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A SUPERMAGIC LABELING OF FINITE COPIES OF CARTESIAN PRODUCT OF CYCLES

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Abstract

A *homomorphism* of a graph H onto a graph G is defined to be a surjective mapping $\psi : V(H) \to V(G)$ such that whenever u, v are adjacent in H, $\psi(u), \psi(v)$ are adjacent in G, that is the induced mapping $\bar{\psi} : E(H) \to E(G)$ satisfying: if e is an edge of H with end vertices u and v, then $\bar{\psi}(e)$ is an edge of G with end vertices $\psi(u)$ and $\psi(v)$. A homomorphism ψ is *harmonious* if $\bar{\psi}$ is a bijection. A triplet $[H, \psi, t]$ is called a *supermagic frame* of G if ψ is a harmonious homomorphism of H onto G and $t : E(H) \to \{1, 2, \dots, |E(H)|\}$ is an injective mapping such that $\sum_{u\in\psi^{-1}(v)} t^*(u)$ is independent of the vertex $v \in V(G)$. Note that $t^*(u)$ is the sum of $t(uw)$ where w is adjacent to u.

In 2000, Ivančo proved that if there is a supermagic frame of a graph G, then G is supermagic. In this paper, we construct a supermagic frame of $m(\geq 2)$ copies of Cartesian product of cycles and apply the Ivančo's result to show that m copies of Cartesian product of cycles is supermagic.

Key words: supermagic frame; Cartesian product of cycles.
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1 Introduction

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$ and let $|V(G)|$ and $|E(G)|$ denote the number of vertices and the number of edges in G, respectively. A *supermagic labeling* of G is an injective f from $E(G)$ into the set of positive integers such that the set of edge labels contains consecutive integers and the *index-mapping* f^* from $V(G)$ into the set of positive integers given by $f^*(v) = \sum\limits_{uv \in E(G)} f(uv)$ is a constant mapping. The graph G is called *supermagic* with *index* λ if there is a supermagic labeling of G such that $f^*(v) = \lambda$ for all $v \in V(G)$.

The concept of supermagic labelings was introduced by Stewart [3]. Several graphs admitted supermagic labelings are shown in [1], [2], [3], [4] and [5].

Let H and G be graphs. A *homomorphism* of H onto G is a surjective mapping $\psi : V(H) \to V(G)$ such that if u, v are adjacent in H, then $\psi(u), \psi(v)$ are adjacent in G. That is, ψ induces a mapping $\bar{\psi} : E(H) \to E(G)$ such that if e is an edge of H, then $\bar{\psi}(e)$ is an edge of G. A homomorphism ψ is *hamonious* if $\bar{\psi}$ is a bijection. A triplet $[H, \psi, t]$ is called a *supermagic frame* of a graph G if ψ is a harmonious homomorphism of H onto G and $t : E(H) \to \{1, 2, \ldots, |E(H)|\}$ is an injective mapping such that $\sum\limits_{u \in \psi^{-1}(v)} t^*(u)$ is independent of the vertex $v \in V(G)$.

In 2000, Ivančo [1] proved the following results.

Proposition 1.1. *[1] Let G be a $d-$regular graph. If G is supermagic, then there exists a supermagic labeling f from $E(G)$ to the set $\{1, 2, \ldots, \frac{d|V(G)|}{2}\}$ with index $\lambda = \frac{d}{2}(1 + \frac{d|V(G)|}{2})$.*

Proposition 1.2. *[1] If there is a supermagic frame of a graph G, then G is supermagic.*

In [3], Stewart proved that M_{2k} is supermagic when $k \equiv 1 \pmod 2$. Later, Singhun et al. [5] proved that nM_{2k}, a graph consists of n copies of M_{2k}, is supermagic if and only if n is odd and $3 \leq k \equiv 1 (\mathrm{mod}\ 2)$. In [1], Ivančo proved that $C_n \times C_n$ is supermagic, by Proposition 1.2. Motivated by these, we would like to construct a supermagic frame for the finite copies of the graph $C_n \times C_n$. This construction implies that the finite copies of $C_n \times C_n$ is supermagic graph.

