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HAMILTONIAN DECOMPOSITION OF COMPLETE TRIPARTITE 3-UNIFORM HYPERGRAPHS

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Abstract

With our definition for complete tripartite 3-uniform hypergraphs which contain two types of edges, we show that complete tripartite 3-uniform hypergraphs with partite sets of equal size $K^{(3)}_{m,m,m}$ is decomposable into (tight) Hamiltonian cycles if and only if $3 \mid m$.

1 Introduction

A *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ consists of a nonempty finite set V of *vertices* with a family $\mathcal{E}$ of subsets of V, called *(hyper)edges.* If each edge has size k, we say that $\mathcal{H}$ is a *k-uniform hypergraph.* A *Hamiltonian decomposition* of a hypergraph is a partition of the set of edges into mutually disjoint Hamiltonian cycles. A *(tight) Hamiltonian cycle* in a k-uniform hypergraph is a cyclic ordering of its vertices such that each consecutive k-tuple of vertices is an edge. This definition was introduced by Katona and Kierstead [4], and we will use this definition of Hamiltonian cycle for this article. The older definition of a Hamiltonian cycle was given by Berge [2]. The Hamiltonian decomposition of complete 3-uniform

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hypergraphs was completely investigated in 1994 by Verrall [8] using Berge's definition. In 2000s, Bailey and Stevens [1], also Meszka and Rosa [7], Xu and Wang [10], decomposed complete k-uniform hypergraphs using Katona-Kierstead's definition and this decomposition problem is still not completed and involving the aids of computer programming.

Our motivation comes from the problem of decomposing complete bipartite 3-uniform hypergraphs. This was first introduced by Jirimutu and Wang [3] and was completed later by Xu and Wang [10]. This leads us to extend "bipartite" to "tripartite" and define a complete tripartite 3-uniform hypergraphs as follows.

Definition 1. The *complete tripartite 3-uniform hypergraph* has the vertex set V partitioned into three subsets V_0, V_1 and V_2 and the edge set $\mathcal{E}$ such that $\mathcal{E} = \{e : e \subseteq V, |e| = 3 \text{ and } |e \cap V_i| < 3 \text{ for all } i \in \{0,1,2\}\}$, and denoted by $K^{(3)}_{m,m,m}$ when $|V_0| = |V_1| = |V_2| = m$.

For convenience, W, $\overline{W}$ and $\overline{\overline{W}}$ are used to denote the vertices of $K^{(3)}_{m,m,m}$ with

$$\begin{aligned}
V_0 &= W = \{0, 1, \ldots, m-1\},\\
V_1 &= \overline{W} = \{\overline{0}, \overline{1}, \ldots, \overline{m-1}\},\\
V_2 &= \overline{\overline{W}} = \{\overline{\overline{0}}, \overline{\overline{1}}, \ldots, \overline{\overline{m-1}}\}.
\end{aligned}$$

Due to Definition 1, we classify edges of $K^{(3)}_{m,m,m}$ into two types:

Type 1 edges are of the form $\{a, \overline{b}, \overline{\overline{c}}\}$ where $a, b, c \in \mathbb{Z}_m$; and

Type 2 edges are of the form $\{x, x', y\}$ in which x and x' are in the same partite set, and y is in a different partite set. Note that there are six forms of $\{x, x', y\}$: $\{a, a', \overline{b}\}, \{a, a', \overline{\overline{c}}\}, \{\overline{b}, \overline{b'}, a\}, \{\overline{b}, \overline{b'}, \overline{\overline{c}}\}, \{\overline{\overline{c}}, \overline{\overline{c'}}, a\}$ and $\{\overline{\overline{c}}, \overline{\overline{c'}}, \overline{b}\}$ where $a, a', b, b', c, c' \in \mathbb{Z}_m$ and $a \neq a', b \neq b', c \neq c'$.

In 2013, Kuhl and Schroeder [5] published their results on Hamiltonian decompositions of complete k-uniform k-partite hypergraphs and completely found solutions for $k = 3$. A complete 3-uniform 3-partite hypergraph defined in [5] by Kuhl and Schroeder consists of all Type 1 edges but no Type 2 edges, so their hypergraph is a subhypergraph of our $K^{(3)}_{m,m,m}$ given by Definition 1. In some cases, we can use some of their results, that is $h(x, y)$ in Section 2, to be a part of our Hamiltonian decompositions of $K^{(3)}_{m,m,m}$.

If $K^{(3)}_{m,m,m}$ has a Hamiltonian decomposition, then the number of edges of $K^{(3)}_{m,m,m}$ which is equal to $\binom{3m}{3} - 3\binom{m}{3}$ must be divisible by $3m$. Thus, the necessary condition is $3 \mid m$. The purpose of this paper is to show that if $3 \mid m$, then $K^{(3)}_{m,m,m}$ has a Hamiltonian decomposition. The proof will be separated into two cases, $m \equiv 0 \pmod 6$ and $m \equiv 3 \pmod 6$ and a special case $m = 3$. In Section 2, we classify four forms of Hamiltonian cycles of $K^{(3)}_{m,m,m}$. These forms will be combined and the combination becomes a Hamiltonian decomposition of $K^{(3)}_{m,m,m}$ in Section 3. Finally, conclusion and discussion will be given in Section 4.

2 Hamiltonian Cycle Constructions

In this section, we provide four forms of a Hamiltonian cycle in $K^{(3)}_{m,m,m}$ to be used through out this article: $C(i,j)$, $C'(i,j)$, ($C_M(i)$ and $C'_M(i)$) and $h(x,y)$. First, let us define a useful notation as follows.

Definition 2. For $x, y \in \mathbb{Z}_m$, $||x-y|| = \min\{(x-y) (\mathrm{mod}\ m), (y-x) (\mathrm{mod}\ m)\}$.

