East-West J. of Mathematics: Vol. 17, No 1 (2015) pp. 48-60

HAMILTONIAN DECOMPOSITION OF COMPLETE TRIPARTITE 3-UNIFORM HYPERGRAPHS

Ratinan Boonklurb[1] **Sirirat Singhun**[2]
and
Sansanee Termtanasombat[3]

[1,3]*Department of Mathematics and Computer Science*
Chulalongkorn University, Bangkok, Thailand, 10330
e-mail: [1]*ratinan.b@chula.ac.th* [3]*sansanee.term@gmail.com*

[2]*Department of Mathematics,*
Ramkhamhaeng University, Bangkok, Thailand, 10240
e-mail: sin_sirirat@ru.ac.th,

Abstract

With our definition for complete tripartite 3-uniform hypergraphs which contain two types of edges, we show that complete tripartite 3-uniform hypergraphs with partite sets of equal size $K^{(3)}_{m,m,m}$ is decomposable into (tight) Hamiltonian cycles if and only if $3 \mid m$.

1 Introduction

A *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ consists of a nonempty finite set V of *vertices* with a family $\mathcal{E}$ of subsets of V, called *(hyper)edges*. If each edge has size k, we say that $\mathcal{H}$ is a *k-uniform hypergraph*. A *Hamiltonian decomposition* of a hypergraph is a partition of the set of edges into mutually disjoint Hamiltonian cycles. A *(tight) Hamiltonian cycle* in a k-uniform hypergraph is a cyclic ordering of its vertices such that each consecutive k-tuple of vertices is an edge. This definition was introduced by Katona and Kierstead [4], and we will use this definition of Hamiltonian cycle for this article. The older definition of a Hamiltonian cycle was given by Berge [2]. The Hamiltonian decomposition of complete 3-uniform

Key words: Hamiltonian decomposition, complete tripartite hypergraph
2010 AMS Mathematics Subject Classification: 05C65
[3]Corresponding author

hypergraphs was completely investigated in 1994 by Verrall [8] using Berge's definition. In 2000s, Bailey and Stevens [1], also Meszka and Rosa [7], Xu and Wang [10], decomposed complete k-uniform hypergraphs using Katona-Kierstead's definition and this decomposition problem is still not completed and involving the aids of computer programming.

Our motivation comes from the problem of decomposing complete bipartite 3-uniform hypergraphs. This was first introduced by Jirimutu and Wang [3] and was completed later by Xu and Wang [10]. This leads us to extend "bipartite" to "tripartite" and define a complete tripartite 3-uniform hypergraphs as follows.

Definition 1. The *complete tripartite 3-uniform hypergraph* has the vertex set V partitioned into three subsets V_0, V_1 and V_2 and the edge set $\mathcal{E}$ such that $\mathcal{E} = \{e : e \subseteq V, |e| = 3 \text{ and } |e \cap V_i| < 3 \text{ for all } i \in \{0,1,2\}\}$, and denoted by $K^{(3)}_{m,m,m}$ when $|V_0| = |V_1| = |V_2| = m$.

For convenience, W, $\overline{W}$ and $\overline{\overline{W}}$ are used to denote the vertices of $K^{(3)}_{m,m,m}$ with

$$\begin{aligned} V_0 &= W = \{0, 1, \ldots, m-1\}, \\ V_1 &= \overline{W} = \{\overline{0}, \overline{1}, \ldots, \overline{m-1}\}, \\ V_2 &= \overline{\overline{W}} = \{\overline{\overline{0}}, \overline{\overline{1}}, \ldots, \overline{\overline{m-1}}\}. \end{aligned}$$

Due to Definition 1, we classify edges of $K^{(3)}_{m,m,m}$ into two types:

Type 1 edges are of the form $\{a, \overline{b}, \overline{\overline{c}}\}$ where $a, b, c \in \mathbb{Z}_m$; and

Type 2 edges are of the form $\{x, x', y\}$ in which x and x' are in the same partite set, and y is in a different partite set. Note that there are six forms of $\{x, x', y\}$: $\{a, a', \overline{b}\}, \{a, a', \overline{\overline{c}}\}, \{\overline{b}, \overline{b'}, a\}, \{\overline{b}, \overline{b'}, \overline{\overline{c}}\}, \{\overline{\overline{c}}, \overline{\overline{c'}}, a\}$ and $\{\overline{\overline{c}}, \overline{\overline{c'}}, \overline{b}\}$ where $a, a', b, b', c, c' \in \mathbb{Z}_m$ and $a \neq a', b \neq b', c \neq c'$.

In 2013, Kuhl and Schroeder [5] published their results on Hamiltonian decompositions of complete k-uniform k-partite hypergraphs and completely found solutions for $k = 3$. A complete 3-uniform 3-partite hypergraph defined in [5] by Kuhl and Schroeder consists of all Type 1 edges but no Type 2 edges, so their hypergraph is a subhypergraph of our $K^{(3)}_{m,m,m}$ given by Definition 1. In some cases, we can use some of their results, that is $h(x, y)$ in Section 2, to be a part of our Hamiltonian decompositions of $K^{(3)}_{m,m,m}$.

If $K^{(3)}_{m,m,m}$ has a Hamiltonian decomposition, then the number of edges of $K^{(3)}_{m,m,m}$ which is equal to $\binom{3m}{3} - 3\binom{m}{3}$ must be divisible by $3m$. Thus, the necessary condition is $3 \mid m$. The purpose of this paper is to show that if $3 \mid m$, then $K^{(3)}_{m,m,m}$ has a Hamiltonian decomposition. The proof will be separated into two cases, $m \equiv 0 \pmod 6$ and $m \equiv 3 \pmod 6$ and a special case $m = 3$. In Section 2, we classify four forms of Hamiltonian cycles of $K^{(3)}_{m,m,m}$. These forms will be combined and the combination becomes a Hamiltonian decomposition of $K^{(3)}_{m,m,m}$ in Section 3. Finally, conclusion and discussion will be given in Section 4.

2 Hamiltonian Cycle Constructions

In this section, we provide four forms of a Hamiltonian cycle in $K^{(3)}_{m,m,m}$ to be used through out this article: $C(i,j)$, $C'(i,j)$, ($C_M(i)$ and $C'_M(i)$) and $h(x,y)$. First, let us define a useful notation as follows.

Definition 2. For $x, y \in \mathbb{Z}_m$, $||x-y|| = \min\{(x-y)(\text{mod } m), (y-x)(\text{mod } m)\}$.

