East-West J. of Mathematics: Vol. 26, No 2 (2025) pp. 117-128
https://doi.org/10.36853/ewjm0415

CATEGORY OF PRESHEAVES OF A LIE GROUPOID AND REPRESENTATION OF LIE GROUPOIDS

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Abstract

A representation of a Lie groupoid $\mathcal{G} := G \rightrightarrows M$ is a Lie groupoid morphism of $\mathcal{G}$ to a Lie groupoid of manifolds. In this article, we provide a representation of a Lie groupoid $\mathcal{G}$ via the functor category of presheaves of $\mathcal{G}$ and their equivalence to $Man_{\mathcal{G}}$, the Lie groupoid of right $\mathcal{G}$-manifolds and right $\mathcal{G}$-manifold morphisms.

Introduction

In recent years the unified framework of Category theory gained much acceptance in every scientific discourse. Physicists have also adopted the approach of interpreting physical states as objects and physical processes as morphisms, thus forming a category which corresponds to a physical system. In order to provide such a realization of a physical system, category theory intertwines with the theory of smooth manifolds, leading to the concept of a Lie categories and Lie groupoids.

One of the reasons behind this surge of interest in Lie groupoids is that it naturally appear in various areas of mathematics and mathematical physics. For example, in foliation theory (as foliation groupoid or monodromy groupoid

Key words: Category, Lie groupoids, Presheaves, Representations.
2000 AMS Mathematics Subject Classification: Primary 22E45, 22E46 Secondary 18A05

for a foliated manifold), theory of orbifolds (as proper, étale Lie groupoids), differentiable stacks (as Morita equivalence classes of Lie groupoids). This makes the geometry of Lie groupoids and their related structures on its own an important direction of inquiry. A Lie groupoid can be viewed as a generalization of either a smooth manifold or a Lie group.

1 Preliminaries

In this section we recall all definitions and results in categories and groupiods needed.

Definition 1.0.1. *(cf. [5]) A category $\mathcal{C}$ consists of the following data:*

1. *A class $\nu\mathcal{C}$ called the class of vertices or objects .*
2. *A class of disjoint sets $\mathcal{C}(a,b)$ one for each pair $(a,b) \in \nu\mathcal{C} \times \nu\mathcal{C}$. An element $f \in \mathcal{C}$ is called a morphism from a to b, written $f : a \to b$; $a =$ dom f called the domain of f and $b =$ cod f called the codomain of f.*
3. *For $a, b, c, \in \nu\mathcal{C}$, a map $\circ : \mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,c)$ such that $(f,g) \to g \circ f$ called the composition of morphisms in $\mathcal{C}$.*
4. *for each $a \in \nu\mathcal{C}$, a unique $1_a \in \mathcal{C}(a,a)$ is called the identity morphism on a.*

These must satisfy the following axioms :

- *The composition is associative : for $f \in \mathcal{C}(a,b), g \in \mathcal{C}(b,c)$ and $h \in \mathcal{C}(c,d)$, we have*
$$h \circ (g \circ f) = (h \circ g) \circ f$$
- *for each $a, b \in \nu\mathcal{C}, f \in \mathcal{C}\ (a,b)$; $f \circ 1_a = f = 1_b \circ f$*

Equivalently we will denote a category by $\mathcal{C} \rightrightarrows \mathcal{X}$ where $\mathcal{C}$ is the set of morphisms, X is the set of objects, and the two arrows indicate the source map $s : \mathcal{C} \to \mathcal{X}$ and target map $t : \mathcal{C} \to \mathcal{X}$ defined by

$$s(x \to y) = x, \quad t(x \to y) = y,$$

alongside these two maps, a category $\mathcal{C}$ comes equipped with the composition map

$$m : \mathcal{C}^{(2)} \to \mathcal{C}, \quad (g,h) \mapsto gh,$$

where $\mathcal{C}^{(2)} = \{(g,h) \in \mathcal{C} \times \mathcal{C} \mid s(g) = t(h)\}$ is the set of all pairs of composable morphisms. Also we denote

$$\mathcal{C}_x = s^{-1}(x), \quad \mathcal{C}^y = t^{-1}(y), \mathcal{C}_{\S}{}^y = \mathcal{C}_x \cap \mathcal{C}^y,$$

Following are some examples of categories:

Example 1.0.1. ***Set***: *The category in which objects are sets and morphisms are functions between the sets.*

If a subcollection $\mathcal{S}$ of objects and morphisms of $\mathcal{C}$, itself constitute a category then $\mathcal{S}$ is called a subcategory of $\mathcal{C}$.