2.1 $C(i,j)$

For $m \equiv 0 \pmod 3$, define a Hamiltonian cycle of $K^{(3)}_{m,m,m}$, $C(i,j)$ by

$$
\begin{aligned}
&C(i,j) = \\
&\quad (a_0+i, \overline{b_0+j}, \overline{\overline{c_0+i+j}}, \overline{\overline{c_1+i+j}}, a_1+i, \overline{b_1+j}, \overline{b_2+j}, \overline{\overline{c_2+i+j}}, a_2+i, \\
&\quad a_3+i, \overline{b_3+j}, \overline{\overline{c_3+i+j}}, \overline{\overline{c_4+i+j}}, a_4+i, \overline{b_4+j}, \overline{b_5+j}, \overline{\overline{c_5+i+j}}, a_5+i, \\
&\quad \ldots, a_{m-3}+i, \overline{b_{m-3}+j}, \overline{\overline{c_{m-3}+i+j}}, \overline{\overline{c_{m-2}+i+j}}, a_{m-2}+i, \overline{b_{m-2}+j}, \\
&\quad \overline{b_{m-1}+j}, \overline{\overline{c_{m-1}+i+j}}, a_{m-1}+i),
\end{aligned}
$$

where $i, j \in \mathbb{Z}_m$, $\{a_0, a_1, \ldots, a_{m-1}\} = \mathbb{Z}_m$, $\{b_0, b_1, \ldots, b_{m-1}\} = \mathbb{Z}_m$, and $\{c_0, c_1, \ldots, c_{m-1}\} = \mathbb{Z}_m$.

Lemma 1. *Let $m \equiv 0 \pmod 3$. Suppose $C(0,0)$ has properties that $c_k - b_k = c_{k'} - b_{k'}$ for all $k, k' \in \mathbb{Z}_m$ with $k \neq k'$, and $||a_{3k-1} - a_{3k}|| \neq ||a_{3k'-1} - a_{3k'}||, ||b_{3k+1} - b_{3k+2}|| \neq ||b_{3k'+1} - b_{3k'+2}||, ||c_{3k} - c_{3k+1}|| \neq ||c_{3k'} - c_{3k'+1}||$ for all $k, k' \in \{0, 1, \ldots, \frac{m}{3} - 1\}$ with $k \neq k'$. Then $\{C(i,j) : i, j \in \mathbb{Z}_m\}$ is a set of m^2 disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

Proof. For edges of the form $\{a, \overline{b}, \overline{\overline{c}}\}$, we will show that if $\{a_k + i, \overline{b_k + j}, \overline{\overline{c_k + i + j}}\} = \{a_{k'} + i', \overline{b_{k'} + j'}, \overline{\overline{c_{k'} + i' + j'}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{a_k + i, \overline{b_k + j}, \overline{\overline{c_k + i + j}}\} = \{a_{k'} + i', \overline{b_{k'} + j'}, \overline{\overline{c_{k'} + i' + j'}}\}$ for some $i, i', j, j', k, k' \in \mathbb{Z}_m$. Then

$$
\begin{aligned}
a_k + i &\equiv a_{k'} + i' \pmod m, \\
b_k + j &\equiv b_{k'} + j' \pmod m, \\
c_k + i + j &\equiv c_{k'} + i' + j' \pmod m.
\end{aligned}
$$

Since $c_k - b_k = c_{k'} - b_{k'}$, we get $i = i'$ and then $a_k = a_{k'}$. Then $j = j'$. Hence, $i = i'$, $j = j'$ and $k = k'$.

For edges of the form $\{a, a', \overline{b}\}$, we will show that if $\{a_{3k-1}+i, a_{3k}+i, \overline{b_{3k}+j}\} = \{a_{3k'-1}+i', a_{3k'}+i', \overline{b_{3k'}+j'}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{a_{3k-1}+i, a_{3k}+i, \overline{b_{3k}+j}\} = \{a_{3k'-1}+i', a_{3k'}+i', \overline{b_{3k'}+j'}\}$ for some $i, i', j, j' \in \mathbb{Z}_m$ and $k, k' \in \{0, 1, \ldots, \frac{m}{3}-1\}$. Then

$$\begin{aligned} a_{3k-1}+i &\equiv a_{3k'-1}+i' \pmod{m}, \\ a_{3k}+i &\equiv a_{3k'}+i' \pmod{m}, \\ b_{3k}+j &\equiv b_{3k'}+j' \pmod{m}, \end{aligned}$$

or

$$\begin{aligned} a_{3k-1}+i &\equiv a_{3k'}+i' \pmod{m}, \\ a_{3k}+i &\equiv a_{3k'-1}+i' \pmod{m}, \\ b_{3k}+j &\equiv b_{3k'}+j' \pmod{m}. \end{aligned}$$

Since $||a_{3k-1}-a_{3k}|| \neq ||a_{3k'-1}-a_{3k'}||$ for all $k \neq k'$ but $a_{3k-1}-a_{3k} \equiv a_{3k'-1}-a_{3k'} \pmod{m}$ or $a_{3k-1}-a_{3k} \equiv a_{3k'}-a_{3k'-1} \pmod{m}$, we have $k = k'$. Then $i = i'$ and $j = j'$.

For other edge-forms: $\{a, a', \overline{\overline{c}}\}$, $\{\overline{b}, \overline{b'}, \overline{\overline{c}}\}$, $\{\overline{b}, \overline{b'}, a\}$, $\{\overline{\overline{c}}, \overline{\overline{c'}}, a\}$, $\{\overline{\overline{c}}, \overline{\overline{c'}}, \overline{b}\}$, we can prove the same result in a similar manner. Thus, all $3m \times m^2$ edges of $\{C(i,j) : i, j \in \mathbb{Z}_m\}$ are distinct and $\{C(i,j) : i, j \in \mathbb{Z}_m\}$ is a set of m^2 disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$. $\square$

Lemma 2. *Let $m \equiv 0 \pmod{3}$. Let $c_i = b_i = x_i$ and $a_i = x_{i+1}$ for all $i \in \mathbb{Z}_m$, where*

$$\begin{aligned} x_{3k} &= \begin{cases} 3k/2 & \text{if } k \text{ is even,} \\ (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases} \\ x_{3k+1} &= 3k+1, \\ x_{3k+2} &= \begin{cases} \lceil m/2 \rceil + 3k/2 & \text{if } k \text{ is even,} \\ \lceil m/2 \rceil + (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases} \end{aligned}$$

and $k \in \{0, 1, \ldots, \frac{m}{3}-1\}$. Then $C(0,0)$ has properties as in Lemma 1. Moreover, $||x-x'|| \equiv 1$ or $2 \pmod{3}$ for all Type 2 edges of the form $\{x, x', y\}$ in $C(0,0)$.