2.1 $C(i,j)$

For $m \equiv 0 \pmod 3$, define a Hamiltonian cycle of $K^{(3)}_{m,m,m}$, $C(i,j)$ by

$$\begin{aligned} &C(i,j) = \\ &\quad (a_0+i, \overline{b_0+j}, \overline{\overline{c_0+i+j}}, \overline{\overline{c_1+i+j}}, a_1+i, \overline{b_1+j}, \overline{b_2+j}, \overline{\overline{c_2+i+j}}, a_2+i, \\ &\quad a_3+i, \overline{b_3+j}, \overline{\overline{c_3+i+j}}, \overline{\overline{c_4+i+j}}, a_4+i, \overline{b_4+j}, \overline{b_5+j}, \overline{\overline{c_5+i+j}}, a_5+i, \\ &\quad \dots, a_{m-3}+i, \overline{b_{m-3}+j}, \overline{\overline{c_{m-3}+i+j}}, \overline{\overline{c_{m-2}+i+j}}, a_{m-2}+i, \overline{b_{m-2}+j}, \\ &\quad \overline{b_{m-1}+j}, \overline{\overline{c_{m-1}+i+j}}, a_{m-1}+i), \end{aligned}$$

where $i, j \in \mathbb{Z}_m$, $\{a_0, a_1, \dots, a_{m-1}\} = \mathbb{Z}_m$, $\{b_0, b_1, \dots, b_{m-1}\} = \mathbb{Z}_m$, and $\{c_0, c_1, \dots, c_{m-1}\} = \mathbb{Z}_m$.

Lemma 1. *Let $m \equiv 0 \pmod 3$. Suppose $C(0,0)$ has properties that $c_k - b_k = c_{k'} - b_{k'}$ for all $k, k' \in \mathbb{Z}_m$ with $k \neq k'$, and $||a_{3k-1} - a_{3k}|| \neq ||a_{3k'-1} - a_{3k'}||, ||b_{3k+1} - b_{3k+2}|| \neq ||b_{3k'+1} - b_{3k'+2}||, ||c_{3k} - c_{3k+1}|| \neq ||c_{3k'} - c_{3k'+1}||$ for all $k, k' \in \{0, 1, \dots, \frac{m}{3} - 1\}$ with $k \neq k'$. Then $\{C(i,j) : i, j \in \mathbb{Z}_m\}$ is a set of m^2 disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

Proof. For edges of the form $\{a, \overline{b}, \overline{\overline{c}}\}$, we will show that if $\{a_k+i, \overline{b_k+j}, \overline{\overline{c_k+i+j}}\} = \{a_{k'}+i', \overline{b_{k'}+j'}, \overline{\overline{c_{k'}+i'+j'}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{a_k+i, \overline{b_k+j}, \overline{\overline{c_k+i+j}}\} = \{a_{k'}+i', \overline{b_{k'}+j'}, \overline{\overline{c_{k'}+i'+j'}}\}$ for some $i, i', j, j', k, k' \in \mathbb{Z}_m$. Then

$$\begin{aligned} a_k + i &\equiv a_{k'} + i' \pmod m, \\ b_k + j &\equiv b_{k'} + j' \pmod m, \\ c_k + i + j &\equiv c_{k'} + i' + j' \pmod m. \end{aligned}$$

Since $c_k - b_k = c_{k'} - b_{k'}$, we get $i = i'$ and then $a_k = a_{k'}$. Then $j = j'$. Hence, $i = i'$, $j = j'$ and $k = k'$.

For edges of the form $\{a, a', \overline{b}\}$, we will show that if $\{a_{3k-1} + i, a_{3k} + i, \overline{b_{3k} + j}\} = \{a_{3k'-1} + i', a_{3k'} + i', \overline{b_{3k'} + j'}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{a_{3k-1} + i, a_{3k} + i, \overline{b_{3k} + j}\} = \{a_{3k'-1} + i', a_{3k'} + i', \overline{b_{3k'} + j'}\}$ for some $i, i', j, j' \in \mathbb{Z}_m$ and $k, k' \in \{0, 1, \ldots, \frac{m}{3} - 1\}$. Then

$$\begin{aligned} a_{3k-1} + i &\equiv a_{3k'-1} + i' \pmod{m}, \\ a_{3k} + i &\equiv a_{3k'} + i' \pmod{m}, \\ b_{3k} + j &\equiv b_{3k'} + j' \pmod{m}, \end{aligned}$$

or

$$\begin{aligned} a_{3k-1} + i &\equiv a_{3k'} + i' \pmod{m}, \\ a_{3k} + i &\equiv a_{3k'-1} + i' \pmod{m}, \\ b_{3k} + j &\equiv b_{3k'} + j' \pmod{m}. \end{aligned}$$

Since $||a_{3k-1} - a_{3k}|| \neq ||a_{3k'-1} - a_{3k'}||$ for all $k \neq k'$ but $a_{3k-1} - a_{3k} \equiv a_{3k'-1} - a_{3k'} \pmod{m}$ or $a_{3k-1} - a_{3k} \equiv a_{3k'} - a_{3k'-1} \pmod{m}$, we have $k = k'$. Then $i = i'$ and $j = j'$.

For other edge-forms: $\{a, a', \overline{\overline{c}}\}$, $\{\overline{b}, \overline{b'}, \overline{\overline{c}}\}$, $\{\overline{b}, \overline{b'}, a\}$, $\{\overline{\overline{c}}, \overline{\overline{c'}}, a\}$, $\{\overline{\overline{c}}, \overline{\overline{c'}}, \overline{b}\}$, we can prove the same result in a similar manner. Thus, all $3m \times m^2$ edges of $\{C(i,j) : i, j \in \mathbb{Z}_m\}$ are distinct and $\{C(i,j) : i, j \in \mathbb{Z}_m\}$ is a set of m^2 disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$. $\square$

Lemma 2. *Let $m \equiv 0 \pmod 3$. Let $c_i = b_i = x_i$ and $a_i = x_{i+1}$ for all $i \in \mathbb{Z}_m$, where*

$$x_{3k} = \begin{cases} 3k/2 & \text{if } k \text{ is even}, \\ (3k+1)/2 & \text{if } k \text{ is odd}, \end{cases}$$

$$x_{3k+1} = 3k + 1,$$

$$x_{3k+2} = \begin{cases} \lceil m/2 \rceil + 3k/2 & \text{if } k \text{ is even}, \\ \lceil m/2 \rceil + (3k+1)/2 & \text{if } k \text{ is odd}, \end{cases}$$

and $k \in \{0, 1, \ldots, \frac{m}{3} - 1\}$. Then $C(0,0)$ has properties as in Lemma 1. Moreover, $||x - x'|| \equiv 1$ or $2 \pmod 3$ for all Type 2 edges of the form $\{x, x', y\}$ in $C(0,0)$.

