda_n} > 0$ on Ω, we have $u_0 \ge 0$ in Ω. Since $g(x, u_0(x)) \ge g(x, 0) \not\equiv 0$ by (fg) and (g.2), we have $g(x, u_0(x)) \not\equiv 0$. By Lemma 4.1, $u_0 > 0$ on $\overline{\Omega}$, so $\lambda_0 \in \Lambda$, and it suffices to put $u_{\lambda_0} = u_0$. □

Remark 4.12. (i) Under the assumptions of Theorem 4.11, (Q_{λ_*}) has at least one positive weak solution $u_{\lambda_*} = \lim_{n\to\infty} u_{\lambda_n}$.

(ii) If $\lambda < \lambda_*$, then (Q_λ) has no positive weak solution.

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