Proof. By this setting, we have $c_k - b_k = 0 = c_{k'} - b_{k'}$ for all $k, k' \in \mathbb{Z}_m$ with $k \neq k'$. For $k \in \{0, 1, \ldots, \frac{m}{3}-1\}$,

$$||a_{3k-1}-a_{3k}|| = ||x_{3k}-x_{3k+1}|| = \begin{cases} (3k+2)/2 & \text{if } k \text{ is even,} \\ (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases}$$

$$||b_{3k+1}-b_{3k+2}|| = ||x_{3k+1}-x_{3k+2}|| = \begin{cases} \lceil m/2 \rceil - (3k+2)/2 & \text{if } k \text{ is even,} \\ \lceil m/2 \rceil - (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases}$$

$$||c_{3k}-c_{3k+1}|| = ||x_{3k}-x_{3k+1}|| = \begin{cases} (3k+2)/2 & \text{if } k \text{ is even,} \\ (3k+1)/2 & \text{if } k \text{ is odd.} \end{cases}$$

Thus, $||a_{3k-1} - a_{3k}|| \neq ||a_{3k'-1} - a_{3k'}||, ||b_{3k+1} - b_{3k+2}|| \neq ||b_{3k'+1} - b_{3k'+2}||$, $||c_{3k} - c_{3k+1}|| \neq ||c_{3k'} - c_{3k'+1}||$ for all $k, k' \in \{0, 1, \ldots, \frac{m}{3} - 1\}$ with $k \neq k'$ and $||x - x'|| \equiv 1$ or $2 \pmod 3$ for all Type 2 edges of the form $\{x, x', y\}$. □

Example 1. Let $m = 6$. The cycle $C(0,0)$ in Lemma 2 is

$$C(0,0) = (1, \overline{0}, \overline{\overline{0}}, \overline{\overline{1}}, 3, \overline{1}, \overline{3}, \overline{\overline{3}}, 2, 4, \overline{2}, \overline{\overline{2}}, \overline{\overline{4}}, 5, \overline{4}, \overline{5}, \overline{\overline{5}}, 0).$$

2.2 $C'(i,j)$

For odd integer m, define a Hamiltonian cycle of $K^{(3)}_{m,m,m}$, $C'(i,j)$ by

$$\begin{aligned} C'(i,j) = \\ (&a_0 + j, a_1 + j, \overline{b_0 + i + j}, \overline{b_1 + i + j}, \overline{\overline{c_0 + 2i + j}}, \overline{\overline{c_1 + 2i + j}}, \\ &a_2 + j, a_3 + j, \overline{b_2 + i + j}, \overline{b_3 + i + j}, \overline{\overline{c_2 + 2i + j}}, \overline{\overline{c_3 + 2i + j}}, \ldots, \\ &a_{m-3} + j, a_{m-2} + j, \overline{b_{m-3} + i + j}, \overline{b_{m-2} + i + j}, \overline{\overline{c_{m-3} + 2i + j}}, \overline{\overline{c_{m-2} + 2i + j}}, \\ &a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-1} + 2i + j}}), \end{aligned}$$

where $i, j \in \mathbb{Z}_m$, $\{a_0, a_1, \ldots, a_{m-1}\} = \mathbb{Z}_m$, $\{b_0, b_1, \ldots, b_{m-1}\} = \mathbb{Z}_m$, and $\{c_0, c_1, \ldots, c_{m-1}\} = \mathbb{Z}_m$.

A similar argument as in the proof of Lemma 1 can be used to prove Lemma 3.

Lemma 3. *For odd integer m, suppose $C'(0,0)$ has properties that $a_0 + c_{m-1} \neq a_{m-1} + c_{m-2} \pmod m$ and $||a_{2k+1} - a_{2k}|| \neq ||a_{2k'+1} - a_{2k'}||, ||b_{2k+1} - b_{2k}|| \neq ||b_{2k'+1} - b_{2k'}||, ||c_{2k+1} - c_{2k}|| \neq ||c_{2k'+1} - c_{2k'}||$ for all $k, k' \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$ with $k \neq k'$. Then $\{C'(i,j) : i, j \in \mathbb{Z}_m\}$ is a set of m^2 disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

Lemma 4. *For odd integer m, let $a_i = b_i = x_i$ for all $i \in \mathbb{Z}_m$, $c_{m-3} = x_0$, $c_{m-2} = x_1$, $c_{m-1} = x_{m-1}$ and $c_i = x_{i+2}$ for all $i \in \{0, 1, \ldots, m-4\}$, where $x_{m-1} = 1, x_{2k} = m - k, x_{2k+1} = k + 2$, and $k \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$. Then $C'(0,0)$ has properties as in Lemma 3. Moreover, $b_{m-1} - a_{m-1} = 0, b_{m-1} - a_0 = 1, c_{m-1} - b_{m-1} = 0, c_{m-2} - b_{m-1} = 1$.*

Proof. By this setting, we have $a_0 + c_{m-1} = 1$ and $a_{m-1} + c_{m-2} = 3$.
For $k \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$,

$$\begin{aligned} ||a_{2k+1} - a_{2k}|| = ||b_{2k+1} - b_{2k}|| &= ||x_{2k+1} - x_{2k}|| \\ &= \min\{2k + 2, m - (2k + 2)\}. \end{aligned}$$

For $k \in \{0, 1, \ldots, \frac{m-1}{2} - 2\}$,

$$\begin{aligned} ||c_{2k+1} - c_{2k}|| &= ||x_{2k+3} - x_{2k+2}|| \\ &= \min\{2k + 4, m - (2k + 4)\} \end{aligned}$$

and $c_{m-2} - c_{m-3} = x_1 - x_0 = 2$.

Since m is odd, $\{||x_{2k+1} - x_{2k}|| : k \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}\} = \{1, 2, \ldots, \frac{m-1}{2}\}$. Thus, $a_0 + c_{m-1} \neq a_{m-1} + c_{m-2} \pmod{m}$ and $||a_{2k+1} - a_{2k}|| \neq ||a_{2k'+1} - a_{2k'}||, ||b_{2k+1} - b_{2k}|| \neq ||b_{2k'+1} - b_{2k'}||, ||c_{2k+1} - c_{2k}|| \neq ||c_{2k'+1} - c_{2k'}||$ for all $k, k' \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$ with $k \neq k'$. $\square$

Example 2. Let $m = 9$. The cycle $C'(0,0)$ in Lemma 4 is

$$C'(0,0) = (0, 2, \overline{0}, \overline{2}, \overline{\overline{8}}, \overline{\overline{3}}, 8, 3, \overline{8}, \overline{3}, \overline{\overline{7}}, \overline{\overline{4}}, 7, 4, \overline{7}, \overline{4}, \overline{\overline{6}}, \overline{\overline{5}}, 6, 5, \overline{6}, \overline{5}, \overline{\overline{0}}, \overline{\overline{2}}, 1, \overline{1}, \overline{\overline{1}}).$$

2.3 $C_M(i)$ and $C'_M(i)$

First, consider the case where m is even. We introduce a technique different from those of 2.1 and 2.2 to construct a family of Hamiltonian cycles which contain no edges of the form $\{a, \overline{b}, \overline{\overline{c}}\}$. This technique requires the knowledge of 1-factors and orthogonal quasigroups.

