East-West J. of Mathematics: Vol. 26, No 1 (2025) pp. 61-68
https://doi.org/10.36853/ewjm0412

A VISION OF THE THEORY OF LOCAL COHOMOLOGY MODULES

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Abstract

This paper is concerned with the relation between the theory of local cohomology modules of a module, defined by an ideal, and associated prime ideals of a module.

1 Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity, and all modules are unitary. Let I be an ideal of R and R-Mod denotes the category of all R-modules and R-homomorphisms. We also denote by $\mathbb{N}_0$ and $\mathbb{N}$ the sets of non-negative and positive integers, respectively.

Local cohomology was introduced by Grothendieck and many people have worked about the understanding of their structure, (non)-vanishing and finiteness properties. For example, Grothendieck's non-vanishing theorem is one of the important theorems in local cohomology. For more details about local cohomology modules, see [3].

Key words: local cohomology modules, graded modules, tame loci, Noetherian ring.
2020 Mathematics Subject Classification: 13D45, 13C13.

Here, we put results for the local cohomology module defined by an ideal. To be more precise, let

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) : I \subseteq \mathfrak{p}\}.$$

The set of elements x of M such that $\operatorname{Supp}_R(Rx) \subseteq V(I)$ is said to be I-torsion submodule of M and is denoted by $\Gamma_I(M)$. It is easy to see that $\Gamma_I(\bullet)$ is a covariant, R-linear functor from the category of R-modules to itself. For an integer i, the local cohomology functor $\mathrm{H}^i_I(\bullet)$ with respect to I is defined to be the i-th right derived functor of $\Gamma_I(\bullet)$.

Also $\mathrm{H}^i_I(M)$, according to [3], is called the i-th local cohomology module of M with respect to I.

Let I be an ideal of R, and let M be an R-module. In [3], the i-th local cohomology module $\mathrm{H}^i_I(M)$ of M with respect to I is defined by

$$\mathrm{H}^i_I(M) = \varinjlim_{t \in \mathbb{N}} \operatorname{Ext}^i_R\left(R/I^t, M\right),$$

for all $0 \leq i \in \mathbb{Z}$.

By [4, Remark 3.5.3 (a)], we have $\mathrm{H}^0_I(M) \cong \Gamma_I(M)$, where we have that

$$\Gamma_I(M) := \left\{m \in M \mid I^t m = 0 \text{ for some } t \in \mathbb{N}\right\},$$

is an R-submodule of the R-module M. We can also to see this definition of the following form.

Definition 1.1. ([4, Definition 3.5.2]) The local cohomology functors, denoted by $\mathrm{H}^i_I(\bullet)$, are the right derived functors of $\Gamma_I(\bullet)$. In other words, if $\mathbf{I}^\bullet$ is an injective resolution of the R-module M, then $\mathrm{H}^i_I(M) \cong \mathrm{H}^i(\Gamma_I(\mathbf{I}^\bullet_M))$ for all $i \geq 0$, where $\mathbf{I}^\bullet_M$ denotes the deleted injective resolution of M.

Local cohomology theory, as in the Definition 1.1, has been a significant tool in commutative algebra and algebraic geometry.

Now, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a standard graded Noetherian ring and let N be a finitely generated graded R-module. Also, assume that

$$R_+ = \bigoplus_{n \in \mathbb{N}} R_n$$

denotes the irrelevant ideal of R. It is well known that for each $i \in \mathbb{N}_0$, $\mathrm{H}^i_{R_+}(N)$ carries a natural grading.

The concept of tameness is the most fundamental concept related to the asymptotic behavior of cohomology. A graded R-module

$$T = \bigoplus_{n \in \mathbb{Z}} T_n$$

is said to be tame, or asymptotically gap free, if either $T_n \neq 0$ for all $n << 0$ or else $T_n = 0$ for all $n << 0$.

Throughout the paper, unless other case stated,

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n$$

is a standard graded Noetherian ring, and

$$R_+ = \bigoplus_{n \in \mathbb{N}} R_n$$

is the irrelevant ideal of R and, moreover, N denote a finitely generated graded R-module.