Next we proceed to recall some basic concepts regarding preadictive categories, preorders and category with subobjects, for a detailed discussion see [8].

Definition 1.0.2. *(cf. [6]) A category $\mathcal{C}$ is called preadditive category or Ab-category if each hom-set $\mathcal{C}(A, B)$ is an additive abelian group and composition is bilinear relative to this addition. i.e.,*

$$(g + g') \circ (f + f') = (g \circ f) + (g \circ f') + (g' \circ f) + (g' \circ f')$$

where $f, f' : A \to B$ and $g, g' : B \to C$.

A preadditive category with a zero object in which every pair of objects admit a biproduct is called an additive category.

1.1 Functors and natural transformations

Structure preserving maps between the categories are called functors. Let $\mathcal{C}$ and $\mathcal{D}$, be two categories, a functor $F : \mathcal{C} \to \mathcal{D}$ consists of two functions: the object function denoted as vF which assigns to each object x of $\mathcal{C}$, an object $vF(x)$ of the category $\mathcal{D}$ and the morphism function denoted by F itself which assigns to each morphism $f : x \to y$ of $\mathcal{C}$, a morphism $F(f) : F(x) \to F(y)$ in $\mathcal{D}$ which preserves identities and composition, ie., $F(1_x) = 1_{F(x)}$ and $F(fg) = F(f)F(g)$. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be v-surjective, v-injective and v-bijective if the object map vF is surjective, injective and bijective respectively. A functor $F : \mathcal{C} \to \mathcal{D}$ is called full, faithful and fully-faithful if the morphism map F is surjective, injective and bijective respectively. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an isomorphism if it is v-bijective and fully-faithful.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of two mappings: the object mapping vF, which assigns to each object x in $\mathcal{C}$ an object $vF(x)$ in $\mathcal{D}$, and the morphism mapping F, which assigns to each morphism $f : x \to y$ in $\mathcal{C}$ a morphism $F(f) : F(x) \to F(y)$ in $\mathcal{D}$. These mappings preserve identities and composition, i.e., $F(1_x) = 1_{F(x)}$ and $F(fg) = F(f)F(g)$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be v-surjective, v-injective, or v-bijective depending on whether the object mapping vF is surjective, injective, or bijective, respectively. It is also classified as full, faithful, or fully-faithful based on whether the morphism mapping F is surjective, injective, or bijective. An isomorphism between $\mathcal{C}$ and $\mathcal{D}$ refers to a functor F that is v-bijective and fully-faithful.

Definition 1.1.1. *Let F and G be two functors between the categories $\mathcal{C}$ and $\mathcal{D}$, then a natural transformation σ from F to G associates each object c of $\mathcal{C}$ a morphism $\sigma(c) : F(c) \to G(c)$ in such a way that for all $f : c \to d$ the following diagram commutes.*

$$\begin{array}{ccccc} c & \quad & F(c) & \xrightarrow{\sigma_c} & G(c) \\ \downarrow f & \quad & F(f)\downarrow & & \downarrow G(f) \\ d & \quad & F(d) & \xrightarrow{\sigma_d} & G(d) \end{array}$$

We shall call $\sigma(c)$ as the component of the natural transformation σ at c. If every component $\sigma(c)$ is an isomorphism in D, then σ is a natural isomorphism and functors F and G are said to be naturally isomorphic. A category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms are natural transformations between the functors is termed as functor category and denote it by $[\mathcal{C}, \mathcal{D}]$. Given a category $\mathcal{C}$, $\mathcal{C}^*$ denotes the functor category $[\mathcal{C}, \mathbf{Set}]$. i.e, $\mathcal{C}^*$ is a category with the object class as the class of all functors from $\mathcal{C}$ to the category **Set** and natural transformations are the morphisms.