Definition 3. Let G be a graph. A *1-factor* of G is a subgraph of G in which every vertex has degree 1. A *1-factorization* of G is a partition of an edge set of G into 1-factors.

Definition 4. $(\mathbb{Z}_n, \circ)$ is a *quasigroup* if
(1) $i \circ j \in \mathbb{Z}_n$ for all $i, j \in \mathbb{Z}_n$, and
(2) $i \circ j \neq i \circ j'$ and $i \circ j \neq i' \circ j$ for all $i, j \in \mathbb{Z}_n$ with $i \neq i', j \neq j'$.
Note that the multiplication table of $(\mathbb{Z}_n, \circ)$ is a *Latin square.*

Definition 5. $(\mathbb{Z}_n, \circ_1)$ and $(\mathbb{Z}_n, \circ_2)$ are *orthogonal* if for $(i,j) \neq (i',j') \in \mathbb{Z}_n^2$, $i \circ_1 j = i' \circ_1 j'$ implies $i \circ_2 j \neq i' \circ_2 j'$.

Lemma 5 ([6]). *There exists a pair of mutually orthogonal Latin squares of order n for every $n \neq 2$ or 6.*

For even integer m, let $M = \{x_0x_1, x_2x_3, x_4x_5, \ldots, x_{m-2}x_{m-1}\}$ be a 1-factor of a graph with $\mathbb{Z}_m$ as a vertex set. By Lemma 5, there exists a pair of orthogonal quasigroups, $(\mathbb{Z}_{m/2}, \circ_1)$ and $(\mathbb{Z}_{m/2}, \circ_2)$ for $m \neq 4$ or 12. For $i \in \mathbb{Z}_{m/2}$, define Hamiltonian cycles of $K^{(3)}_{m,m,m}$, $C_M(i)$ and $C'_M(i)$, by

$$\begin{aligned} C_M(i) = (&x_0, x_1, \overline{x_{2(i\circ_1 0)}}, \overline{x_{2(i\circ_1 0)+1}}, \overline{\overline{x_{2(i\circ_2 0)}}}, \overline{\overline{x_{2(i\circ_2 0)+1}}},\\ &x_2, x_3, \overline{x_{2(i\circ_1 1)}}, \overline{x_{2(i\circ_1 1)+1}}, \overline{\overline{x_{2(i\circ_2 1)}}}, \overline{\overline{x_{2(i\circ_2 1)+1}}}, \ldots,\\ &x_{m-2}, x_{m-1}, \overline{x_{2(i\circ_1 \frac{m-2}{2})}}, \overline{x_{2(i\circ_1 \frac{m-2}{2})+1}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})}}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})+1}}}) \end{aligned}$$

and

$$\begin{aligned} C'_M(i) = (&x_1, x_0, \overline{x_{2(i\circ_1 0)+1}}, \overline{x_{2(i\circ_1 0)}}, \overline{\overline{x_{2(i\circ_2 0)+1}}}, \overline{\overline{x_{2(i\circ_2 0)}}},\\ &x_3, x_2, \overline{x_{2(i\circ_1 1)+1}}, \overline{x_{2(i\circ_1 1)}}, \overline{\overline{x_{2(i\circ_2 1)+1}}}, \overline{\overline{x_{2(i\circ_2 1)}}}, \ldots,\\ &x_{m-1}, x_{m-2}, \overline{x_{2(i\circ_1 \frac{m-2}{2})+1}}, \overline{x_{2(i\circ_1 \frac{m-2}{2})}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})+1}}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})}}}). \end{aligned}$$

Example 3. Let $m = 6$. The multiplication tables of orthogonal quasigroups $(\mathbb{Z}_3, \circ_1)$ and $(\mathbb{Z}_3, \circ_2)$ are as follows.

$\circ_1$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\circ_2$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Let $M = \{x_0x_1, x_2x_3, x_4x_5\} = \{03, 14, 25\}$. Then

$$\begin{aligned}
C_M(0) &= (0, 3, \overline{0}, \overline{3}, \overline{\overline{0}}, \overline{\overline{3}}, 1, 4, \overline{1}, \overline{4}, \overline{\overline{1}}, \overline{\overline{4}}, 2, 5, \overline{2}, \overline{5}, \overline{\overline{2}}, \overline{\overline{5}}),\\
C_M(1) &= (0, 3, \overline{1}, \overline{4}, \overline{\overline{2}}, \overline{\overline{5}}, 1, 4, \overline{2}, \overline{5}, \overline{\overline{0}}, \overline{\overline{3}}, 2, 5, \overline{0}, \overline{3}, \overline{\overline{1}}, \overline{\overline{4}}),\\
C_M(2) &= (0, 3, \overline{2}, \overline{5}, \overline{\overline{1}}, \overline{\overline{4}}, 1, 4, \overline{0}, \overline{3}, \overline{\overline{2}}, \overline{\overline{5}}, 2, 5, \overline{1}, \overline{4}, \overline{\overline{0}}, \overline{\overline{3}}),\\
C'_M(0) &= (3, 0, \overline{3}, \overline{0}, \overline{\overline{3}}, \overline{\overline{0}}, 4, 1, \overline{4}, \overline{1}, \overline{\overline{4}}, \overline{\overline{1}}, 5, 2, \overline{5}, \overline{2}, \overline{\overline{5}}, \overline{\overline{2}}),\\
C'_M(1) &= (3, 0, \overline{4}, \overline{1}, \overline{\overline{5}}, \overline{\overline{2}}, 4, 1, \overline{5}, \overline{2}, \overline{\overline{3}}, \overline{\overline{0}}, 5, 2, \overline{3}, \overline{0}, \overline{\overline{4}}, \overline{\overline{1}}),\\
C'_M(2) &= (3, 0, \overline{5}, \overline{2}, \overline{\overline{4}}, \overline{\overline{1}}, 4, 1, \overline{3}, \overline{0}, \overline{\overline{5}}, \overline{\overline{2}}, 5, 2, \overline{4}, \overline{1}, \overline{\overline{3}}, \overline{\overline{0}}).
\end{aligned}$$

For $m = 4$ and 12, there are no orthogonal quasigroups $(\mathbb{Z}_{m/2}, \circ_1)$ and $(\mathbb{Z}_{m/2}, \circ_2)$, therefore, $C_M(i)$ and $C'_M(i)$ will be constructed by the following way.