Proof. By this setting, we have $c_k - b_k = 0 = c_{k'} - b_{k'}$ for all $k, k' \in \mathbb{Z}_m$ with $k \neq k'$. For $k \in \{0, 1, \ldots, \frac{m}{3} - 1\}$,

$$||a_{3k-1} - a_{3k}|| = ||x_{3k} - x_{3k+1}|| = \begin{cases} (3k+2)/2 & \text{if } k \text{ is even}, \\ (3k+1)/2 & \text{if } k \text{ is odd}, \end{cases}$$

$$||b_{3k+1} - b_{3k+2}|| = ||x_{3k+1} - x_{3k+2}|| = \begin{cases} \lceil m/2 \rceil - (3k+2)/2 & \text{if } k \text{ is even}, \\ \lceil m/2 \rceil - (3k+1)/2 & \text{if } k \text{ is odd}, \end{cases}$$

$$||c_{3k} - c_{3k+1}|| = ||x_{3k} - x_{3k+1}|| = \begin{cases} (3k+2)/2 & \text{if } k \text{ is even}, \\ (3k+1)/2 & \text{if } k \text{ is odd}. \end{cases}$$

Thus, $||a_{3k-1} - a_{3k}|| \neq ||a_{3k'-1} - a_{3k'}||, ||b_{3k+1} - b_{3k+2}|| \neq ||b_{3k'+1} - b_{3k'+2}||$, $||c_{3k} - c_{3k+1}|| \neq ||c_{3k'} - c_{3k'+1}||$ for all $k, k' \in \{0, 1, \ldots, \frac{m}{3} - 1\}$ with $k \neq k'$ and $||x - x'|| \equiv 1$ or $2 \pmod 3$ for all Type 2 edges of the form $\{x, x', y\}$. □

Example 1. Let $m = 6$. The cycle $C(0,0)$ in Lemma 2 is

$$C(0,0) = (1, \overline{0}, \overline{\overline{0}}, \overline{\overline{1}}, 3, \overline{1}, \overline{3}, \overline{\overline{3}}, 2, 4, \overline{2}, \overline{\overline{2}}, \overline{\overline{4}}, 5, \overline{4}, \overline{5}, \overline{\overline{5}}, 0).$$

2.2 $C'(i,j)$

For odd integer m, define a Hamiltonian cycle of $K^{(3)}_{m,m,m}$, $C'(i,j)$ by

$$\begin{aligned} C'(i,j) = &\\ (&a_0 + j, a_1 + j, \overline{b_0 + i + j}, \overline{b_1 + i + j}, \overline{\overline{c_0 + 2i + j}}, \overline{\overline{c_1 + 2i + j}},\\ &a_2 + j, a_3 + j, \overline{b_2 + i + j}, \overline{b_3 + i + j}, \overline{\overline{c_2 + 2i + j}}, \overline{\overline{c_3 + 2i + j}}, \ldots,\\ &a_{m-3} + j, a_{m-2} + j, \overline{b_{m-3} + i + j}, \overline{b_{m-2} + i + j}, \overline{\overline{c_{m-3} + 2i + j}}, \overline{\overline{c_{m-2} + 2i + j}},\\ &a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-1} + 2i + j}}), \end{aligned}$$

where $i, j \in \mathbb{Z}_m$, $\{a_0, a_1, \ldots, a_{m-1}\} = \mathbb{Z}_m$, $\{b_0, b_1, \ldots, b_{m-1}\} = \mathbb{Z}_m$, and $\{c_0, c_1, \ldots, c_{m-1}\} = \mathbb{Z}_m$.

A similar argument as in the proof of Lemma 1 can be used to prove Lemma 3.

Lemma 3. *For odd integer m, suppose $C'(0,0)$ has properties that $a_0 + c_{m-1} \neq a_{m-1} + c_{m-2} \pmod m$ and $||a_{2k+1} - a_{2k}|| \neq ||a_{2k'+1} - a_{2k'}||, ||b_{2k+1} - b_{2k}|| \neq ||b_{2k'+1} - b_{2k'}||, ||c_{2k+1} - c_{2k}|| \neq ||c_{2k'+1} - c_{2k'}||$ for all $k, k' \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$ with $k \neq k'$. Then $\{C'(i,j) : i, j \in \mathbb{Z}_m\}$ is a set of m^2 disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

Lemma 4. *For odd integer m, let $a_i = b_i = x_i$ for all $i \in \mathbb{Z}_m$, $c_{m-3} = x_0$, $c_{m-2} = x_1$, $c_{m-1} = x_{m-1}$ and $c_i = x_{i+2}$ for all $i \in \{0, 1, \ldots, m-4\}$, where $x_{m-1} = 1, x_{2k} = m - k, x_{2k+1} = k + 2$, and $k \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$. Then $C'(0,0)$ has properties as in Lemma 3. Moreover, $b_{m-1} - a_{m-1} = 0, b_{m-1} - a_0 = 1, c_{m-1} - b_{m-1} = 0, c_{m-2} - b_{m-1} = 1$.*

Proof. By this setting, we have $a_0 + c_{m-1} = 1$ and $a_{m-1} + c_{m-2} = 3$.
For $k \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$,

$$\begin{aligned} ||a_{2k+1} - a_{2k}|| = ||b_{2k+1} - b_{2k}|| &= ||x_{2k+1} - x_{2k}|| \\ &= \min\{2k + 2, m - (2k + 2)\}. \end{aligned}$$

For $k \in \{0, 1, \ldots, \frac{m-1}{2} - 2\}$,

$$\begin{aligned} ||c_{2k+1} - c_{2k}|| &= ||x_{2k+3} - x_{2k+2}|| \\ &= \min\{2k + 4, m - (2k + 4)\} \end{aligned}$$

and $c_{m-2} - c_{m-3} = x_1 - x_0 = 2$.

Since m is odd, $\{||x_{2k+1}-x_{2k}|| : k \in \{0,1,\dots,\frac{m-1}{2}-1\}\} = \{1,2,\dots,\frac{m-1}{2}\}$. Thus, $a_0 + c_{m-1} \neq a_{m-1} + c_{m-2} \pmod{m}$ and $||a_{2k+1} - a_{2k}|| \neq ||a_{2k'+1} - a_{2k'}||, ||b_{2k+1} - b_{2k}|| \neq ||b_{2k'+1} - b_{2k'}||, ||c_{2k+1} - c_{2k}|| \neq ||c_{2k'+1} - c_{2k'}||$ for all $k, k' \in \{0,1,\dots,\frac{m-1}{2}-1\}$ with $k \neq k'$. $\square$

Example 2. Let $m = 9$. The cycle $C'(0,0)$ in Lemma 4 is

$$C'(0,0) = (0,2,\overline{0},\overline{2},\overline{\overline{8}},\overline{\overline{3}},8,3,\overline{8},\overline{3},\overline{\overline{7}},\overline{\overline{4}},7,4,\overline{7},\overline{4},\overline{\overline{6}},\overline{\overline{5}},6,5,\overline{6},\overline{5},\overline{\overline{0}},\overline{\overline{2}},1,\overline{1},\overline{\overline{1}}).$$

2.3 $C_M(i)$ and $C'_M(i)$

First, consider the case where m is even. We introduce a technique different from those of 2.1 and 2.2 to construct a family of Hamiltonian cycles which contain no edges of the form $\{a,\overline{b},\overline{\overline{c}}\}$. This technique requires the knowledge of 1-factors and orthogonal quasigroups.