Here, we use properties of commutative algebra and homological algebra for the development of the results (see [1] and [11]).

2 Associated prime ideals

In this section, we assume that the base ring R_0 is semi-local and study the stability of the set $\left\{\mathrm{Ass}_{R_0}(\mathrm{H}^i_{R_+}(N)_n)\right\}_{n \in \mathbb{Z}}$ when $n \to -\infty$. To this end, we need to consider tameness and Artinianness of graded R-modules

$$\Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^i_{R_+}(N)) \ , \text{ for all maximal ideal } \ \mathfrak{m}_0 \text{ of } R_0.$$

Definition 2.1. Let $T = \bigoplus_{n \in \mathbb{Z}} T_n$ be a graded R-module. Then, the following statements hold.

(1) If T is finitely generated, then in view of [7], one can see that $T_n = 0$, for all $n << 0$, T_n is a finitely generated R_0-module for all $n \in \mathbb{Z}$ and there exists $X \subseteq \mathrm{Spec}(R_0)$ such that $\mathrm{Ass}_{R_0}(T_n) = X$, for all $n >> 0$.

(2) Following [2, Definition 4.1], T is called tame, or asymptotically gap free, if there exists an integer n_0 such that either $T_n = 0$ for all $n < n_0$ or, $T_n \neq 0$ for all $n < n_0$. One can see that any Noetherian or Artinian graded R-module is tame.

(3) We say that $\{\mathrm{Ass}_{R_0}(T_n)\}_{n \in \mathbb{Z}}$ is asymptotically stable (when $n \to -\infty$) if there exists an integer n_0 and $X \subseteq \mathrm{Spec}(R_0)$ such that $\mathrm{Ass}_{R_0}(T_n) = X$ for all $n < n_0$.

(4) For each $i \in \mathbb{N}_0$, it is straightforward to see that

$$\mathrm{Ass}_R(\mathrm{H}^i_{R_+}(N)) = \left\{ \mathfrak{p}_0 + R_+ \mid \mathfrak{p}_0 \in \bigcup_{n\mathbb{Z}} \mathrm{Ass}_{R_0}(\mathrm{H}^i_{R_+}(N)_n) \right\}.$$

Definition 2.2. *Let I be an ideal of R and T be an R-module. Then T is said to be I-cofinite if* $\operatorname{Supp}(T) \subseteq V(I)$ *and* $\operatorname{Ext}^i_R(R/I, T)$ *is a finite R-module for all $i \in \mathbb{N}_0$, where*

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}.$$

Lemma 2.3. ([5, Theorem 2.5]) *Let I be an ideal of R with* $\dim(R/I) \leq 1$ *and N be a finitely generated R-module. Then $\mathrm{H}^i_I(N)$ is I-cofinite, for all $i \in \mathbb{N}_0$.*

We have now the following result.

Proposition 2.4. *Let* $\dim(R_0) = 2$. *Then, we have that*

$$\operatorname{Ass}_R(\mathrm{H}^i_{R_+}(N))$$

is a finite set, for all $i \in \mathbb{N}_0$.

Proof. Let

$$x \in \bigcap_{\mathfrak{m}_0 \in \max(R_0)} \mathfrak{m}_0 \setminus \bigcup_{\mathfrak{p}_0 \in \min(R_0)} \mathfrak{p}_0$$

and

$$A := \{\mathfrak{p}_0 + R_+ \mid \mathfrak{p}_0 \in \operatorname{Spec}(R_0) \text{ and } x \in \mathfrak{p}_0\}.$$

By our hypotheses, A is a finite set, $\operatorname{ht}(xR_0) = 1$ and $\dim((R_0)_x) \leq 1$. Using [6, Paragraph 2(4)] and Lemma 2.3, to deduce that

$$\mathrm{H}^i_{R_+}(N)_x \cong \mathrm{H}^i_{(R_x)_+}(N_x) \text{ is } (R_x)_+ - \text{cofinite.}$$

So,

$$\operatorname{Ass}_{R_x}(\mathrm{H}^i_{R_+}(N)_x) \text{ is a finite set.}$$

Now, the result follows by using the facts that

$$\operatorname{Ass}_R(\mathrm{H}^i_{R_+}(N)) \subseteq \left\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p}R_x \in \operatorname{Ass}_{R_x}(\mathrm{H}^i_{R_+}(N)_x)\right\} \bigcup A.$$

□

3 The results

We have now the following proposition.

