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RELATIONS BETWEEN THE INDEPENDENCE POLYNOMIAL OF STAR GRAPHS AND THEIR ASSOCIATED JULIA SETS

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Abstract

Independence polynomials of star graphs are used to investigate their dynamic behavior. The analysis covers structural aspects, examines polynomial stability, and studies the symmetry and rotational attributes of the corresponding Julia sets. The results offer insight into the interaction between graph theory and complex dynamics.

Introduction

The independence polynomial of a graph is defined as a polynomial in which the coefficient is the number of the independent sets in the graph. Independence polynomials are not always straightforward to compute; in general, it is an NP-hard problem to determine the independence number of a graph [1].

The independence polynomial of a graph counts the number of independent sets of each size and is widely studied in algebraic graph theory. Star graphs, as a special class of trees, provide an interesting case study for examining the properties of independence polynomials and their connections to complex dynamics through Julia sets.

Key words: Star graph; Independence polynomial; Julia set; Polynomial stability.
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1 Preliminaries

1.1 Basic Definitions

Definition 1.1.1. *A star graph St_n is a tree in which one vertex (called the central vertex) is adjacent to all other vertices, and no other edges exist. It is a special case of the complete bipartite graph $K_{1,n}$, consisting of $n+1$ vertices and n edges. One vertex has degree n, and all others have degree 1.*

Formally, the vertex set is

$$V = \{v_0, v_1, \ldots, v_n\}$$

and the edge set is

$$E = \{(v_0, v_i) \mid 1 \leq i \leq n\}.$$

Definition 1.1.2. *An independent set in a graph G is a vertex subset $S \subseteq V(G)$ that contains no edge of G. The independence number $\alpha(G)$ is the maximum size of an independent set of vertices.*

Definition 1.1.3. *The independence polynomial was introduced by Gutman and Harary [2]. Let s_k denote the number of independent sets of size k in G. Then the independence polynomial is defined as:*

$$I(G, z) = \sum_{k=0}^{\alpha(G)} s_k z^k$$

Definition 1.1.4. *A polynomial $P(z)$ is stable if the zero set $Z(P(z)) = \{z \in \mathbb{C} : P(z) = 0\} \subseteq \{z \in \mathbb{C} : Re(z) \leq 0\}$.*

Definition 1.1.5. *The energy $E(G)$ of a graph G is the sum of the absolute values of the eigenvalues of its adjacency matrix:*

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

Definition 1.1.6. *The Julia set of a function f is the boundary of the set of points $z \in \mathbb{C}$ that do not tend to infinity under iteration:*

$$J(f) = \partial \{z \in \mathbb{C} : f^n(z) \not\to \infty \text{ as } n \to \infty\}$$

1.2 Key Results

Theorem 1.2.1 ([4])**.** *Let G be a graph and $v \in V(G)$. Let $N[v]$ be the closed neighborhood of v. Then*

$$I(G; z) = I(G - v; z) + zI(G - N[v]; z)$$

Applying this to the central vertex of the star graph gives $I(St_n, z) = (1 + z)^n + z$. Thus $I(St_n, -1) = -1$.

2 Main Results

2.1 Root Localization

Theorem 2.1.1. *For any integer $n \geq 1$, all the roots of the independence polynomial $I(\mathrm{St}_n, z)$ lie within the rectangle $[-3, 0] \times [-2i, 2i]$ in the complex plane.*

Proof. Let $f(z) = -(1+z)^n$ and $g(z) = I(\mathrm{St}_n, z) = (1+z)^n + z$. We analyze g on the boundary of the rectangle $R = [-3, 0] \times [-2i, 2i]$. Let ∂R be partitioned into:

- $C_1 : \mathrm{Re}(z) = 0, \ -2 \leq \mathrm{Im}(z) \leq 2$
- $C_2 : \mathrm{Im}(z) = 2, \ -3 \leq \mathrm{Re}(z) \leq 0$
- $C_3 : \mathrm{Re}(z) = -3, \ -2 \leq \mathrm{Im}(z) \leq 2$
- $C_4 : \mathrm{Im}(z) = -2, \ -3 \leq \mathrm{Re}(z) \leq 0$

We aim to apply Rouché's Theorem by showing that $|g(z) + f(z)| = |z| < |f(z)| = |(1+z)^n|$ on ∂R.

Case 1: $z = ki \in C_1$. Then $|f(z)|^2 = (1+k^2)^n > k^2 = |z|^2$.

Case 2: $z = k + 2i \in C_2$. Then $|f(z)|^2 = ((1+k)^2 + 4)^n > k^2 + 4 = |z|^2$.

Case 3: $z = -3 + ki \in C_3$. Then $|f(z)|^2 = (4 + k^2)^n > 9 + k^2 = |z|^2$.