Definition 3. Let G be a graph. A *1-factor* of G is a subgraph of G in which every vertex has degree 1. A *1-factorization* of G is a partition of an edge set of G into 1-factors.

Definition 4. $(\mathbb{Z}_n, \circ)$ is a *quasigroup* if
(1) $i \circ j \in \mathbb{Z}_n$ for all $i, j \in \mathbb{Z}_n$, and
(2) $i \circ j \neq i \circ j'$ and $i \circ j \neq i' \circ j$ for all $i, j \in \mathbb{Z}_n$ with $i \neq i', j \neq j'$.
Note that the multiplication table of $(\mathbb{Z}_n, \circ)$ is a *Latin square.*

Definition 5. $(\mathbb{Z}_n, \circ_1)$ and $(\mathbb{Z}_n, \circ_2)$ are *orthogonal* if for $(i,j) \neq (i',j') \in \mathbb{Z}_n^2$, $i \circ_1 j = i' \circ_1 j'$ implies $i \circ_2 j \neq i' \circ_2 j'$.

Lemma 5 ([6]). *There exists a pair of mutually orthogonal Latin squares of order n for every $n \neq 2$ or 6.*

For even integer m, let $M = \{x_0x_1, x_2x_3, x_4x_5, \dots, x_{m-2}x_{m-1}\}$ be a 1-factor of a graph with $\mathbb{Z}_m$ as a vertex set. By Lemma 5, there exists a pair of orthogonal quasigroups, $(\mathbb{Z}_{m/2}, \circ_1)$ and $(\mathbb{Z}_{m/2}, \circ_2)$ for $m \neq 4$ or 12. For $i \in \mathbb{Z}_{m/2}$, define Hamiltonian cycles of $K^{(3)}_{m,m,m}$, $C_M(i)$ and $C'_M(i)$, by

$$\begin{aligned}C_M(i) = (&x_0, x_1, \overline{x_{2(i\circ_1 0)}}, \overline{x_{2(i\circ_1 0)+1}}, \overline{\overline{x_{2(i\circ_2 0)}}}, \overline{\overline{x_{2(i\circ_2 0)+1}}},\\ &x_2, x_3, \overline{x_{2(i\circ_1 1)}}, \overline{x_{2(i\circ_1 1)+1}}, \overline{\overline{x_{2(i\circ_2 1)}}}, \overline{\overline{x_{2(i\circ_2 1)+1}}}, \dots,\\ &x_{m-2}, x_{m-1}, \overline{x_{2(i\circ_1 \frac{m-2}{2})}}, \overline{x_{2(i\circ_1 \frac{m-2}{2})+1}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})}}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})+1}}})\end{aligned}$$

and

$$\begin{aligned}C'_M(i) = (&x_1, x_0, \overline{x_{2(i\circ_1 0)+1}}, \overline{x_{2(i\circ_1 0)}}, \overline{\overline{x_{2(i\circ_2 0)+1}}}, \overline{\overline{x_{2(i\circ_2 0)}}},\\ &x_3, x_2, \overline{x_{2(i\circ_1 1)+1}}, \overline{x_{2(i\circ_1 1)}}, \overline{\overline{x_{2(i\circ_2 1)+1}}}, \overline{\overline{x_{2(i\circ_2 1)}}}, \dots,\\ &x_{m-1}, x_{m-2}, \overline{x_{2(i\circ_1 \frac{m-2}{2})+1}}, \overline{x_{2(i\circ_1 \frac{m-2}{2})}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})+1}}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})}}}).\end{aligned}$$

Example 3. Let $m = 6$. The multiplication tables of orthogonal quasigroups $(\mathbb{Z}_3, \circ_1)$ and $(\mathbb{Z}_3, \circ_2)$ are as follows.

$\circ_1$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\circ_2$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Let $M = \{x_0x_1, x_2x_3, x_4x_5\} = \{03, 14, 25\}$. Then

$$\begin{aligned}
C_M(0) &= (0, 3, \overline{0}, \overline{3}, \overline{\overline{0}}, \overline{\overline{3}}, 1, 4, \overline{1}, \overline{4}, \overline{\overline{1}}, \overline{\overline{4}}, 2, 5, \overline{2}, \overline{5}, \overline{\overline{2}}, \overline{\overline{5}}),\\
C_M(1) &= (0, 3, \overline{1}, \overline{4}, \overline{\overline{2}}, \overline{\overline{5}}, 1, 4, \overline{2}, \overline{5}, \overline{\overline{0}}, \overline{\overline{3}}, 2, 5, \overline{0}, \overline{3}, \overline{\overline{1}}, \overline{\overline{4}}),\\
C_M(2) &= (0, 3, \overline{2}, \overline{5}, \overline{\overline{1}}, \overline{\overline{4}}, 1, 4, \overline{0}, \overline{3}, \overline{\overline{2}}, \overline{\overline{5}}, 2, 5, \overline{1}, \overline{4}, \overline{\overline{0}}, \overline{\overline{3}}),\\
C'_M(0) &= (3, 0, \overline{3}, \overline{0}, \overline{\overline{3}}, \overline{\overline{0}}, 4, 1, \overline{4}, \overline{1}, \overline{\overline{4}}, \overline{\overline{1}}, 5, 2, \overline{5}, \overline{2}, \overline{\overline{5}}, \overline{\overline{2}}),\\
C'_M(1) &= (3, 0, \overline{4}, \overline{1}, \overline{\overline{5}}, \overline{\overline{2}}, 4, 1, \overline{5}, \overline{2}, \overline{\overline{3}}, \overline{\overline{0}}, 5, 2, \overline{3}, \overline{0}, \overline{\overline{4}}, \overline{\overline{1}}),\\
C'_M(2) &= (3, 0, \overline{5}, \overline{2}, \overline{\overline{4}}, \overline{\overline{1}}, 4, 1, \overline{3}, \overline{0}, \overline{\overline{5}}, \overline{\overline{2}}, 5, 2, \overline{4}, \overline{1}, \overline{\overline{3}}, \overline{\overline{0}}).
\end{aligned}$$

For $m = 4$ and 12, there are no orthogonal quasigroups $(\mathbb{Z}_{m/2}, \circ_1)$ and $(\mathbb{Z}_{m/2}, \circ_2)$, therefore, $C_M(i)$ and $C'_M(i)$ will be constructed by the following way.