Proposition 3.1. *Let $S_1, S_2, \ldots, S_k$ be multiplicative closed subsets of R_0 with*

$$\operatorname{Spec}(R_0) = \bigcup_{j=1}^{k} \left\{\mathfrak{p}_0 \in \operatorname{Spec}(R_0) \mid \mathfrak{p}_0 \bigcap S_j = \emptyset\right\}.$$

Then the following hold:

(1) *If for all* $j = 1, \dots k$, $\mathrm{Ass}_{S_j^{-1}R_0}(\mathrm{H}^i_{S_j^{-1}R_+}(S_j^{-1}R, S_j^{-1}N)_n)$ *is asymptotically stable, then so is*
$$\mathrm{Ass}_{R_0}(\mathrm{H}^i_{R_+}(N)_n).$$

(2) *If for all* $j = 1, \dots, k$, *we have that* $\mathrm{H}^i_{S_j^{-1}R_+}(S_j^{-1}R, S_j^{-1}N)$ *is tame, then so is* $\mathrm{H}^i_{R_+}(N)$.

Proof. One can use the same argument as used in the [8, Lemma 2.2.1] to prove the claim. □

Proposition 3.2. ([9, Paragraph 18 Lemma 2]) *Let* A *be a ring and let* N *be an* A*-module. Let* $x \in A$ *be both* A*-regular and* N*-regular, and assume that* $xA = 0$. *Then,* $\mathrm{Hom}_A(A, N) = 0$ *and* $\mathrm{Ext}^{n+1}_A(A, N) \cong \mathrm{Ext}^n_{A/xA}(A, N/xN)$, *for all* $n \in \mathbb{N}_0$.

Using the above Lemma we have the following, which will be used in the proof of the next theorem.

Proposition 3.3. *Let* $x \in R_0$ *be both* R*-regular and* N*-regular. Then*
$$\mathrm{H}^{i+1}_{R_+}(R/xR, N) \cong \mathrm{H}^i_{(R/xR)_+}(R/xR, N/xN),$$
for all $i \in \mathbb{N}_0$.

Lemma 3.4. ([10, Corollary 1.5]) *Let* N *be* I*-cofinite. Then for every maximal ideal* $\mathfrak{m}$ *of* R, $\Gamma_{\mathfrak{m}}(N)$ *is Artinian and* I*-cofinite.*

4 Application

We presented now the following result.

Theorem 4.1. *Assume that* $\dim(R_0) \leq 2$, $\mathrm{depth}(R_0) > 0$ *and* $\Gamma_{\mathfrak{m}_0R}(R) = 0 = \Gamma_{\mathfrak{m}_0R}(N)$ *for all* $\mathfrak{m}_0 \in \max(R_0)$. *Then, it follows that the graded* R*-module* $\Gamma_{\mathfrak{m}_0R}(\mathrm{H}^i_{R_+}(N))$ *is tame, for all* $i \in N_0$ *and all maximal ideal* $\mathfrak{m}_0$ *of* R_0.

Proof. Using Proposition 3.1, (2), we may assume that $(R_0, \mathfrak{m}_0)$ is local. In view of Lemma 2.3, and Lemma 3.4, the assertion holds for $\dim(R_0) \leq 1$. So let $\dim(R_0) = 2$. By Proposition 2.4, and Definition 2.1 (1) and (4), the set
$$A := \left(\bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}(\mathrm{H}^i_{R_+}(N)_n) \bigcup \mathrm{Ass}_{R_0}(N) \bigcup \mathrm{Ass}_{R_0}(R)\right) \setminus \{\mathfrak{m}_0\}$$
is finite. Therefore, there is some $x \in \mathfrak{m}_0 \setminus A$. Hence,
$$\dim(R_0/xR_0) = 1\ ,$$

x is R, and N-regular and moreover

$$\mathrm{H}^i_{R_+}(N)_n/\Gamma_{\mathfrak{m}_0}(\mathrm{H}^i_{R_+}(N)_n) - \text{regular},$$

for all $n \in \mathbb{Z}$.