For $m = 4$, let $M = \{x_0x_1, x_2x_3\}$. Then

$$\begin{aligned}
C_M(0) &= (x_0, x_1, \overline{x_0}, \overline{x_1}, \overline{\overline{x_0}}, \overline{\overline{x_1}}, x_2, x_3, \overline{x_2}, \overline{x_3}, \overline{\overline{x_2}}, \overline{\overline{x_3}}),\\
C_M(1) &= (x_0, x_1, \overline{x_3}, \overline{x_2}, \overline{\overline{x_3}}, \overline{\overline{x_2}}, x_2, x_3, \overline{x_1}, \overline{x_0}, \overline{\overline{x_1}}, \overline{\overline{x_0}}),\\
C'_M(0) &= (x_1, x_0, \overline{x_1}, \overline{x_0}, \overline{\overline{x_2}}, \overline{\overline{x_3}}, x_3, x_2, \overline{x_3}, \overline{x_2}, \overline{\overline{x_0}}, \overline{\overline{x_1}}),\\
C'_M(1) &= (x_1, x_0, \overline{x_2}, \overline{x_3}, \overline{\overline{x_1}}, \overline{\overline{x_0}}, x_3, x_2, \overline{x_0}, \overline{x_1}, \overline{\overline{x_3}}, \overline{\overline{x_2}}).
\end{aligned}$$

For $m = 12$, let $(\mathbb{Z}_{m/4}, \circ_3)$ and $(\mathbb{Z}_{m/4}, \circ_4)$ be orthogonal quasigroups. For $i \in \mathbb{Z}_{m/2}$, define $C_M(i)$ and $C'_M(i)$ by

$$\begin{aligned}
C_M(i) = (&x_0, x_1, \overline{b_0(i)}, \overline{b_1(i)}, \overline{\overline{c_0(i)}}, \overline{\overline{c_1(i)}},\\
&x_2, x_3, \overline{b_2(i)}, \overline{b_3(i)}, \overline{\overline{c_2(i)}}, \overline{\overline{c_3(i)}}, \ldots,\\
&x_{m-2}, x_{m-1}, \overline{b_{m-2}(i)}, \overline{b_{m-1}(i)}, \overline{\overline{c_{m-2}(i)}}, \overline{\overline{c_{m-1}(i)}})
\end{aligned}$$

and

$$\begin{aligned}
C'_M(i) = (&x_1, x_0, \overline{b'_0(i)}, \overline{b'_1(i)}, \overline{\overline{c'_0(i)}}, \overline{\overline{c'_1(i)}},\\
&x_3, x_2, \overline{b'_2(i)}, \overline{b'_3(i)}, \overline{\overline{c'_2(i)}}, \overline{\overline{c'_3(i)}}, \ldots,\\
&x_{m-1}, x_{m-2}, \overline{b'_{m-2}(i)}, \overline{b'_{m-1}(i)}, \overline{\overline{c'_{m-2}(i)}}, \overline{\overline{c'_{m-1}(i)}}),
\end{aligned}$$

where for $j, k \in \{0, 1, \ldots, \frac{m}{4} - 1\}$,

$$
\begin{aligned}
&b_{2k}(j) = b_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = b'_{2k+1}(j) = b'_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = x_{2(j \circ_3 k)},\\
&c_{2k}(j) = c_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = c'_{2k+1}(j + \tfrac{m}{4}) = c'_{\frac{m}{2}+2k}(j) = x_{2(j \circ_4 k)},\\
&b_{2k+1}(j) = b_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = b'_{2k}(j) = b'_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = x_{2(j \circ_3 k)+1},\\
&c_{2k+1}(j) = c_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = c'_{2k}(j + \tfrac{m}{4}) = c'_{\frac{m}{2}+2k+1}(j) = x_{2(j \circ_4 k)+1},\\
&b_{2k+1}(j + \tfrac{m}{4}) = b_{\frac{m}{2}+2k}(j) = b'_{2k}(j + \tfrac{m}{4}) = b'_{\frac{m}{2}+2k+1}(j) = x_{\frac{m}{2}+2(j \circ_3 k)},\\
&c_{2k+1}(j + \tfrac{m}{4}) = c_{\frac{m}{2}+2k}(j) = c'_{2k}(j) = c'_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = x_{\frac{m}{2}+2(j \circ_4 k)},\\
&b_{2k}(j + \tfrac{m}{4}) = b_{\frac{m}{2}+2k+1}(j) = b'_{2k+1}(j + \tfrac{m}{4}) = b'_{\frac{m}{2}+2k}(j) = x_{\frac{m}{2}+2(j \circ_3 k)+1},\\
&c_{2k}(j + \tfrac{m}{4}) = c_{\frac{m}{2}+2k+1}(j) = c'_{2k+1}(j) = c'_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = x_{\frac{m}{2}+2(j \circ_4 k)+1}.
\end{aligned}
$$

Lemma 6. *For even integer m, given a 1-factor M of a graph with $\mathbb{Z}_m$ as a vertex set, $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$ is a set of m disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

Proof. Let $M = \{x_0x_1, x_2x_3, \ldots, x_{m-2}x_{m-1}\}$. Consider the case where $m \notin \{4, 12\}$. For edges of the form $\{a, a', \overline{b}\}$, we will show that if $\{x_{2k}, x_{2k+1}, \overline{x_{2(i\circ_1 k)+j}}\} = \{x_{2k'}, x_{2k'+1}, \overline{x_{2(i'\circ_1 k')+j'}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{x_{2k}, x_{2k+1}, \overline{x_{2(i\circ_1 k)+j}}\} = \{x_{2k'}, x_{2k'+1}, \overline{x_{2(i'\circ_1 k')+j'}}\}$ for some $i, i', k, k' \in \mathbb{Z}_{m/2}$ and $j, j' \in \{0, 1\}$. Then

$$
\begin{aligned}
2k &= 2k',\\
2(i \circ_1 k) + j &= 2(i' \circ_1 k') + j'.
\end{aligned}
$$

That is $k = k'$, $j = j'$ and $i \circ_1 k = i' \circ_1 k$. Since $(\mathbb{Z}_{m/2}, \circ_1)$ is a quasigroup, $i = i'$.

The proof for edges of the form $\{a, a', \overline{\overline{c}}\}$ can be done in the same way.