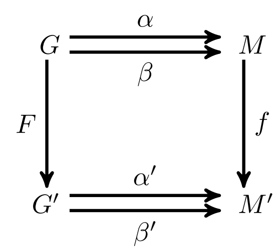
1.2 Groupoids and Lie groupoids.

A groupoid is a category such that every arrow has an inverse. In other words a groupoid consist of two sets $\mathbf{G}$ and $\mathbf{M}$ called the groupoid and the base respectively, together with two maps α and β from $\mathbf{G}$ to $\mathbf{M}$ called the source projection and target projection, a map $\mathbf{1}$: $\mathbf{M} \to \mathbf{G}$ with $x \mapsto 1_x$ called the object inclusion map, and a partial multiplication $(h, g) \mapsto hg$ in $\mathbf{G}$ defined on the set $\mathbf{G} * \mathbf{G} = \{(h, g) \in \mathbf{G} \times \mathbf{G} / \alpha(h) = \beta(g)\}$ all subject to the following conditionsm (For a formal definition see cf. [5]).

1. $\alpha(hg) = \alpha(g)$ and $\beta(hg) = \beta(h)$ $\forall$ $(h, g) \in \mathbf{G} * \mathbf{G}$.
2. $j(hg) = (jh)g$ $\forall$ j,h,g $\in \mathbf{G}$ such that $\alpha(j)= \beta(h)$ and $\alpha(h)= \beta(g)$.
3. $\alpha(1_x)= \beta(1_x) = x \quad \forall\ x \in \mathbf{M}$.
4. $g1_{\alpha(g)} = g$ and $1_{\beta(g)}g = g \quad \forall\ g \in \mathbf{G}$
5. Each $g \in \mathbf{G}$ has a 2 sided inverse g^{-1} such that $\alpha(g^{-1})= \beta(g)$ and $\beta(g^{-1})= \alpha(g)$ and $g^{-1}g = 1_{\alpha(g)}$, $gg^{-1} = 1_{\beta(g)}$.

A groupoid $\mathbf{G}$ with base $\mathbf{M}$ is often denoted by $\mathbf{G} \rightrightarrows \mathbf{M}$.

Definition 1.2.1. *Let $\mathbf{G}$ and $\mathbf{G}'$ be groupoids on $\mathbf{M}$ and $\mathbf{M}'$ respectively. A morphism between them is a pair of maps F: $\mathbf{G} \to \mathbf{G}'$, f: $\mathbf{M} \to \mathbf{M}'$ such that $\alpha' \circ F = f \circ \alpha$, $\beta' \circ F = f \circ \beta$ and $F(hg) = F(h)F(g)$ for all $(h, g) \in \mathbf{G} * \mathbf{G}$ (cf. [5]).*



We also say F is a morphism over f. If $\mathbf{M} = \mathbf{M}'$ and f $= id_{\mathbf{M}}$ we say that F is a morphism over $\mathbf{M}$ or F is a base-preserving morphism.

Definition 1.2.2. *(cf.[5]) A Lie groupoid is a groupoid $\mathcal{G} := G \rightrightarrows M$ together with smooth structures on $\mathbf{G}$ and $\mathbf{M}$ such that the maps $\alpha,\ \beta : \mathbf{G} \to \mathbf{M}$ are surjective submersions, the object inclusion map $x \mapsto 1_x$, $\mathbf{M} \to \mathbf{G}$ is smooth, and the partial multiplication $\mathbf{G} * \mathbf{G} \to \mathbf{G}$ is smooth.*

In the context of Lie groupoids, a surjective submersion refers to a smooth map between two Lie groupoids where the map is both surjective (every element in the target groupoid has a pre-image in the source groupoid) and a submersion (the differential of the map is surjective at every point). Specifically, for a Lie groupoid $\mathbf{G} \rightrightarrows \mathbf{M}$, where α and β are the source and target maps respectively, a surjective submersion condition is often imposed on the pair of maps (α, β) : $\mathbf{G}_1 \to \mathbf{M} \times \mathbf{M}$, where $(\mathbf{G}_1)$ is the set of composable pairs of arrows.

Example 1.2.1.