For $m = 4$, let $M = \{x_0x_1, x_2x_3\}$. Then

$$\begin{aligned}
C_M(0) &= (x_0, x_1, \overline{x_0}, \overline{x_1}, \overline{\overline{x_0}}, \overline{\overline{x_1}}, x_2, x_3, \overline{x_2}, \overline{x_3}, \overline{\overline{x_2}}, \overline{\overline{x_3}}),\\
C_M(1) &= (x_0, x_1, \overline{x_3}, \overline{x_2}, \overline{\overline{x_3}}, \overline{\overline{x_2}}, x_2, x_3, \overline{x_1}, \overline{x_0}, \overline{\overline{x_1}}, \overline{\overline{x_0}}),\\
C'_M(0) &= (x_1, x_0, \overline{x_1}, \overline{x_0}, \overline{\overline{x_2}}, \overline{\overline{x_3}}, x_3, x_2, \overline{x_3}, \overline{x_2}, \overline{\overline{x_0}}, \overline{\overline{x_1}}),\\
C'_M(1) &= (x_1, x_0, \overline{x_2}, \overline{x_3}, \overline{\overline{x_1}}, \overline{\overline{x_0}}, x_3, x_2, \overline{x_0}, \overline{x_1}, \overline{\overline{x_3}}, \overline{\overline{x_2}}).
\end{aligned}$$

For $m = 12$, let $(\mathbb{Z}_{m/4}, \circ_3)$ and $(\mathbb{Z}_{m/4}, \circ_4)$ be orthogonal quasigroups. For $i \in \mathbb{Z}_{m/2}$, define $C_M(i)$ and $C'_M(i)$ by

$$\begin{aligned}
C_M(i) = (&x_0, x_1, \overline{b_0(i)}, \overline{b_1(i)}, \overline{\overline{c_0(i)}}, \overline{\overline{c_1(i)}},\\
&x_2, x_3, \overline{b_2(i)}, \overline{b_3(i)}, \overline{\overline{c_2(i)}}, \overline{\overline{c_3(i)}}, \ldots,\\
&x_{m-2}, x_{m-1}, \overline{b_{m-2}(i)}, \overline{b_{m-1}(i)}, \overline{\overline{c_{m-2}(i)}}, \overline{\overline{c_{m-1}(i)}})
\end{aligned}$$

and

$$\begin{aligned}
C'_M(i) = (&x_1, x_0, \overline{b'_0(i)}, \overline{b'_1(i)}, \overline{\overline{c'_0(i)}}, \overline{\overline{c'_1(i)}},\\
&x_3, x_2, \overline{b'_2(i)}, \overline{b'_3(i)}, \overline{\overline{c'_2(i)}}, \overline{\overline{c'_3(i)}}, \ldots,\\
&x_{m-1}, x_{m-2}, \overline{b'_{m-2}(i)}, \overline{b'_{m-1}(i)}, \overline{\overline{c'_{m-2}(i)}}, \overline{\overline{c'_{m-1}(i)}}),
\end{aligned}$$

where for $j, k \in \{0, 1, \dots, \frac{m}{4} - 1\}$,

$$\begin{aligned}
b_{2k}(j) &= b_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = b'_{2k+1}(j) = b'_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = x_{2(j \circ_3 k)},\\
c_{2k}(j) &= c_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = c'_{2k+1}(j + \tfrac{m}{4}) = c'_{\frac{m}{2}+2k}(j) = x_{2(j \circ_4 k)},\\
b_{2k+1}(j) &= b_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = b'_{2k}(j) = b'_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = x_{2(j \circ_3 k)+1},\\
c_{2k+1}(j) &= c_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = c'_{2k}(j + \tfrac{m}{4}) = c'_{\frac{m}{2}+2k+1}(j) = x_{2(j \circ_4 k)+1},\\
b_{2k+1}(j + \tfrac{m}{4}) &= b_{\frac{m}{2}+2k}(j) = b'_{2k}(j + \tfrac{m}{4}) = b'_{\frac{m}{2}+2k+1}(j) = x_{\frac{m}{2}+2(j \circ_3 k)},\\
c_{2k+1}(j + \tfrac{m}{4}) &= c_{\frac{m}{2}+2k}(j) = c'_{2k}(j) = c'_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = x_{\frac{m}{2}+2(j \circ_4 k)},\\
b_{2k}(j + \tfrac{m}{4}) &= b_{\frac{m}{2}+2k+1}(j) = b'_{2k+1}(j + \tfrac{m}{4}) = b'_{\frac{m}{2}+2k}(j) = x_{\frac{m}{2}+2(j \circ_3 k)+1},\\
c_{2k}(j + \tfrac{m}{4}) &= c_{\frac{m}{2}+2k+1}(j) = c'_{2k+1}(j) = c'_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = x_{\frac{m}{2}+2(j \circ_4 k)+1}.
\end{aligned}$$

Lemma 6. *For even integer m, given a 1-factor M of a graph with $\mathbb{Z}_m$ as a vertex set, $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$ is a set of m disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

Proof. Let $M = \{x_0x_1, x_2x_3, \dots, x_{m-2}x_{m-1}\}$. Consider the case where $m \notin \{4, 12\}$. For edges of the form $\{a, a', \overline{b}\}$, we will show that if $\{x_{2k}, x_{2k+1}, \overline{x_{2(i \circ_1 k)+j}}\} = \{x_{2k'}, x_{2k'+1}, \overline{x_{2(i' \circ_1 k')+j'}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{x_{2k}, x_{2k+1}, \overline{x_{2(i \circ_1 k)+j}}\} = \{x_{2k'}, x_{2k'+1}, \overline{x_{2(i' \circ_1 k')+j'}}\}$ for some $i, i', k, k' \in \mathbb{Z}_{m/2}$ and $j, j' \in \{0, 1\}$. Then

$$\begin{aligned}
2k &= 2k',\\
2(i \circ_1 k) + j &= 2(i' \circ_1 k') + j'.
\end{aligned}$$

That is $k = k'$, $j = j'$ and $i \circ_1 k = i' \circ_1 k$. Since $(\mathbb{Z}_{m/2}, \circ_1)$ is a quasigroup, $i = i'$.

The proof for edges of the form $\{a, a', \overline{\overline{c}}\}$ can be done in the same way.