2 Result

Let nG, where $n \geq 2$, be a graph consisting of n copies of G. In this section, let $m \geq 2$ and G be the graph $m(C_n \times C_n)$, where $n \geq 3$. For $k = 1, 2, 3, \ldots, m$, the vertex set of the k^{th} copy of G is the set $\{[v_r^k, v_s^k] \mid r = 1, 2, 3, \ldots, n$ and $s = 1, 2, 3, \ldots, n\}$ and the edge set of the k^{th} copy of G is the set $\{[v_i^k, v_j^k][v_i^k, v_{j+1}^k],$

$[v_i^k, v_j^k][v_{i+1}^k, v_j^k] \mid i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, n\}$. Thus, the vertices of the k^{th} copy of G are the following.

$$\begin{array}{cccc}
[v_1^k, v_1^k], & [v_1^k, v_2^k], & \ldots, & [v_1^k, v_n^k] \\
[v_2^k, v_1^k], & [v_2^k, v_2^k], & \ldots, & [v_2^k, v_n^k] \\
 & \vdots & & \vdots \\
[v_n^k, v_1^k], & [v_n^k, v_2^k], & \ldots, & [v_n^k, v_n^k]
\end{array}$$

Figure 1 shows the k^{th} copy of G with vertices and edges. Then, G is a 4-regular graph and it can be seen that G can be decomposed into $2mn$ copies of cycle of length n.

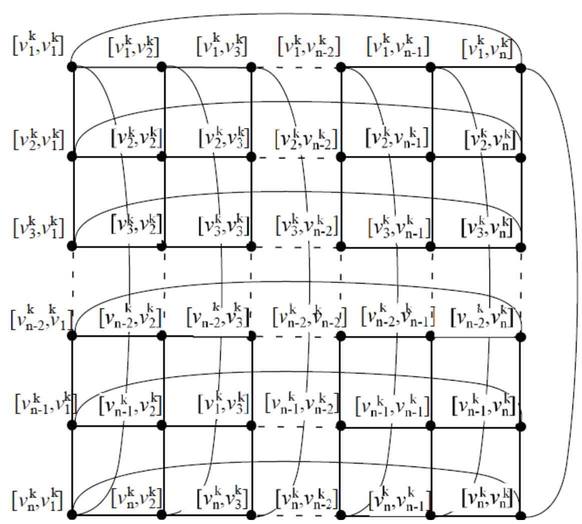

Figure 1: The k^{th} copy of $C_n \times C_n$ with vertices and edges.

Let $m \geq 2$, $n \geq 3$ and H be the graph $2mnC_n$, the $2mn$ copies of cycle C_n. We first show that there is a surjective mapping from each cycle of H to each cycle of G in Lemma 2.1. Then, by using the surjective mapping in Lemma 2.1, we show that there is a harmonius homomorphism of H onto G in Lemma 2.2. Later, the mapping t is defined and the values of t^* are obtained in Lemma 2.3. By Lemma 2.1-2.3, we conclude in Lemma 2.4 that $[H, \psi, t]$ is a supermagic frame of H onto G. Then, by Proposition 1.2, G is supermagic.

Lemma 2.1. *Let $m \geq 2, n \geq 3$ and G denote the graph $m(C_n \times C_n)$ for $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. Let G_t, where $t = 2(k-1)n+i$, be a subgraph of G induced by $\{[v_i^k, v_j^k] \mid j = 1, 2, \ldots, n\}$ and G_t where $t = (2k-1)n+i$, be a subgraph of G induced by $\{[v_j^k, v_i^k] \mid j = 1, 2, \ldots, n\}$. For $i = 1, 2, \ldots, 2mn$,*

let H_i be a cycle with the vertex set $\{u_0^i, u_1^i, u_2^i, \ldots, u_{n-1}^i\}$ and the edge set $\{u_j^i u_{j+1}^i | j = 0, 1, \ldots, n-1\}$, where the subscripts are taken under modulo n. Let $\psi_i : V(H_i) \to V(G_i)$ be given by
for $i = 1, 2, \ldots, mn$ and $j = 0, 1, \ldots, n-1$,