For edges of the form $\{\overline{b}, \overline{b'}, \overline{\overline{c}}\}$, we will show that if $\{\overline{x_{2(i\circ_1 k)}}, \overline{x_{2(i\circ_1 k)+1}}, \overline{\overline{x_{2(i\circ_2 k)+j}}}\} = \{\overline{x_{2(i'\circ_1 k')}}, \overline{x_{2(i'\circ_1 k')+1}}, \overline{\overline{x_{2(i'\circ_2 k')+j'}}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{\overline{x_{2(i\circ_1 k)}}, \overline{x_{2(i\circ_1 k)+1}}, \overline{\overline{x_{2(i\circ_2 k)+j}}}\} = \{\overline{x_{2(i'\circ_1 k')}}, \overline{x_{2(i'\circ_1 k')+1}}, \overline{\overline{x_{2(i'\circ_2 k')+j'}}}\}$ for some $i, i', k, k' \in \mathbb{Z}_{m/2}$ and $j, j' \in \{0, 1\}$. Then

$$
\begin{aligned}
i \circ_1 k &= i' \circ_1 k',\\
i \circ_2 k &= i' \circ_2 k',\\
j &= j'.
\end{aligned}
$$

Since $(\mathbb{Z}_{m/2}, \circ_1)$ and $(\mathbb{Z}_{m/2}, \circ_2)$ are orthogonal quasigroups, we have $i = i'$ and $k = k'$.

The proof for edges of the forms $\{\overline{b}, \overline{b'}, a\}$, $\{\overline{\overline{c}}, \overline{\overline{c'}}, a\}$ and $\{\overline{\overline{c}}, \overline{\overline{c'}}, \overline{b}\}$ can also be done in the same way. Thus, all $3m \times m$ edges of $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$

are distinct and $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$ is a set of m disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.

For $m = 4$, it is easy to see that $C_M(0)$, $C_M(1)$, $C'_M(0)$ and $C'_M(1)$ are mutually disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.

For $m = 12$, consider edges of the form $\{a, a', \overline{b}\}$: $e_1 = \{x_{2k}, x_{2k+1}, \overline{x_i}\}$ and $e_2 = \{x_{\frac{m}{2}+2k}, x_{\frac{m}{2}+2k+1}, \overline{x_i}\}$, where $k \in \mathbb{Z}_{m/4}$ and $i \in \mathbb{Z}_m$. Note that $\{2(j \circ_3 k), 2(j \circ_3 k) + 1, \frac{m}{2} + 2(j \circ_3 k), \frac{m}{2} + 2(j \circ_3 k) + 1 : j, k \in \mathbb{Z}_{m/4}\} = \mathbb{Z}_m$ by means of a quasigroup.

If $i = 2(j \circ_3 k)$, then $e_1 \in C_M(j)$ and $e_2 \in C'_M(j + \frac{m}{4})$.

If $i = 2(j \circ_3 k) + 1$, then $e_1 \in C'_M(j)$ and $e_2 \in C_M(j + \frac{m}{4})$.

If $i = \frac{m}{2} + 2(j \circ_3 k)$, then $e_1 \in C'_M(j + \frac{m}{4})$ and $e_2 \in C_M(j)$.

If $i = \frac{m}{2} + 2(j \circ_3 k) + 1$, then $e_1 \in C_M(j + \frac{m}{4})$ and $e_2 \in C'_M(j)$.

Thus, each edge of the form $\{a, a', \overline{b}\}$ is in a unique Hamiltonian cycle. Also use this way to show the same result for edges of the form $\{a, a', \overline{\overline{c}}\}$.

For edges of the form $\{\overline{b}, \overline{b'}, \overline{\overline{c}}\}$: $\{\overline{x_{2(j\circ_3 k)}}, \overline{x_{2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\}$ (or $\{\overline{x_{\frac{m}{2}+2(j\circ_3 k)}}, \overline{x_{\frac{m}{2}+2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\}$), we will show that if $\{\overline{x_{2(j\circ_3 k)}}, \overline{x_{2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\} = \{\overline{x_{2(j'\circ_3 k')}}, \overline{x_{2(j'\circ_3 k')+1}}, \overline{\overline{x_{i'}}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{\overline{x_{2(j\circ_3 k)}}, \overline{x_{2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\} = \{\overline{x_{2(j'\circ_3 k')}}, \overline{x_{2(j'\circ_3 k')+1}}, \overline{\overline{x_{i'}}}\}$ for some $j, j', k, k' \in \mathbb{Z}_{m/4}$ and $i \in \mathbb{Z}_m$. Then

$$\begin{aligned} j \circ_3 k &= j' \circ_3 k', \\ i &= i'. \end{aligned}$$

There are four possibilities for i: $2(j \circ_4 k)$, $2(j \circ_4 k) + 1$, $\frac{m}{2} + 2(j \circ_4 k)$ or $\frac{m}{2} + 2(j \circ_4 k) + 1$ (also for i': $2(j' \circ_4 k')$, $2(j' \circ_4 k') + 1$, $\frac{m}{2} + 2(j' \circ_4 k')$ or $\frac{m}{2} + 2(j' \circ_4 k') + 1$). Since $i = i'$, in any cases, we have $j \circ_4 k = j' \circ_4 k'$. The orthogonality of $(\mathbb{Z}_{m/4}, \circ_3)$ and $(\mathbb{Z}_{m/4}, \circ_4)$ implies $j = j'$ and $k = k'$.

Edges of the forms $\{\overline{b}, \overline{b'}, a\}$, $\{\overline{\overline{c}}, \overline{\overline{c'}}, a\}$ and $\{\overline{\overline{c}}, \overline{\overline{c'}}, \overline{b}\}$ can be showed in a similar manner. This completes the proof. □

2.4 $h(x, y)$

For $(x, y) \in \mathbb{Z}_m^2$, define a Hamiltonian cycle of $K^{(3)}_{m,m,m}$, $h(x, y)$ by

$$h(x, y) = (0, \overline{x}, \overline{\overline{x + y}}, m - 1, \overline{m - 1 + x}, \overline{\overline{m - 1 + x + y}}, \ldots, 1, \overline{1 + x}, \overline{\overline{1 + x + y}}).$$

Kuhl and Schroeder [5] define a *difference type* of each edge of the form $\{a, \overline{b}, \overline{\overline{c}}\}$ to be $(b - a, c - b)$ in modulo m. There are m edges with a specific difference type. $h(x, y)$ has $3m$ edges and contains all edges of difference types (x, y), $(x + 1, y)$ and $(x, y + 1)$.

Note that all m^3 edges in $K^{(3)}_{m,m,m}$ are classified into m^2 distinct difference types.