It follows that,

$$\Gamma_{\mathfrak{m}_0}(\mathrm{H}^i_{R_+}(N)_n) = \Gamma_{xR_0}(\mathrm{H}^i_{R_+}(N)_n),$$

for all $n \in \mathbb{Z}$ and hence

$$\Gamma_{\mathfrak{m}_0}(\mathrm{H}^i_{R_+}(N)) = \Gamma_{xR_0}(\mathrm{H}^i_{R_+}(N)).$$

Now, consider the exact sequence

$$0 \to M \stackrel{x}{\to} M \to M/xM \to 0$$

to get the following exact sequence

$$0 \to \mathrm{H}^i_{R_+}(N)/x\mathrm{H}^i_{R_+}(N) \to \mathrm{H}^{i+1}_{R_+}(R/xR, N) \to \mathrm{H}^{i+1}_{R_+}(N).$$

Application of the functor $\Gamma_{\mathfrak{m}_0 R}(\bullet)$ to this sequence induces the following exact sequence

$$0 \to \Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^i_{R_+}(N)/x\mathrm{H}^i_{R_+}(N)) \to \Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^{i+1}_{R_+}(R/xR, N)) \to \Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^{i+1}_{R_+}(N)).$$

In view of Proposition 3.3, we have that

$$\mathrm{H}^{i+1}_{R_+}(R/xR, N) \cong \mathrm{H}^i_{(R/xR)_+}(R/xR, N/xN).$$

As

$$\dim((R/xR)_0) = 1,$$

by Lemmas 2.3 and 3.4, we have that,

$$\Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^{i+1}_{R_+}(R/xR, N)),$$

is Artinian. Therefore,

$$\Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^i_{R_+}(N)/x\mathrm{H}^i_{R_+}(N)),$$

is Artinian. Hence,

$$\Theta = \Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^i_{R_+}(N)) + x\mathrm{H}^i_{R_+}(N)/x\mathrm{H}^i_{R_+}(N)$$

is Artinian and consequently, Θ is tame. It follows that either $\Theta_n = 0$ for all $n << 0$ or $\Theta_n \neq 0$ for all $n << 0$. In the first case

$$\Big(\Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^i_{R_+}(N)) + x\mathrm{H}^i_{R_+}(N)\Big)_n \subseteq x\mathrm{H}^i_{R_+}(N)_n$$

for all $n << 0$. Then,

$$\Gamma_{xR_0}(\mathrm{H}^i_{R_+}(N)_n) = \left(\Gamma_{\mathfrak{m}_0}(\mathrm{H}^i_{R_+}(N)_n)\right) \subseteq x\mathrm{H}^i_{R_+}(N)_n.$$

It follows that

$$\Gamma_{xR_0}(\mathrm{H}^i_{R_+}(N)_n) = x\Gamma_{xR_0}(\mathrm{H}^i_{R_+}(N)_n),$$

for all $n << 0$.

Now, in view of Nakayama's Lemma, we get

$$\Gamma_{xR_0}(\mathrm{H}^i_{R_+}(N)_n) = \Gamma_{\mathfrak{m}_0}(\mathrm{H}^i_{R_+}(N)_n) = 0,$$

for all $n << 0$.

In the second case,

$$\left(\Gamma_{\mathfrak{m}_0}(\mathrm{H}^i_{R_+}(N)_n)\right) \text{ is not contained } \ x\mathrm{H}^i_{R_+}(N)_n,$$

for all $n << 0$.

This implies that

$$\Gamma_{\mathfrak{m}_0}\left(\mathrm{H}^i_{R_+}(N)_n\right) \neq 0,$$

for all $n << 0$, as desired. □

We finished the article with the following conclusion.

5 Conclusion

In this article, we can to relate the theory of commutative algebra, to the theory of local cohomology modules. With the results of the article, we show the importance of local cohomology theory as a study tool within of the commutative algebra theory.

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