$$\psi_i(u_j^i) = [v_l^k, v_{l+j}^k] \text{ when } k = \lceil \tfrac{i}{n} \rceil \text{ and } l \equiv i (\text{mod } n),$$

for $i = mn+1, mn+2, \ldots, 2mn$ and $j = 0, 1, \ldots, n-1$,

$$\psi_i(u_j^i) = [v_{l+j}^k, v_l^k] \text{ when } k = \lceil \tfrac{1+2mn-i}{n} \rceil \text{ and } l \equiv 2mn + 1 - i (\text{mod } n).$$

(Note that for the subscript l under modulo n, we use n instead of 0.) Then, ψ_i is surjective for $i = 1, 2, \ldots, 2mn$.

Proof We claim that ψ_i is surjective. Let $i \in \{1, 2, \ldots, 2mn\}$ and v be a vertex of G_i. Then, $v = [v_r^k, v_s^k]$ for some $k \in \{1, 2, \ldots, m\}$ and $r, s \in \{1, 2, \ldots, n\}$.

If $r \leq s$, then $s = r + j$ for some $j \in \{0, 1, \ldots, n-1\}$. Choose $i \in \{1, 2, \ldots, mn\}$ such that $k = \lceil \frac{i}{n} \rceil$ and $i \equiv r$ (mod n). Then $\psi_i(u_j^i)$=$[v_r^k, v_{r+j}^k]$=$[v_r^k, v_s^k]$.

If $r > s$, then $r = s+j$ for some $j \in \{0, 1, 2, \ldots, n-1\}$. Choose $i \in \{mn+1, mn+2, \ldots, 2mn\}$ such that $k = \lceil \frac{1+2mn-i}{n} \rceil$ and $s \equiv 2mn + 1 - i$ (mod n). Then, $\psi_i(u_j^i)$=$[v_{s+j}^k, v_s^k]$=$[v_r^k, v_s^k]$.

Therefore, ψ_i is surjective. □

Lemma 2.2. *Let G denote the graph $m(C_n \times C_n)$ and H_i be the graph defined as in Lemma 2.1. Let $H = \bigcup_{i=1}^{2mn} H_i$. Then, there is a harmonius homomorphism of H and G.*

Proof Let G denote the graph $m(C_n \times C_n)$, and ψ_i be defined as in Lemma in 2.1. Define $\psi : V(H) \to V(G)$ by

$$\psi(u_j^i) = \psi_i(u_j^i) \text{ for } u_j^i \in V(H_i) \subseteq V(H)$$

where $i \in \{1, 2, \ldots, 2mn\}$ and $j \in \{0, 1, \ldots, n-1\}$.
First, we claim that ψ is surjective.

Let v be a vertex of G. Then, $v \in G_i$ for some $i \in \{1, 2, \ldots, 2mn\}$. Since ψ_i is surjective, there is a vertex u_j^i for some $j \in \{0, 1, \ldots, n-1\}$ for H_i such that $\psi_i(u_j^i) = v$. Then,

$$\psi(u_j^i) = \psi_i(u_j^i) = v.$$

This shows that ψ is surjective.
Next, we claim that if uv is an edge of H, then $\psi(u)\psi(v)$ is an edge of G.

Let $u_b^a u_d^c$ be an edge of H where $a, c \in \{1, 2, \ldots, 2mn\}$ and $b, d \in \{0, 1, \ldots, n-1\}$. Then, $a = c$ and $d = b + 1$.

If $a \in \{1, 2, \ldots, mn\}$, then

$$\psi(u_b^a)\psi(u_d^c) = \psi_a(u_b^a)\psi_c(u_d^c) = [v_l^k, v_{l+b}^k][v_l^k, v_{l+b+1}^k],$$

where $k = \left\lceil \frac{a}{n} \right\rceil$ and $l \equiv a \pmod{n}$.