For edges of the form $\{\overline{b}, \overline{b'}, \overline{\overline{c}}\}$, we will show that if $\{\overline{x_{2(i \circ_1 k)}}, \overline{x_{2(i \circ_1 k)+1}}, \overline{\overline{x_{2(i \circ_2 k)+j}}}\} = \{\overline{x_{2(i' \circ_1 k')}}, \overline{x_{2(i' \circ_1 k')+1}}, \overline{\overline{x_{2(i' \circ_2 k')+j'}}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{\overline{x_{2(i \circ_1 k)}}, \overline{x_{2(i \circ_1 k)+1}}, \overline{\overline{x_{2(i \circ_2 k)+j}}}\} = \{\overline{x_{2(i' \circ_1 k')}}, \overline{x_{2(i' \circ_1 k')+1}}, \overline{\overline{x_{2(i' \circ_2 k')+j'}}}\}$ for some $i, i', k, k' \in \mathbb{Z}_{m/2}$ and $j, j' \in \{0, 1\}$. Then

$$\begin{aligned}
i \circ_1 k &= i' \circ_1 k',\\
i \circ_2 k &= i' \circ_2 k',\\
j &= j'.
\end{aligned}$$

Since $(\mathbb{Z}_{m/2}, \circ_1)$ and $(\mathbb{Z}_{m/2}, \circ_2)$ are orthogonal quasigroups, we have $i = i'$ and $k = k'$.

The proof for edges of the forms $\{\overline{b}, \overline{b'}, a\}$, $\{\overline{\overline{c}}, \overline{\overline{c'}}, a\}$ and $\{\overline{\overline{c}}, \overline{\overline{c'}}, \overline{b}\}$ can also be done in the same way. Thus, all $3m \times m$ edges of $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$

are distinct and $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$ is a set of m disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.

For $m = 4$, it is easy to see that $C_M(0)$, $C_M(1)$, $C'_M(0)$ and $C'_M(1)$ are mutually disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.

For $m = 12$, consider edges of the form $\{a, a', \overline{b}\}$: $e_1 = \{x_{2k}, x_{2k+1}, \overline{x_i}\}$ and $e_2 = \{x_{\frac{m}{2}+2k}, x_{\frac{m}{2}+2k+1}, \overline{x_i}\}$, where $k \in \mathbb{Z}_{m/4}$ and $i \in \mathbb{Z}_m$. Note that $\{2(j \circ_3 k), 2(j \circ_3 k) + 1, \frac{m}{2} + 2(j \circ_3 k), \frac{m}{2} + 2(j \circ_3 k) + 1 : j, k \in \mathbb{Z}_{m/4}\} = \mathbb{Z}_m$ by means of a quasigroup.

If $i = 2(j \circ_3 k)$, then $e_1 \in C_M(j)$ and $e_2 \in C'_M(j + \frac{m}{4})$.
If $i = 2(j \circ_3 k) + 1$, then $e_1 \in C'_M(j)$ and $e_2 \in C_M(j + \frac{m}{4})$.
If $i = \frac{m}{2} + 2(j \circ_3 k)$, then $e_1 \in C'_M(j + \frac{m}{4})$ and $e_2 \in C_M(j)$.
If $i = \frac{m}{2} + 2(j \circ_3 k) + 1$, then $e_1 \in C_M(j + \frac{m}{4})$ and $e_2 \in C'_M(j)$.

Thus, each edge of the form $\{a, a', \overline{b}\}$ is in a unique Hamiltonian cycle. Also use this way to show the same result for edges of the form $\{a, a', \overline{\overline{c}}\}$.

For edges of the form $\{\overline{b}, \overline{b'}, \overline{\overline{c}}\}$: $\{\overline{x_{2(j\circ_3 k)}}, \overline{x_{2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\}$ (or $\{\overline{x_{\frac{m}{2}+2(j\circ_3 k)}}, \overline{x_{\frac{m}{2}+2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\}$), we will show that if $\{\overline{x_{2(j\circ_3 k)}}, \overline{x_{2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\} = \{\overline{x_{2(j'\circ_3 k')}}, \overline{x_{2(j'\circ_3 k')+1}}, \overline{\overline{x_{i'}}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{\overline{x_{2(j\circ_3 k)}}, \overline{x_{2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\} = \{\overline{x_{2(j'\circ_3 k')}}, \overline{x_{2(j'\circ_3 k')+1}}, \overline{\overline{x_{i'}}}\}$ for some $j, j', k, k' \in \mathbb{Z}_{m/4}$ and $i \in \mathbb{Z}_m$. Then

$$\begin{aligned} j \circ_3 k &= j' \circ_3 k', \\ i &= i'. \end{aligned}$$

There are four possibilities for i: $2(j \circ_4 k)$, $2(j \circ_4 k) + 1$, $\frac{m}{2} + 2(j \circ_4 k)$ or $\frac{m}{2} + 2(j \circ_4 k) + 1$ (also for i': $2(j' \circ_4 k')$, $2(j' \circ_4 k') + 1$, $\frac{m}{2} + 2(j' \circ_4 k')$ or $\frac{m}{2} + 2(j' \circ_4 k') + 1$). Since $i = i'$, in any cases, we have $j \circ_4 k = j' \circ_4 k'$. The orthogonality of $(\mathbb{Z}_{m/4}, \circ_3)$ and $(\mathbb{Z}_{m/4}, \circ_4)$ implies $j = j'$ and $k = k'$.

Edges of the forms $\{\overline{b}, \overline{b'}, a\}$, $\{\overline{\overline{c}}, \overline{\overline{c'}}, a\}$ and $\{\overline{\overline{c}}, \overline{\overline{c'}}, \overline{b}\}$ can be showed in a similar manner. This completes the proof. □

2.4 $h(x, y)$

For $(x, y) \in \mathbb{Z}_m^2$, define a Hamiltonian cycle of $K^{(3)}_{m,m,m}$, $h(x, y)$ by

$$h(x, y) = (0, \overline{x}, \overline{\overline{x+y}}, m-1, \overline{m-1+x}, \overline{\overline{m-1+x+y}}, \ldots, 1, \overline{1+x}, \overline{\overline{1+x+y}}).$$

Kuhl and Schroeder [5] define a *difference type* of each edge of the form $\{a, \overline{b}, \overline{\overline{c}}\}$ to be $(b - a, c - b)$ in modulo m. There are m edges with a specific difference type. $h(x, y)$ has $3m$ edges and contains all edges of difference types (x, y), $(x + 1, y)$ and $(x, y + 1)$.

Note that all m^3 edges in $K^{(3)}_{m,m,m}$ are classified into m^2 distinct difference types.