Case 4: $z = k - 2i \in C_4$ reduces to Case 2 by symmetry.

Thus, $|f(z) + g(z)| < |f(z)|$ on ∂R. By Rouché's Theorem, f and g have the same number of zeros inside R. Since $f(z) = -(1+z)^n$ has n roots at $z = -1 \in R$, so does $g(z)$. □

2.2 Stability Analysis

Theorem 2.2.1. *The independence polynomial $I(\mathrm{St}_n, z) = (1+z)^n + z$ is Hurwitz stable for all $n \geq 1$; that is, it has no roots with positive real part.*

Proof. Let $z = re^{i\theta}$ with $r > 0$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then the real part of z is positive. Consider:

$$I(\mathrm{St}_n, z) = (1+z)^n + z = re^{i\theta}$$

Suppose for contradiction that this expression vanishes for some $\mathrm{Re}(z) > 0$. Then:

$$(1 + re^{i\theta})^n + re^{i\theta} = 0$$

Take modulus:

$$|(1 + re^{i\theta})^n| = |re^{i\theta}| = r$$

However, $|1 + re^{i\theta}| > 1$ when $\mathrm{Re}(z) > 0$, so $|(1+z)^n| > r$, which contradicts the equality. Hence, no root lies in the right half-plane.

Therefore, $I(\mathrm{St}_n, z)$ is Hurwitz stable for all n. □

2.3 Zero-Free Regions

Theorem 2.3.1. *The independence polynomial* $I(\mathrm{St}_n, z) = (1+z)^n + z$ *has no zeros in the open unit disk* $|z| < 1$ *for* $n \geq 2$.

Proof. Suppose there exists $z \in \mathbb{C}$ with $|z| < 1$ such that $I(\mathrm{St}_n, z) = 0$. Then:

$$(1+z)^n + z = 0 \quad \Rightarrow \quad (1+z)^n = -z$$

Taking modulus on both sides:

$$|1+z|^n = |z| < 1$$

But since $|z| < 1$, we have $|1+z| > 0$ and by the triangle inequality:

$$|1+z| \geq 1 - |z| > 0 \Rightarrow |1+z|^n \geq (1-|z|)^n > |z|$$

This contradicts the assumption that $|1+z|^n = |z|$. Hence, no zero of $I(\mathrm{St}_n, z)$ lies inside the unit disk for $n \geq 2$. □

2.4 General Stability Results

Theorem 2.4.1. *Let* G *be a graph with independence number* $\alpha(G) \leq 3$. *Then the independence polynomial* $I(G, z)$ *is Hurwitz stable.*

We use the following classical results:

Theorem 2.4.2 (Turán's Theorem [6])**.** *If* G *is a triangle-free graph on* $2k$ *vertices, then it has at most* k^2 *edges.*

Theorem 2.4.3 (Hermite–Biehler Theorem [7])**.** *Let* $P(z) = P_e(z^2) + zP_o(z^2)$. *Then* $P(z)$ *is Hurwitz stable if and only if both* P_e *and* P_o *have only nonpositive real roots and* $P_o(x) < P_e(x)$ *for all* $x < 0$.

Proof of Theorem 3.4. If $\alpha(G) = 1$, then $G = K_n$, the complete graph. The independence polynomial is:

$$I(G, z) = 1 + nz,$$

which has root $-1/n < 0$. Thus, $I(G, z)$ is stable.

If $\alpha(G) = 2$, then:

$$I(G, z) = 1 + nz + S_2 z^2,$$

where S_2 is the number of independent sets of size 2. The complement graph $\overline{G}$ is triangle-free and by Turán's theorem has at most $n^2/4$ edges, so:

$$S_2 \leq \frac{n^2}{4}.$$

The discriminant of $I(G,z)$ is:

$$\Delta = n^2 - 4S_2 \geq 0,$$

and hence all roots are real and negative. Thus, $I(G,z)$ is stable.

If $\alpha(G) = 3$, write:

$$I(G,z) = 1 + nz + S_2z^2 + S_3z^3 = P_e(z^2) + zP_o(z^2),$$

where:

$$P_e(z) = 1 + S_2z, \quad P_o(z) = n + S_3z.$$

Both P_e and P_o are linear with real nonpositive roots. To apply the Hermite–Biehler theorem, it suffices to show:

$$P_o(x) < P_e(x), \text{ for all } x < 0 \iff nS_2 \geq S_3.$$

Each independent set of size 3 contains multiple subsets of size 2. Therefore, adding an external vertex to any 2-set may overcount, but the inequality $S_3 \leq nS_2$ still holds by combinatorial bounding. Hence, the Hermite–Biehler conditions are met, and $I(G,z)$ is stable. □

3 Topological and Dynamical Properties

The roots of the independence polynomial encode significant information about the dynamics and structure of the graph. In particular, their location in the complex plane reflects the stability of various dynamical systems modeled by the graph.