Lemma 7 ([5]). *Let $m \equiv 0 \pmod 3$ and $\mathcal{A}_0 = \{(x, y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod 3\}$. Then $\{h(x, y) : (x, y) \in \mathcal{A}_0\}$ is a set of $m^2/3$ disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

3 Main Results

Let H be a subhypergraph of $K^{(3)}_{m,m,m}$. Let $n_1(H)$ and $n_2(H)$ denote the number of Type 1 and Type 2 edges in H, respectively. Each Hamiltonian cycle in Section 2, $C(i, j)$, $C'(i, j)$, ($C_M(i)$ and $C'_M(i)$) and $h(x, y)$ can be regarded as a subhypergraph of $K^{(3)}_{m,m,m}$. We count the number of Type 1 edges and Type 2 edges for each of the four forms of Hamiltonian cycle in Section 2 and overall edges in $K^{(3)}_{m,m,m}$ as shown in the following table.

H	$n_1(H)$	$n_2(H)$	condition
$K^{(3)}_{m,m,m}$	m^3	$3m^3 - 3m^2$	$-$
$C(i, j)$	m	$2m$	$m \equiv 0 \pmod 3$
$C'(i, j)$	3	$3m - 3$	m is odd
$C_M(i), C'_M(i)$	0	$3m$	m is even
$h(x, y)$	$3m$	0	$-$

Let $C(0, 0)$ be a Hamiltonian cycle in Lemma 2 and $C'(0, 0)$ be a Hamiltonian cycle in Lemma 4. We obtain several results as follows.

3.1 $m = 3$

For $m = 3$, $n_1(K^{(3)}_{3,3,3}) = 27$ and $n_2(K^{(3)}_{3,3,3}) = 54$. The sets of Hamiltonian cycles $\mathcal{C}_1 = \{C(i, j) : i, j \in \mathbb{Z}_m\}$ and $\mathcal{C}_2 = \{C'(i, j) : i, j \in \mathbb{Z}_m\}$ both have m^2 Hamiltonian cycles. We calculate the number of edges in $\mathcal{C}_1$, $\sum_{H\in\mathcal{C}_1} n_1(H) = m^3 = 27$ and $\sum_{H\in\mathcal{C}_1} n_2(H) = 2m^3 = 54$, and the number of edges in $\mathcal{C}_2$, $\sum_{H\in\mathcal{C}_2} n_1(H) = 3m^2 = 27$ and $\sum_{H\in\mathcal{C}_2} n_2(H) = 3m^3 - 3m^2 = 54$. By Lemma 1 and Lemma 3, we can conclude that $\mathcal{C}_1$ and $\mathcal{C}_2$ are both Hamiltonian decompositions of $K^{(3)}_{3,3,3}$.

Example 4. Let $C(0, 0) = (0, \overline{0}, \overline{\overline{0}}, \overline{\overline{1}}, 1, \overline{1}, \overline{2}, \overline{\overline{2}}, 2)$. Then the Hamiltonian decomposition $\mathcal{C}_1$ of $K^{(3)}_{3,3,3}$ obtained from Section 2.1 is shown below.

$$C(0, 0) = (0, \overline{0}, \overline{\overline{0}}, \overline{\overline{1}}, 1, \overline{1}, \overline{2}, \overline{\overline{2}}, 2),$$
$$C(0, 1) = (0, \overline{1}, \overline{\overline{1}}, \overline{\overline{2}}, 1, \overline{2}, \overline{0}, \overline{\overline{0}}, 2),$$
$$C(0, 2) = (0, \overline{2}, \overline{\overline{2}}, \overline{\overline{0}}, 1, \overline{0}, \overline{1}, \overline{\overline{1}}, 2),$$

$$
\begin{aligned}
C(1,0) &= (1,\overline{0},\overline{\overline{1}},\overline{\overline{2}},2,\overline{1},\overline{2},\overline{\overline{0}},0),\\
C(1,1) &= (1,\overline{1},\overline{\overline{2}},\overline{\overline{0}},2,\overline{2},\overline{0},\overline{\overline{1}},0),\\
C(1,2) &= (1,\overline{2},\overline{\overline{0}},\overline{\overline{1}},2,\overline{0},\overline{1},\overline{\overline{2}},0),\\
C(2,0) &= (2,\overline{0},\overline{\overline{2}},\overline{\overline{0}},0,\overline{1},\overline{2},\overline{\overline{1}},1),\\
C(2,1) &= (2,\overline{1},\overline{\overline{0}},\overline{\overline{1}},0,\overline{2},\overline{0},\overline{\overline{2}},1),\\
C(2,2) &= (2,\overline{2},\overline{\overline{1}},\overline{\overline{2}},0,\overline{0},\overline{1},\overline{\overline{0}},1).
\end{aligned}
$$

3.2 $m \equiv 0 \pmod 6$

For $m \equiv 0 \pmod 6$, we have two families $\mathcal{H}_1$ and $\mathcal{H}_2$ of Hamiltonian cycles forming two Hamiltonian decompositions of $K^{(3)}_{m,m,m}$. First,

$$\mathcal{H}_1 = \{C(i,j) : i,j \in \mathbb{Z}_m\} \cup \{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_1\},$$

where $\mathcal{F}_1$ is a 1-factorization of a graph G with $V(G) = \mathbb{Z}_m = [\overline{0}] \cup [\overline{1}] \cup [\overline{2}]$ and $E(G) = \{uv : u, v \in \mathbb{Z}_m, ||u - v|| \equiv 0 \pmod 3\}$. G is isomorphic to $3K_{m/3}$, three copies of $K_{m/3}$. Each component consists of vertices in the same class of modulo 3. Next,

$$\mathcal{H}_2 = \{h(x,y) : (x,y) \in \mathcal{A}_0\} \cup \{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_2\},$$

where $\mathcal{F}_2$ is a 1-factorization of K_m with $\mathbb{Z}_m$ as a vertex set and $\mathcal{A}_0 = \{(x,y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod 3\}$.

Since K_{2n} is factorizable into $2n-1$ 1-factors [9], we have $|\mathcal{F}_1| = m/3-1$ and $|\mathcal{F}_2| = m-1$. Then we calculate the number of edges in $\mathcal{H}_1$, $\sum_{H \in \mathcal{H}_1} n_1(H) = m^2 \times m = m^3$ and $\sum_{H \in \mathcal{H}_1} n_2(H) = m^2 \times 2m + m(m/3-1) \times 3m = 3m^3 - 3m^2$ and the number of edges in $\mathcal{H}_2$, $\sum_{H \in \mathcal{H}_2} n_1(H) = m^2/3 \times 3m = m^3$ and $\sum_{H \in \mathcal{H}_2} n_2(H) = m(m-1) \times 3m = 3m^3 - 3m^2$.