Lemma 7 ([5]). *Let $m \equiv 0 \pmod 3$ and $\mathcal{A}_0 = \{(x,y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod 3\}$. Then $\{h(x,y) : (x,y) \in \mathcal{A}_0\}$ is a set of $m^2/3$ disjoint Hamiltonian cycles of $K_{m,m,m}^{(3)}$.*

3 Main Results

Let H be a subhypergraph of $K_{m,m,m}^{(3)}$. Let $n_1(H)$ and $n_2(H)$ denote the number of Type 1 and Type 2 edges in H, respectively. Each Hamiltonian cycle in Section 2, $C(i,j)$, $C'(i,j)$, ($C_M(i)$ and $C'_M(i)$) and $h(x,y)$ can be regarded as a subhypergraph of $K_{m,m,m}^{(3)}$. We count the number of Type 1 edges and Type 2 edges for each of the four forms of Hamiltonian cycle in Section 2 and overall edges in $K_{m,m,m}^{(3)}$ as shown in the following table.

H	$n_1(H)$	$n_2(H)$	condition
$K_{m,m,m}^{(3)}$	m^3	$3m^3 - 3m^2$	$-$
$C(i,j)$	m	$2m$	$m \equiv 0 \pmod 3$
$C'(i,j)$	3	$3m-3$	m is odd
$C_M(i), C'_M(i)$	0	$3m$	m is even
$h(x,y)$	$3m$	0	$-$

Let $C(0,0)$ be a Hamiltonian cycle in Lemma 2 and $C'(0,0)$ be a Hamiltonian cycle in Lemma 4. We obtain several results as follows.

3.1 $m = 3$

For $m = 3$, $n_1(K_{3,3,3}^{(3)}) = 27$ and $n_2(K_{3,3,3}^{(3)}) = 54$. The sets of Hamiltonian cycles $\mathcal{C}_1 = \{C(i,j) : i,j \in \mathbb{Z}_m\}$ and $\mathcal{C}_2 = \{C'(i,j) : i,j \in \mathbb{Z}_m\}$ both have m^2 Hamiltonian cycles. We calculate the number of edges in $\mathcal{C}_1$, $\sum_{H\in\mathcal{C}_1} n_1(H) = m^3 = 27$ and $\sum_{H\in\mathcal{C}_1} n_2(H) = 2m^3 = 54$, and the number of edges in $\mathcal{C}_2$, $\sum_{H\in\mathcal{C}_2} n_1(H) = 3m^2 = 27$ and $\sum_{H\in\mathcal{C}_2} n_2(H) = 3m^3 - 3m^2 = 54$. By Lemma 1 and Lemma 3, we can conclude that $\mathcal{C}_1$ and $\mathcal{C}_2$ are both Hamiltonian decompositions of $K_{3,3,3}^{(3)}$.

Example 4. Let $C(0,0) = (0, \overline{0}, \overline{\overline{0}}, \overline{\overline{1}}, 1, \overline{1}, \overline{2}, \overline{\overline{2}}, 2)$. Then the Hamiltonian decomposition $\mathcal{C}_1$ of $K_{3,3,3}^{(3)}$ obtained from Section 2.1 is shown below.

$$C(0,0) = (0, \overline{0}, \overline{\overline{0}}, \overline{\overline{1}}, 1, \overline{1}, \overline{2}, \overline{\overline{2}}, 2),$$
$$C(0,1) = (0, \overline{1}, \overline{\overline{1}}, \overline{\overline{2}}, 1, \overline{2}, \overline{0}, \overline{\overline{0}}, 2),$$
$$C(0,2) = (0, \overline{2}, \overline{\overline{2}}, \overline{\overline{0}}, 1, \overline{0}, \overline{1}, \overline{\overline{1}}, 2),$$

$$
\begin{aligned}
C(1,0) &= (1, \overline{0}, \overline{\overline{1}}, \overline{\overline{2}}, 2, \overline{1}, \overline{2}, \overline{\overline{0}}, 0), \\
C(1,1) &= (1, \overline{1}, \overline{\overline{2}}, \overline{\overline{0}}, 2, \overline{2}, \overline{0}, \overline{\overline{1}}, 0), \\
C(1,2) &= (1, \overline{2}, \overline{\overline{0}}, \overline{\overline{1}}, 2, \overline{0}, \overline{1}, \overline{\overline{2}}, 0), \\
C(2,0) &= (2, \overline{0}, \overline{\overline{2}}, \overline{\overline{0}}, 0, \overline{1}, \overline{2}, \overline{\overline{1}}, 1), \\
C(2,1) &= (2, \overline{1}, \overline{\overline{0}}, \overline{\overline{1}}, 0, \overline{2}, \overline{0}, \overline{\overline{2}}, 1), \\
C(2,2) &= (2, \overline{2}, \overline{\overline{1}}, \overline{\overline{2}}, 0, \overline{0}, \overline{1}, \overline{\overline{0}}, 1).
\end{aligned}
$$

3.2 $m \equiv 0 \pmod 6$

For $m \equiv 0 \pmod 6$, we have two families $\mathcal{H}_1$ and $\mathcal{H}_2$ of Hamiltonian cycles forming two Hamiltonian decompositions of $K^{(3)}_{m,m,m}$. First,

$$\mathcal{H}_1 = \{C(i,j) : i, j \in \mathbb{Z}_m\} \cup \{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_1\},$$

where $\mathcal{F}_1$ is a 1-factorization of a graph G with $V(G) = \mathbb{Z}_m = [\overline{0}] \cup [\overline{1}] \cup [\overline{2}]$ and $E(G) = \{uv : u, v \in \mathbb{Z}_m, ||u - v|| \equiv 0 \pmod 3\}$. G is isomorphic to $3K_{m/3}$, three copies of $K_{m/3}$. Each component consists of vertices in the same class of modulo 3. Next,

$$\mathcal{H}_2 = \{h(x,y) : (x,y) \in \mathcal{A}_0\} \cup \{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_2\},$$

where $\mathcal{F}_2$ is a 1-factorization of K_m with $\mathbb{Z}_m$ as a vertex set and $\mathcal{A}_0 = \{(x,y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod 3\}$.

Since K_{2n} is factorizable into $2n-1$ 1-factors [9], we have $|\mathcal{F}_1| = m/3-1$ and $|\mathcal{F}_2| = m - 1$. Then we calculate the number of edges in $\mathcal{H}_1$, $\sum_{H \in \mathcal{H}_1} n_1(H) = m^2 \times m = m^3$ and $\sum_{H \in \mathcal{H}_1} n_2(H) = m^2 \times 2m + m(m/3 - 1) \times 3m = 3m^3 - 3m^2$ and the number of edges in $\mathcal{H}_2$, $\sum_{H \in \mathcal{H}_2} n_1(H) = m^2/3 \times 3m = m^3$ and $\sum_{H \in \mathcal{H}_2} n_2(H) = m(m-1) \times 3m = 3m^3 - 3m^2$.

We make some observations.