1. *Any manifold $\mathbf{M}$ can be regarded as a Lie groupoid $\mathbf{M} \rightrightarrows \mathbf{M}$ with $\alpha = \beta = id_{\mathbf{M}}$ and every morphism is an identity morphism. A groupoid in which every morphism is an identity morphism will be called a base groupoid.*

2. *Let $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$ be a smooth action of a Lie group $\mathbf{G}$ on a manifold $\mathbf{M}$. Give the product manifold $\mathbf{G} \times \mathbf{M}$ the structure of a Lie groupoid on $\mathbf{M}$ in the following way.*

   - *α be the projection into the second factor of $\mathbf{G} \times \mathbf{M}$, β be the group action itself.*
   - *The object inclusion map is $x \mapsto 1_x = (1, x)$.*
   - *partial multiplication is $(g_2, y)\ (g_1, x) = (g_2 g_1, x)$ which is defines iff $y = g_1 x$.*
   - *The inverse of (g, x) is (g^{-1}, gx).*

   With this structure we denote $\mathbf{G} \times \mathbf{M}$ by $\mathbf{G} \triangleleft \mathbf{M}$ and call it the action groupoid of $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$.

Definition 1.2.3. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two Lie groupoids with base* $\mathbf{M}$ *and* $\mathbf{M}'$ *respectively. A Lie groupoid morphism is the groupoid morphism (F, f) with both F and f are smooth. In other words the morphism between two Lie groupoids is a smooth functor between them. It is an isomorphism if both F and f are diffeomorphisms (cf. [5]).*

1.3 Representation of Lie groupoids.

A representation of the Lie groupoid $\mathcal{G} = G \rightrightarrows M$ is a Lie groupoid morphism from $\mathcal{G}$ to a Lie groupoid of smooth manifolds.

Example 1.3.1. *Let (E, π, M) be a vector bundle over a smooth manifold M. There always exists a Lie groupoid with base M and whose morphisms $x \to x'$ are the linear isomorphisms $E_x \to E_{x'}$, where E_x and $E_{x'}$ are the fibers of the vector bundle E over the points x and x' respectively. This groupoid associated with E is called a frame groupoid (or linear frame groupid), and is denoted by $\phi(E)$. A representation of $\mathcal{G} := G \rightrightarrows M$ is a Lie groupoid morphism from $\mathcal{G}$ to the frame groupoid of a smooth vector bundle over the same base manifold M, any such representation assigns each arrow $g : x \to x'$ of $\mathcal{G}$, to a linear isomorphism between the corresponding fibers of the vector bundle such that the composition of arrows is respected.*

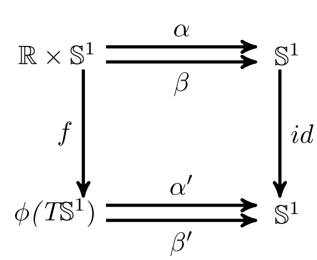
Example 1.3.2. *Consider the action $\star$ of the Lie group $G = \mathbb{R}$ on the smooth manifold $M = \mathbb{S}^1$ defined by $\star(r, z) = e^{2\pi i r} z$. Let $\mathbb{R} \times \mathbb{S}^1 \rightrightarrows \mathbb{S}^1$ be the corresponding action groupoid with source map as the projection into the second component and target map as the Lie group action. Now consider the tangent bundle $T\mathbb{S}^1 \to \mathbb{S}^1$ and define a map $f : \mathbb{R} \times \mathbb{S}^1 \to \phi(T\mathbb{S}^1)$ by*

$$f(r, z) = h_{z, e^{2\pi i r} z}$$

where $h_{z, e^{2\pi i r} z}$ is the isomorphism of the tangent space $T_z\, \mathbb{S}^1$ to $T_{ze^{2\pi i r}} \mathbb{S}^1$ given by

$$h_{z, e^{2\pi i r} z}(z, v) = (ze^{2\pi i r}, v).$$

Diagramatically,



Then f is a smooth mapping satisfying $id \circ \alpha = \alpha' \circ f$ and $id \circ \beta = \beta' \circ f$. Also

$$\begin{aligned} f((r_1, z_1)(r_2, z_2)) &= f(r_1 + r_2, z_2) \\ &= h_{z_2, e^{2\pi i(r_1+r_2)} z_2} \\ &= h_{z_2, e^{2\pi i r_1} e^{2\pi i r_2} z_2} \\ &= h_{z_2, e^{2\pi i r_1} z_1} \\ &= h_{z_1, e^{2\pi i r_1} z_1} h_{z_2, e^{2\pi i r_2} z_2} \\ &= f((r_1, z_1)) f((r_2, z_2)) \end{aligned}$$

whenever the composition $(r_1, z_1)(r_2, z_2)$ is defined. Therefore f is a Lie groupoid morphism and hence defines a representation of the action groupoid $\mathbb{R} \times \mathbb{S}^1 \rightrightarrows \mathbb{S}^1$.