We make some observations.

1. For any two 1-factors M and M' in K_m, we see that if M and M' are disjoint, then $C_M(i)$ and $C_{M'}(i)$ are also disjoint for all $i \in \mathbb{Z}_{m/2}$.

2. For all Type 2 edges $\{x, x', y\}$ in $\{C(i,j) : i,j \in \mathbb{Z}_m\}$, $||x - x'|| \equiv 1$ or $2 \pmod 3$.

3. For all Type 2 edges $\{x, x', y\}$ in $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_1\}$, $||x - x'|| \equiv 0 \pmod 3$.

4. $\{h(x,y) : (x,y) \in \mathcal{A}_0\}$ contains only Type 1 edges.

5. $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_2\}$ contains only Type 2 edges.

By these observations, Lemma 1, Lemma 6 and Lemma 7, we see that $\mathcal{H}_1$ and $\mathcal{H}_2$ are both Hamiltonian decompositions of $K^{(3)}_{m,m,m}$, where $m \equiv 0 \pmod 6$.

Example 5. For $m = 6$, the Hamiltonian decomposition $\mathcal{H}_1$ consists of $C(i, j)$, where $i, j \in \mathbb{Z}_6$ with $C(0, 0)$ in Example 1 and $C_M(i)$, $C'_M(i)$ where $i \in \mathbb{Z}_3$ and $M = \{03, 14, 25\}$ in Example 3. The Hamiltonian decomposition $\mathcal{H}_2$ consists of $\{h(x, y) : (x, y) \in \mathcal{A}_0\}$ and $C_M(i)$, $C'_M(i)$ where $i \in \mathbb{Z}_3$ and $M \in \{\{01, 25, 34\}, \{02, 31, 45\}, \{03, 42, 51\}, \{04, 53, 12\}, \{05, 14, 23\}\}$.

3.3 $m \equiv 3 \pmod 6$

For $m \equiv 3 \pmod 6$, let

$$\mathcal{H}_3 = \{C'(i, j) : i, j \in \mathbb{Z}_m\} \cup \{h(x, y) : (x, y) \in \mathcal{A}_0, x \neq y\}$$

where $\mathcal{A}_0 = \{(x, y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod 3\}$. We calculate the number of edges in $\mathcal{H}_3$, $\sum_{H\in\mathcal{H}_3} n_1(H) = m^2 \times 3 + (m^2/3 - m) \times 3m = m^3$ and $\sum_{H\in\mathcal{H}_3} n_2(H) = m^2 \times (3m - 3) = 3m^3 - 3m^2$.

To show that $\mathcal{H}_3$ is a Hamiltonian decomposition of $K^{(3)}_{m,m,m}$, we must show that $\{C'(i, j) : i, j \in \mathbb{Z}_m\}$ contains all edges of difference types (x, y), $(x+1, y)$, and $(x, y + 1)$ for all $x, y \in \mathbb{Z}_m$ with $x = y$.

Three edges of the form $\{a, \overline{b}, \overline{\overline{c}}\}$ in $C'(i, j)$ for each $i, j \in \mathbb{Z}_m$ have difference types

$$\begin{aligned} &(b_{m-1} - a_{m-1} + i, c_{m-2} - b_{m-1} + i) = (i, i + 1) \\ &\text{for the edge } \{a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-2} + 2i + j}}\}, \\ &(b_{m-1} - a_{m-1} + i, c_{m-1} - b_{m-1} + i) = (i, i) \\ &\text{for the edge } \{a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-1} + +2i + j}}\}, \\ &(b_{m-1} - a_0 + i, c_{m-1} - b_{m-1} + i) = (i + 1, i) \\ &\text{for the edge } \{a_0 + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-1} + 2i + j}}\}. \end{aligned}$$

Since each i, j corresponds to m different values of $\mathbb{Z}_m$, $\{C'(i, j) : i, j \in \mathbb{Z}_m\}$ contains all edges of difference type (i, i), $(i + 1, i)$ and $(i, i + 1)$ as desired.

Thus, $\mathcal{H}_3$ is a Hamiltonian decompositions of $K^{(3)}_{m,m,m}$, where $m \equiv 3 \pmod 6$. The following theorem concludes all the results.

Theorem 1. *$K^{(3)}_{m,m,m}$ is decomposable into Hamiltonian cycles if and only if $3 \mid m$.*

4 Discussion

If $3 \nmid m$, it is reasonable to consider a Hamiltonian decomposition of $K^{(3)}_{m,m,m} - I$ where I is a 1-factor of $K^{(3)}_{m,m,m}$. When m is even, $m \neq 4$ and $3 \nmid m$, $K^{(3)}_{m,m,m} - I$ has a Hamiltonian decomposition by a combination of Hamiltonian cycles $h(x, y)$ retrieved from [5] and $C_M(i), C'_M(i)$. Thus, the case of m is odd with $3 \nmid m$ and the case of $m = 4$ are still open for investigating the existence of Hamiltonian decomposition of $K^{(3)}_{m,m,m} - I$ where $K^{(3)}_{m,m,m}$ is given by Definition 1.

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References

[1] R.F. Bailey and B. Stevens, Hamiltonian decompositions of complete k-uniform hypergraphs, *Discrete Math.* 310 (2010), 3088–3095.

[2] C. Berge, *Graphes et Hypergraphes*, Dunod, Paris, 1970.

[3] Jirimutu and J. Wang, Hamiltonian decomposition of complete bipartite r-hypergraphs, *Acta Math. Appl. Sin.* 17 (2001), 563–566.

[4] G.Y. Katona and H.A. Kierstead, Hamiltonian chains in hypergraphs, *J. Graph Theory.* 30 (1999), 205–212.

[5] J. Kuhl and M.W. Schroeder, Hamilton cycle decompositions of k-uniform k-partite hypergraphs, *Australas. J. Combin.* 56 (2013), 23–37.

[6] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, Boca Raton, 1997.

[7] M. Meszka and A. Rosa, Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles, *Australas. J. Combin.* 45 (2009), 291–302.

[8] H. Verrall, Hamilton decompositions of complete 3-uniform hypergraphs, *Discrete Math.* 132 (1994), 333–348.

[9] W.D. Wallis, *Introduction to Combinatorial Designs*, 2nd edition, Chapman & Hall/CRC, 2007.

[10] B. Xu and J. Wang, On the Hamiltonian cycle decompositions of complete 3-uniform hypergraphs, *Electron. Notes Discrete Math.* 11 (2002), 722–733.