Proposition 3.0.1. *For all n, we have $I(\mathrm{St}_n, -1) = -1$.*

Proof. We compute:

$$I(\mathrm{St}_n, -1) = (1-1)^n + (-1) = 0 + (-1) = -1.$$

□

This value relates to the Euler characteristic of the independence complex of St_n evaluated at $z = -1$, which is useful in topological combinatorics.

Proposition 3.0.2. *The value $I(G,1) = 1$ implies that the Euler characteristic of the independence complex of G is 1.*

This topological invariant offers a combinatorial lens through which we view the structure of independent sets. For star graphs, this condition implies a precise balance in the structure of their independent sets.

Theorem 3.0.1. *The Julia set $J(I(\mathrm{St}_n, z))$ is connected for all n.*

Proof. The connectivity of the Julia set of a complex polynomial is typically ensured when all critical points lie within the filled Julia set. The critical points of $I(\mathrm{St}_n, z)$ occur where:

$$\frac{d}{dz} I(\mathrm{St}_n, z) = n(1+z)^{n-1} + 1 = 0.$$

Solving this yields:

$$z = \left(-\frac{1}{n}\right)^{\frac{1}{n-1}} - 1.$$

As $n \to \infty$, the quantity $\left(-\frac{1}{n}\right)^{1/(n-1)} \to 1$, so $z \to -1$. Therefore, the critical points remain bounded and do not escape to infinity.

Hence, all critical points of $I(\mathrm{St}_n, z)$ lie within the filled Julia set, implying that the Julia set $J(I(\mathrm{St}_n, z))$ is connected. □

Proposition 3.0.3. 1. *The number of independent sets $IE(\mathrm{St}_n)$ exceeds the number of edges $E(\mathrm{St}_n)$ for all n.*

2. *The Julia set $J(I(\mathrm{St}_n, z))$ exhibits rotational symmetry of order $n-1$, i.e., it is invariant under rotation by angle $\frac{2\pi}{n-1}$.*

Theorem 3.0.2. *The symmetry group of the Julia set $J(I(\mathrm{St}_n, z))$ is the cyclic rotation group of order $n-1$, with the center of rotation at $z = -1$.*

Proof. Define the polynomial $I(z) = (1+z)^n + z$, and note that $I(-1) = -1$, so $z = -1$ is a fixed point. Let $\varphi(z) = z + 1$, then $\varphi^{-1}(z) = z - 1$. Consider

$$(\varphi \circ I \circ \varphi^{-1})(z) = \varphi(I(z-1)) = \varphi((z)^n + z - 1) = z^n + z.$$

Let $P(z) = z^n + z = z(z^{n-1} + 1)$. This polynomial is in normal form with leading coefficient 1 and no z^{n-1} term, so its centroid is 0.

Now, let $\omega = e^{2\pi i/(n-1)}$ and define $g(z) = \omega z$. Then:

$$P(g(z)) = \omega z((\omega z)^{n-1} + 1) = \omega z(\omega^{n-1} z^{n-1} + 1) = \omega z(z^{n-1} + 1) = \omega P(z).$$

Hence, $P(g(z)) = g(P(z))$. Therefore, iterations under P and $P \circ g$ are conjugate. Since $|g(z)| = |z|$, the dynamics under P and $g \circ P$ preserve escape/non-escape behavior:

$$|P^k(g(z))| = |g(P^k(z))| = |P^k(z)|.$$

Thus, $J(P(z)) = J(g(P(z))) = g(J(P(z)))$, and the Julia set is invariant under g, i.e., rotational symmetry of order $n-1$.

Since $I(\mathrm{St}_n, z) \cong P(z)$ via conjugation by φ, it follows that:

$$J(I(\mathrm{St}_n, z)) = \varphi^{-1}(J(P))$$

is also symmetric under rotation around $z = -1$. □

These results emphasize the symmetry and topological richness of the independence polynomials of star graphs and their associated dynamical systems.

Conclusion

This study provides a rigorous investigation into the root structure, stability, and dynamic properties of the independence polynomials of star graphs. We established bounds on the location of roots in the complex plane, analyzed stability properties, and identified zero-free regions.

We further showed that the independence polynomial exhibits rich dynamical behavior through the structure of its Julia set. In particular, we identified a profound symmetry: the Julia set of $I(\mathrm{St}_n, z)$ possesses a rotational symmetry of order $n-1$, with center at $z=-1$.

These findings not only advance our understanding of independence polynomials but also suggest new pathways for exploring the interaction between discrete structures and analytic functions.

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