Definition 1.3.1. *Let (V, ρ) be a representation of a Lie groupoid $\mathcal{G}$. A sub-representation of $\mathcal{G}$ is a subbundle V' of V that is preserved by the groupoid action. i.e, for each arrow g: x to y in $\mathcal{G}$ $\rho_g|_{V'_x}$ is an isomorphism fron V'_x to V'_y.*

2 Lie groupoids and category of presheaves

A presheaf on a category $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to the category of sets, **Set**. In simpler terms, it assigns a set to each object in $\mathcal{C}$ and a function to each morphism, respecting the composition and identities of $\mathcal{C}$. Think of it as a way to organize and relate data associated with the objects and morphisms of $\mathcal{C}$. A natural transformation between two presheaves (functors) provides a way to relate them in a way that respects the structure of the category $\mathcal{C}$. By taking presheaves on $\mathcal{C}$ as objects and natural transformations as morphisms, we have the category of presheves, $PSh(\mathcal{C})$. This category plays a crucial role in various areas of mathematics. In essence, the category of presheaves on $\mathcal{C}$ provides a framework for studying how data and relationships on $\mathcal{C}$ can be organized and manipulated in a functorial way.

In this section, we discuss the category of presheaves of Lie groupoids. Let $\mathcal{G}$ and $\mathbf{M}$ be smooth manifolds and $\mathcal{G} \rightrightarrows \mathbf{M}$ a Lie groupoid where the maps α, $\beta : \mathcal{G} \to \mathbf{M}$ are surjective submertions, the object inclusion map $x \mapsto 1_x$, $\mathbf{M} \to \mathcal{G}$ is smooth, and the partial multiplication $\mathcal{G} * \mathcal{G} \to \mathcal{G}$ is smooth. A presheaf of $\mathcal{G} \rightrightarrows \mathbf{M}$ is an $\mathcal{A}$-valued functor $P : \mathcal{G}_M^{op} \to \mathcal{A}$, these presheaves together with natural transformation between such presheaves constitute the category of presheaves $PSh(\mathcal{G}_M^{op}, \mathcal{A})$ of the Lie groupoid.

Definition 2.0.1 (Action of Lie groupoid on a manifold)**.** *A right action of a Lie groupoid $\mathcal{G} : \mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ on a manifold M consists a pair of smooth maps*

$(a_{\mathcal{G}}, \mu)$, *where* $a_{\mathcal{G}} : M \to \mathcal{G}_0$ *(called anchor map) and* $\mu : M \times_{a_{\mathcal{G}}, \mathcal{G}_0, t} \mathcal{G}_1 \to M$ *satisfying*

1. $\mu(m, 1_{a_{\mathcal{G}}(m)}) = m$ *for all* $m \in M$,
2. $a_{\mathcal{G}}(\mu(m, g)) = s(g)$ *for all* $(m, g) \in M \times_{a_{\mathcal{G}}, \mathcal{G}_0, t} \mathcal{G}_1$, *and*
3. $\mu(\mu(m, g), g') = \mu(m, m \circ m')$ *for all* $(m, g, g') \in M \times_{a_{\mathcal{G}}, \mathcal{G}_0, t} \mathcal{G}_1 \times_{s, \mathcal{G}_0, t} \mathcal{G}_1$.

In a similar way, we can define the left action of $\mathcal{G}$ on M.