Since $[v_l^k, v_{l+b}^k][v_l^k, v_{l+b+1}^k]$ is an edge of G in the k^{th} copy, $\psi(u_b^a)\psi(u_d^c)$ is an edge of G.

If $a \in \{mn+1, mn+2, \ldots, 2mn\}$, then

$$\psi(u_b^a)\psi(u_d^c) = \psi_a(u_b^a)\psi_c(u_d^c) = [v_{l+b}^k, v_l^k][v_{l+b+1}^k, v_l^k],$$

where $k = \left\lceil \frac{1+2mn-a}{n} \right\rceil$ and $l \equiv 2mn+1-a \pmod{n}$.

Since $[v_{l+b}^k, v_l^k][v_{l+b+1}^k, v_l^k]$ is an edge of G in the k^{th} copy, $\psi(u_b^a)\psi(u_d^c)$ is an edge of G.

Finally, we claim that the induced mapping $\bar{\psi}$ is bijective.

Let $u_b^a u_d^c$ and $u_y^x u_s^r$ be edges of H such that

$$\bar{\psi}(u_b^a u_d^c) = \bar{\psi}(u_y^x u_s^r).$$

Then, $a = c$, $x = r$, $d = b+1$, $s = y+1$ and a, x are both in $\{1, 2, \ldots, mn\}$ or $\{mn+1, mn+2, \ldots, 2mn\}$.

Suppose that a, x are both in the set $\{1, 2, \ldots, mn\}$. Then,

$$[v_{l_1}^{k_1}, v_{l_1+b}^{k_1}][v_{l_1}^{k_1}, v_{l_1+b+1}^{k_1}] = [v_{l_2}^{k_2}, v_{l_2+y}^{k_2}][v_{l_2}^{k_2}, v_{l_2+y+1}^{k_2}]$$

where $k_1 = \left\lceil \frac{a}{n} \right\rceil$, $l_1 \equiv a \pmod{n}$, $k_2 = \left\lceil \frac{x}{n} \right\rceil$ and $l_2 \equiv x \pmod{n}$.

Since $\bar{\psi}(u_b^a u_d^c)$ and $\bar{\psi}(u_y^x u_s^r)$ are edges of G, $k_1 = k_2$.

If $[v_{l_1}^{k_1}, v_{l_1+b}^{k_1}] = [v_{l_2}^{k_2}, v_{l_2+y+1}^{k_2}]$ and $[v_{l_1}^{k_1}, v_{l_1+b+1}^{k_1}] = [v_{l_2}^{k_2}, v_{l_2+y}^{k_2}]$, then $l_1 = l_2$ and $b = y+1$ and $b+1 = y$, which is a contradiction.

Therefore, $[v_{l_1}^{k_1}, v_{l_1+b}^{k_1}] = [v_{l_2}^{k_2}, v_{l_2+y}^{k_2}]$ and $[v_{l_1}^{k_1}, v_{l_1+b+1}^{k_1}] = [v_{l_2}^{k_2}, v_{l_2+y+1}^{k_2}]$. That is, $l_1 = l_2$, $b = y$ and $d = s$. Then, $u_b^a u_d^c = u_y^x u_s^r$.

Suppose that a, x are both in the set $\{mn+1, mn+2, \ldots, 2mn\}$. Then,

$$[v_{l_1+b}^{k_1}, v_{l_1}^{k_1}][v_{l_1+b+1}^{k_1}, v_{l_1}^{k_1}] = [v_{l_2+y}^{k_2}, v_{l_2}^{k_2}][v_{l_2+y+1}^{k_2}, v_{l_2}^{k_2}]$$

where $k_1 = \left\lceil \frac{1+2mn-a}{n} \right\rceil$, $l_1 \equiv 1+2mn-a \pmod{n}$, $k_2 = \left\lceil \frac{1+2mn-x}{n} \right\rceil$ and $l_2 \equiv 1+2mn-x \pmod{n}$.