1. For any two 1-factors M and M' in K_m, we see that if M and M' are disjoint, then $C_M(i)$ and $C_{M'}(i)$ are also disjoint for all $i \in \mathbb{Z}_{m/2}$.
2. For all Type 2 edges $\{x, x', y\}$ in $\{C(i,j) : i, j \in \mathbb{Z}_m\}$, $||x - x'|| \equiv 1$ or $2 \pmod 3$.
3. For all Type 2 edges $\{x, x', y\}$ in $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_1\}$, $||x - x'|| \equiv 0 \pmod 3$.
4. $\{h(x,y) : (x,y) \in \mathcal{A}_0\}$ contains only Type 1 edges.
5. $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_2\}$ contains only Type 2 edges.

By these observations, Lemma 1, Lemma 6 and Lemma 7, we see that $\mathcal{H}_1$ and $\mathcal{H}_2$ are both Hamiltonian decompositions of $K^{(3)}_{m,m,m}$, where $m \equiv 0 \pmod 6$.

Example 5. For $m = 6$, the Hamiltonian decomposition $\mathcal{H}_1$ consists of $C(i,j)$, where $i,j \in \mathbb{Z}_6$ with $C(0,0)$ in Example 1 and $C_M(i)$, $C'_M(i)$ where $i \in \mathbb{Z}_3$ and $M = \{03, 14, 25\}$ in Example 3. The Hamiltonian decomposition $\mathcal{H}_2$ consists of $\{h(x,y) : (x,y) \in \mathcal{A}_0\}$ and $C_M(i)$, $C'_M(i)$ where $i \in \mathbb{Z}_3$ and $M \in \{\{01, 25, 34\}, \{02, 31, 45\}, \{03, 42, 51\}, \{04, 53, 12\}, \{05, 14, 23\}\}$.

3.3 $m \equiv 3 \pmod 6$

For $m \equiv 3 \pmod 6$, let

$$\mathcal{H}_3 = \{C'(i,j) : i,j \in \mathbb{Z}_m\} \cup \{h(x,y) : (x,y) \in \mathcal{A}_0, x \neq y\}$$

where $\mathcal{A}_0 = \{(x,y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod 3\}$. We calculate the number of edges in $\mathcal{H}_3$, $\sum_{H \in \mathcal{H}_3} n_1(H) = m^2 \times 3 + (m^2/3 - m) \times 3m = m^3$ and $\sum_{H \in \mathcal{H}_3} n_2(H) = m^2 \times (3m - 3) = 3m^3 - 3m^2$.

To show that $\mathcal{H}_3$ is a Hamiltonian decomposition of $K^{(3)}_{m,m,m}$, we must show that $\{C'(i,j) : i,j \in \mathbb{Z}_m\}$ contains all edges of difference types (x,y), $(x+1,y)$, and $(x,y+1)$ for all $x,y \in \mathbb{Z}_m$ with $x = y$.

Three edges of the form $\{a, \overline{b}, \overline{\overline{c}}\}$ in $C'(i,j)$ for each $i,j \in \mathbb{Z}_m$ have difference types

$$\begin{aligned}
&(b_{m-1} - a_{m-1} + i, c_{m-2} - b_{m-1} + i) = (i, i+1)\\
&\text{for the edge } \{a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-2} + 2i + j}}\},\\
&(b_{m-1} - a_{m-1} + i, c_{m-1} - b_{m-1} + i) = (i, i)\\
&\text{for the edge } \{a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-1} + +2i + j}}\},\\
&(b_{m-1} - a_0 + i, c_{m-1} - b_{m-1} + i) = (i+1, i)\\
&\text{for the edge } \{a_0 + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-1} + 2i + j}}\}.
\end{aligned}$$

Since each i,j corresponds to m different values of $\mathbb{Z}_m$, $\{C'(i,j) : i,j \in \mathbb{Z}_m\}$ contains all edges of difference type (i,i), $(i+1,i)$ and $(i,i+1)$ as desired.

Thus, $\mathcal{H}_3$ is a Hamiltonian decompositions of $K^{(3)}_{m,m,m}$, where $m \equiv 3 \pmod 6$. The following theorem concludes all the results.

Theorem 1. *$K^{(3)}_{m,m,m}$ is decomposable into Hamiltonian cycles if and only if $3 \mid m$.*

4 Discussion

If $3 \nmid m$, it is reasonable to consider a Hamiltonian decomposition of $K^{(3)}_{m,m,m} - I$ where I is a 1-factor of $K^{(3)}_{m,m,m}$. When m is even, $m \neq 4$ and $3 \nmid m$, $K^{(3)}_{m,m,m} - I$ has a Hamiltonian decomposition by a combination of Hamiltonian cycles $h(x,y)$ retrieved from [5] and $C_M(i), C'_M(i)$. Thus, the case of m is odd with $3 \nmid m$ and the case of $m = 4$ are still open for investigating the existence of Hamiltonian decomposition of $K^{(3)}_{m,m,m} - I$ where $K^{(3)}_{m,m,m}$ is given by Definition 1.

Acknowledgement

This research is supported by Science Achievement Scholarship of Thailand (SAST). The authors would like to thank Professor Chis Rodger for his suggestion in the case of $m \equiv 3 \pmod 6$ during CIMPA research school on Graphs, Codes, and Designs at Ramkhumhaeng University, Bangkok, Thailand.

References

[1] R.F. Bailey and B. Stevens, Hamiltonian decompositions of complete k-uniform hypergraphs, *Discrete Math.* 310 (2010), 3088–3095.

[2] C. Berge, *Graphes et Hypergraphes*, Dunod, Paris, 1970.

[3] Jirimutu and J. Wang, Hamiltonian decomposition of complete bipartite r-hypergraphs, *Acta Math. Appl. Sin.* 17 (2001), 563–566.

[4] G.Y. Katona and H.A. Kierstead, Hamiltonian chains in hypergraphs, *J. Graph Theory.* 30 (1999), 205–212.

[5] J. Kuhl and M.W. Schroeder, Hamilton cycle decompositions of k-uniform k-partite hypergraphs, *Australas. J. Combin.* 56 (2013), 23–37.

[6] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, Boca Raton, 1997.

[7] M. Meszka and A. Rosa, Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles, *Australas. J. Combin.* 45 (2009), 291–302.

[8] H. Verrall, Hamilton decompositions of complete 3-uniform hypergraphs, *Discrete Math.* 132 (1994), 333–348.

[9] W.D. Wallis, *Introduction to Combinatorial Designs*, 2nd edition, Chapman & Hall/CRC, 2007.

[10] B. Xu and J. Wang, On the Hamiltonian cycle decompositions of complete 3-uniform hypergraphs, *Electron. Notes Discrete Math.* 11 (2002), 722–733.