Example 2.0.1. *For any Lie groupoid* $\mathcal{G}$, *the pair* (s, m) *where* $s : \mathcal{G}_1 \to \mathcal{G}_0$ *(the source map) and* $m : \mathcal{G}_1 \times_{s, \mathcal{G}_0, t} \mathcal{G}_0 \to \mathcal{G}_1$ *(the composition map), gives a right action of* $\mathcal{G}_0$ *on* $\mathcal{G}_1$. *Similarly the pair* (t, m) *where* $t : \mathcal{G}_1 \to \mathcal{G}_0$ *(the target map) and* $m : \mathcal{G}_0 \times_{s, \mathcal{G}_0, t} \mathcal{G}_1 \to \mathcal{G}_1$, *gives a left action of* $\mathcal{G}_0$ *on* $\mathcal{G}_1$.

For a Lie groupoid $\mathcal{G}$, a presheaf is a functor $P : \mathcal{G}^{op} \to \mathbf{Diff}$ where **Diff**, the category of smooth manifolds. A category whose objects are presheaves of $\mathcal{G}$, and morphisms the natural transformations between such presheaves is the category of presheaves, we denote this as $\mathcal{G}^*$.

Lemma 2.0.1. *Let* $\mathcal{G} \rightrightarrows \mathbf{M}$, *be a Lie groupoid. Then there is a natural isomorphism* η *of Lie groupoid* $\mathcal{G}$ *and the category of presheaves* $\mathcal{G}^*$.

Proof. Given Lie groupoid $\mathcal{G} \rightrightarrows \mathbf{M}$, it is easy to observe that for each $x \in \mathbf{M}$ and $g \in \mathcal{G}$ there exists contravariant functor $P : \mathcal{G}^{op} \to \mathbf{Diff}$ carries x to a smooth manifold and g to a smooth map between soomth manifolds. Since every $g \in \mathcal{G}$ is invertible the functor P is a natural isomorphism and hence $\mathcal{G}$ and $\mathcal{G}^*$ are naturaly equivalent. □

Also given a Lie groupoid of $\mathcal{G}$, the set of smooth manifolds M where $\mathcal{G}$ can act (from the right), and the smooth maps between such manifolds where the action of $\mathcal{G}$ is preserved is a category whose objects are right $\mathcal{G}$-manifolds and morphisms are right $\mathcal{G}$-manifold morphisms. We denote this category by $Man_{\mathcal{G}}$.

The following lemma shows that for any Lie groupoid $\mathcal{G}$, the functor category $\mathcal{G}^*$ is equivalent to the category $Man_{\mathcal{G}}$.

Theorem 2.0.2. *Let* $\mathcal{G}$ *be a Lie groupoid. The functor category* $\mathcal{G}^*$ *is categorically equivalent to the catgeory* $Man_{\mathcal{G}}$ *of the right* $\mathcal{G}$*-manifolds.*

Proof. For a Lie groupoid $\mathcal{G}$, consider the functor category $\mathcal{G}^*$ whose objects are the functors $\{P^i : \mathcal{G}^{op} \to \mathbf{Diff}\}$ of all functors from $\mathcal{G}^{op}$ to **Diff** and morphisms are $\{\nu^{ij} : P^i \to P^j\}$ of natural transformations between functors. Consider the map $F : \mathcal{G}^* \to Man_{\mathcal{G}}$ given as follows. For an object $P : \mathcal{G}^{op} \to \mathbf{Diff}$ in $\mathcal{G}^*$, the object map F_0 is

$$F_0(P) = \bigsqcup_{x \in \mathcal{G}_0} P(x),$$

the disjoint union of $P(x)$. Then $F_0(P)$ is a smooth manifold as it is the total space of the bundle $(\bigsqcup_{x\in\mathcal{G}_0} P(x), p, \mathcal{G}_0)$, where p is the projection. $F_0(P) \in Man_{\mathcal{G}}$, since for each $m \in F_0(P)$, then $m \in P(x)$ for $x \in \mathcal{G}_0$, define the anchor map $a_P : F_0(P) \to \mathcal{G}_0$ by projecting each element in $P(x)$ to x and $\mu_P : F_0(P) \times_{a_P,\mathcal{G}_0,t} \mathcal{G}_1 \to F_0(P)$ by

$$\mu_P(m, g) = P(g^{op})(m).$$

(1) Let $m \in F_0(P)$, then $m \in P(x)$ for some $x \in \mathcal{G}_0$. Then,

$$\mu_P(m, 1_{a_P(m)}) = \mu_P(m, 1_x) = P(1_x^{op})(m) = P(1_x)(m) = 1_{P(x)}(m) = m$$

(2) Let $m \in F_0(P)$ and $g : x \to y$ in $\mathcal{G}_1$ such that $a_P(m) = t(g)$. Then, $a_P(m) = y$ and thus $m \in P(y)$. Therefore,

$$a_P(\mu_P(m, g)) = a_P(P(g^{op})(m)) = x = s(g),$$

since $P(g^{op})(m) \in P(x)$.