Since $\bar{\psi}(u_b^a u_d^c)$ and $\bar{\psi}(u_y^x u_s^r)$ are edges of G, $k_1 = k_2$.

If $[v_{l_1+b}^{k_1}, v_{l_1}^{k_1}] = [v_{l_2+y+1}^{k_2}, v_{l_2}^{k_2}]$ and $[v_{l_1+b+1}^{k_1}, v_{l_1}^{k_1}] = [v_{l_2+y}^{k_2}, v_{l_2}^{k_2}]$, then $l_1 = l_2$ and $b = y+1$ and $y = b+1$, which is a contradiction.

Therefore, $[v_{l_1+b}^{k_1}, v_{l_1}^{k_1}] = [v_{l_2+y}^{k_2}, v_{l_2}^{k_2}]$ and $[v_{l_1+b+1}^{k_1}, v_{l_1}^{k_1}] = [v_{l_2+y+1}^{k_2}, v_{l_2}^{k_2}]$. That is, $l_1 = l_2$, $b = y$, and $d = s$. Then, $u_b^a u_d^c = u_y^x u_s^r$.

Since $|E(H)| = |E(G)|$, $\bar{\psi}$ is surjective. Hence, $\bar{\psi}$ is bijective.

Therefore, ψ is harmonious homomorphism of H onto G. □

Lemma 2.3. *Let H be the graph defined as in Lemma 2.2 and $t : E(H) \to \{1, 2, \ldots, 2mn^2\}$ be defined by*

$$t(u_j^i u_{j+1}^i) = \begin{cases} 2jmn + i & \text{if } j \equiv 0 \pmod{2}, \\ 1 + 2(j+1)mn - i & \text{if } j \equiv 1 \pmod{2}, \end{cases}$$

for $i = 1, 2, \ldots, 2mn$ *and* $j = 0, 1, \ldots, n-1$.
Then, t *is injective. Moreover, the index-mapping* t^* *of* t *satisfies the following equation.*

$$t^*(u_j^i) = \begin{cases} 4jmn + 1 & \textit{if } j \neq 0 \textit{ and } n \equiv 0, 1 \pmod 2, \\ 1 + 2mn^2 & \textit{if } j = 0 \textit{ and } n \equiv 0 \pmod 2, \\ 2mn(n-1) + 2i & \textit{if } j = 0 \textit{ and } n \equiv 1 \pmod 2. \end{cases}$$

Proof We claim that t is injective. Let $u_b^a u_d^c$ and $u_y^x u_s^r$ be edges of H such that

$$t(u_b^a u_d^c) = t(u_y^x u_s^r).$$

Then, $a = c, x = r, d = b + 1$ and $s = y + 1$.

If $b \equiv 0 \pmod 2$ and $y \equiv 1 \pmod 2$, then

$$2bmn + a = 1 + 2(y + 1)mn - x.$$

Hence, $2b = 2(y + 1)$ and $1 - x = a$. Then, $b = y + 1$ and $x = 1 - a$. Since $a \in \{1, 2, \ldots, 2mn\}$, x is not a positive integer, which is a contradiction.

If $b \equiv 1 \pmod 2$ and $y \equiv 0 \pmod 2$, then

$$1 + 2(b + 1)mn - a = 2ymn - x.$$

Hence, $2(b + 1) = 2y$ and $1 - x = a$. Then, $b = y - 1$ and $x = 1 - a$. Since $a \in \{1, 2, \ldots, 2mn\}$, x is not a positive integer, which is a contradiction.

Therefore, $b, y \equiv 0 \pmod 2$ or $b, y \equiv 1 \pmod 2$.

Suppose that $b, y \equiv 0 \pmod 2$. Then, $2bmn + a = 2ymn + x$, that is, $y = b$ and $a = x$. Hence, $d = s$. Then, $u_b^a u_d^c = u_y^x u_s^r$.

Suppose that $b, y \equiv 1 \pmod 2$. Then, $1+2(b+1)mn-a = 1+2(y+1)mn-x$, that is, $y = b$ and $a = x$. Hence, $d = s$. Then, $u_b^a u_d^c = u_y^x u_s^r$.