(3) For all $(m, g, g') \in F_0(P) \times_{a_P,\mathcal{G}_0,t} \mathcal{G}_1 \times_{s,\mathcal{G}_0,t} \mathcal{G}_1$,

$$\begin{aligned}
\mu_P(\mu_P(m, g), g') &= \mu_p(P(g^{op}(m), g') \\
&= P((g')^{op})(P(g^{op})(m)) \\
&= (P((g')^{op}) \circ P(g^{op}))(m) \\
&= P((g \circ g')^{op})(m) \\
&= \mu_P(m, g \circ g').
\end{aligned}$$

Thus, $F_0(P)$ is a right $\mathcal{G}$- manifold.
Now, consider a natural transformation $\nu^{12} : P^1 \to P^2$ between the functors $P^1, P^2 : \mathcal{G}^{op} \to$ **Diff**. For each $x \in \mathcal{G}_0$, $\nu^{12}(x) : P^1(x) \to P^2(x)$ is a smooth map, and each arrow $g : x \to y$ in $\mathcal{G}_1$, the following diagram commutes.

$$\begin{array}{ccc}
P^1(x) & \xrightarrow{\nu^{12}(x)} & P^2(x) \\
{\scriptstyle P^1(g)}\downarrow & & \downarrow{\scriptstyle P^2(g)} \\
P^1(y) & \xrightarrow[\nu^{12}(y)]{} & P^2(y)
\end{array}$$

That is, $\nu^{12}(y) \circ P^1(g) = P^2(g) \circ \nu^{12}(x)$. We want to show that $F_1(\nu^{12})$ is a right $\mathcal{G}$-manifold morphism. Define $F_1(\nu^{12}) : F_0(P^1) \to F_0(P^2)$ by

$$F_1(\nu^{12})(m) = \nu^{12}(x)(m)$$

for $m \in F_0(P^1)$. Since $m \in F_0(P^1)$, $m \in P^1(x)$ for some $x \in \mathcal{G}_0$. Then $\nu^{12}(x) : P^1(x) \to P^2(x)$ maps each element in $P^1(x)$ to an element in $P^2(x)$. Thus, $\nu^{12}(x)(m) \in P^2(x) \subset F_0(P^2)$. Therefore, $F_1(\nu^{12})$ is well defined.

Since $F_0(P^1)$ and $F_0(P^2)$ are right $\mathcal{G}$-manifolds with action (a_{P^1}, μ_{P^1}) and (a_{P^2}, ν_{P^2}), for $m \in P^1(x) \subset F_0(P^1)$,

(1) $a_{P^2}(F_1(\nu^{12})(m)) = a_{P^2}(\nu^{12}(x)(m)) = x = a_{P^1}(m)$.

(2) Let $m \in F_0(P^1)$ and $g : x \to y$ with $a_{P^1}(m) = t(g)$. Then $m \in P^1_0(y)$ and

$$\begin{aligned} F_1(\nu^{12})(\mu_{P^1}(m,g)) &= F_1(\nu^{12})(P^1(g^{op})(m)) \\ &= \nu^{12}(x)(P^1(g^{op})(m)) \\ &= P^2(g^{op})(\nu^{12}(y)(m)) \\ &= P^2(g^{op})(F_1(\nu^{12})(m)) \\ &= \mu_{P^2}(F_1(\nu^{12})(m), g) \end{aligned}$$

(using the fact that ν^{12} is a natural transformation).

Thus $F_1(\nu^{12})$ is a right $\mathcal{G}$-manifold morphism. Moreover, for two composable morphisms $\nu^{12} : P^1 \to P^2$ and $\nu^{23} : P_2 \to P_3$ in $\mathcal{G}^*$,

$$F_1(\nu^{12} \circ \nu^{23}) = F_1(\nu^{12}) \circ F_1(\nu^{23})$$

and for each object P in $\mathcal{G}^*$, $F_1(1_P) = 1_{F_0(P)}$. Hence $F = (F_1, F_0)$ is a functor.