Next, we will show that

$$t^*(u_j^i) = \begin{cases} 4jmn + 1 & \text{if } j \neq 0 \text{ and } n \equiv 0, 1 (\text{mod } 2), \\ 1 + 2mn^2 & \text{if } j = 0 \text{ and } n \equiv 0 \pmod 2, \\ 2mn(n-1) + 2i & \text{if } j = 0 \text{ and } n \equiv 1 \pmod 2. \end{cases}$$

For $j \neq 0, t^*(u_j^i) = t(u_j^i u_{j+1}^i) + t(u_{j-1}^i u_j^i)$.

If $n \equiv 0$ (mod 2), then

$$\begin{aligned} t^*(u_j^i) &= 2j(mn) + i \quad + \quad 1 + 2(j - 1 + 1)mn - i \\ &= 4j(mn) + 1. \end{aligned}$$

If $n \equiv 1 \pmod 2$, then

$$\begin{aligned} t^*(u_j^i) &= 1 + 2(j + 1)mn - i + 2(j - 1)mn + i \\ &= 4j(mn) + 1. \end{aligned}$$

For $j = 0$ and $n \equiv 0 \pmod 2$,

$$\begin{aligned} t^*(u_j^i) = t^*(u_0^i) &= t(u_0^i u_1^i) + t(u_{n-1}^i u_0^i) \\ &= (0+i) \ + \ (1 + 2(n-1+1))mn - i \\ &= 1 + 2nm^2. \end{aligned}$$

For $j = 0$ and $n \equiv 1 \pmod 2$,

$$\begin{aligned} t^*(u_j^i) = t^*(u_0^i) &= t(u_0^i u_1^i) + t(u_{n-1}^i u_0^i) \\ &= (0+i) \ + \ (2(n-1)(mn)) + i \\ &= 2(mn)(n-1) + 2i. \end{aligned}$$

This completes the proof. □

Lemma 2.4. *Let G denote the graph $m(C_n \times C_n)$, ψ and t be mappings defined as in Lemma 2.2 and Lemma 2.3, respectively. Let $\psi^{-1}(v)$ be the inverse image of v under ψ. Then, for each $v \in V(G)$,*

$$\sum_{u \in \psi^{-1}(v)} t^*(u) = 2 + 4mn^2.$$

Proof Let $v \in V(G)$.

Case (i) : v is the vertex on the main diagonal. That is, v is in the form $[v_r^k, v_r^k]$ for some $k \in \{1, 2, \ldots, m\}$ and $r \in \{1, 2, \ldots, n\}$. Then,

$$\psi^{-1}([v_r^k, v_r^k]) = \{u_0^{(k-1)n+r}, u_0^{2mn-(k-1)n-(r-1)}\}.$$

If $n \equiv 0 \pmod 2$, then

$$\begin{aligned} \sum_{u \in \psi^{-1}(v)} t^*(u) &= t^*(u_0^{(k-1)n+r}) + t^*(u_0^{2mn-(k-1)n-(r-1)}) \\ &= (1 + 2mn^2) + (1 + 2mn^2) \\ &= 2 + 4mn^2. \end{aligned}$$

If $n \equiv 1 \pmod 2$, then

$$\begin{aligned} \sum_{u \in \psi^{-1}(v)} t^*(u) &= t^*(u_0^{(k-1)n+r}) + t^*(u_0^{2mn-(k-1)n-(r-1)}) \\ &= 2(mn)(n-1) + 2((k-1)n + r) + \\ &\quad\ 2(mn)(n-1) + 2(2mn - (k-1)n - (r-1)) \\ &= 2 + 4mn^2. \end{aligned}$$

Case (ii) : v is the vertex above the main diagonal. That is, v is in the form $[v_r^k, v_{r+s}^k]$ for some $k \in \{1, 2, \ldots, m\}$ and $r, s \in \{1, 2, \ldots, n\}$. Then,

$$\psi^{-1}([v_r^k, v_{r+s}^k]) = \{u_s^{(k-1)n+r}, u_{n-s}^{2mn-(k-1)n-(r-1)-s}\}$$

and

$$\begin{aligned} \sum_{u \in \psi^{-1}(v)} t^*(u) &= t^*(u_s^{(k-1)n+r}) + t^*(u_{n-s}^{2mn-(k-1)n-(r-1)-s}) \\ &= (4smn + 1) + (4(n-s)(mn) + 1) \\ &= 2 + 4mn^2. \end{aligned}$$

Case (iii) : v is the vertex under the main diagonal. That is, v is in the form $[v_{r+s}^k, v_r^k]$ for some $k \in \{1, 2, \ldots, m\}$ and $r, s \in \{1, 2, \ldots, n\}$. Then,

$$\psi^{-1}([v_{r+s}^k, v_s^k]) = \{u_{n-s}^{(k-1)n+r}, u_s^{2mn-(k-1)n-(r-1)+s}\}$$

and

$$\begin{aligned} \sum_{u \in \psi^{-1}(v)} t^*(u) &= t^*(u_{n-s}^{(k-1)n+r}) + t^*(u_s^{2mn-(k-1)n-(r-1)+s}) \\ &= (4(n-s)(mn)+1) + 4(s)(mn) + 1 \\ &= 2 + 4mn^2. \end{aligned}$$

This completes the proof. □

Theorem 2.5. *The graph $m(C_n \times C_n)$ is supermagic for any integer $n \geq 3$ and $m \geq 2$.*

Proof Let G denote the graph $m(C_n \times C_n)$ and $G_1, G_2, \ldots, G_{2mn}$ be subgraphs of G defined as in Lemma 2.1. Then, we can see that $G_1, G_2, \ldots, G_{mn}$ are horizontal cycles of G and $G_{mn+1}, G_{mn+2}, \ldots, G_{2mn}$ are vertical cycles of G and $G_1, G_2, \ldots, G_{2mn}$ form a decomposition of G into pairwise edge-disjoint cycles. For $i = 1, 2, \ldots, 2mn$, let H_i be a graph, and ψ_i be a mapping defined as in Lemma 2.1. Let H be a graph defined as in Lemma 2.2. Then, by Lemma 2.3, there is a harmonious homomorphism ψ (defined as in the proof of Lemma 2.2) of H onto G. Let $t : E(H) \to \{1, 2, \ldots, 2mn^2\}$ be a mapping as in Lemma 2.3. By Lemma 2.3, t is injective and the index mapping t^* satisfies

$$t^*(u_j^i) = \begin{cases} 4jmn + 1 & \text{if } j \neq 0 \text{ and } n \equiv 0, 1 (\text{mod } 2), \\ 1 + 2mn^2 & \text{if } j = 0 \text{ and } n \equiv 0 \ (\text{mod } 2), \\ 2mn(n-1) + 2i & \text{if } j = 0 \text{ and } n \equiv 1 \ (\text{mod } 2). \end{cases}$$

By Lemma 2.4, $\sum_{u \in \psi^{-1}(v)} t^*(u) = 2 + 4mn^2$ for each $v \in V(G)$. Then, $[H, \psi, t]$ is a supermagic frame of G . By Proposition 1.2, G is supermagic. □

Example 2.6. Figure 2 shows supermagic labeling of $2(C_3 \times C_3)$.

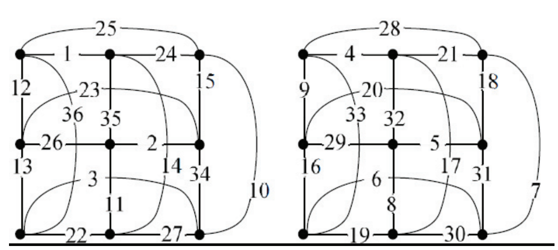

Figure 2: The supermagic labeling of $2(C_3 \times C_3)$ with $\lambda = 74$.

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