Consider the map $G : Man_{\mathcal{G}} \to \mathcal{G}^*$ given as follows. Let M be a right $\mathcal{G}$-manifold with action (a_M, μ_M). Then, $G_0(M) : \mathcal{G}^{op} \to \mathbf{Diff}$ is defined by

$$G_0(M)(x) = a_M^{-1}(x)$$

and for $g : x \to y$, $G_0(M)(g^{op}) : G_0(M)(y) \to G_0(M)(x)$ by

$$G_0(M)(g^{op})(m) = \mu_M(m, g),$$

where $m \in G_0(M)(y)$. Then $G_0(M)$ is a functor, and thus object map is defined. Let M and M' be two right $\mathcal{G}$-manifolds with actions (a_M, μ_M) and $(a_{M'}, \mu_{M'})$, respectively, and $f : M \to M'$ be a right $\mathcal{G}$-manifold morphism. To show that $G_1(f) : G_0(M) \to G_0(M')$ is a natural transformation; for each object $x \in \mathcal{G}_0$, define $G_1(f)(x) : G_0(M)(x) \to G_0(M')(x)$ by

$$G_1(f)(x)(p) = f(p)$$

for $p \in G_0(M)(x)$. Since f is smooth, $G_1(f)(x)$ is also smooth. Consider an arrow $g : x \to y$ in $\mathcal{G}_1$ and the following diagram.

$$\begin{array}{ccc} G_0(M)(y) & \xrightarrow{G_1(f)(y)} & G_0(M')(y) \\ {\scriptstyle G_0(M)(g^{op})}\downarrow & & \downarrow{\scriptstyle G_0(M')(g^{op})} \\ G_0(M)(x) & \xrightarrow[G_1(f)(x)]{} & G_0(M')(x) \end{array}$$

Then for $m \in G_0(M)(y) = a_M^{-1}(y)$,

$$\begin{aligned} (G_1(f)(x) \circ G_0(M)(g^{op}))(m) &= G_1(f)(x)(\mu_M(m, g)) \\ &= f(\mu_M(m, g)) \\ &= \mu_{M'}(f(m), g) \\ &= G_0(M')(g^{op})(f(m)) \\ &= (G_0(M')(g^{op}) \circ G_1(f)(y))(m) \end{aligned} \tag{1}$$

Thus $G_1(f)$ is a natural transformation, and for two composable morphisms $f : M \to M'$ and $f' : M' \to M''$,

$$G_1(f \circ f') = G_1(f) \circ G_1(f').$$

Also, for each object M in $Man_{\mathcal{G}}$, $G_1(1_M) = 1_{G_0(M)}$. Hence $G = (G_1, G_0)$ is a functor.

For any functor $P : \mathcal{G}^{op} \to \mathbf{Diff}$, $(G \circ F)(P) = P$ and thus $G \circ F$ is naturally isomorphic to the identity functor $I_{\mathcal{G}^*}$ on $\mathcal{G}^*$. Moreover, for any right $\mathcal{G}$-manifold M, $(F \circ G)(M) \cong M$ and thus $F \circ G$ is naturally isomorphic to the identity functor $I_{Man_{\mathcal{G}}}$ on $Man_{\mathcal{G}}$. Hence $\mathcal{G}^*$ and $Man_{\mathcal{G}}$ are equivalent. $\square$ In the light of Lemma 4.1 and Theorem 4.1, it is seen that

$$\Psi := \eta \circ F : \mathcal{G} \to Man_{\mathcal{G}}$$

is a fully-faitfull representation of the Lie groupoid $\mathcal{G}$.

CONCLUDING REMARK

A representation of a Lie groupoid $\mathcal{G}$ is a Lie groupod morphism of $\mathcal{G}$ to some bundle of manifolds (see[9]). In this paper we described the category of preseaves $\mathcal{G}^*$ of a Lie groupoid $\mathcal{G}$ and via this functor category obtained a representation of $\mathcal{G}$ on manifolds.

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