

# HAMILTONIAN DECOMPOSITION OF COMPLETE TRIPARTITE 3-UNIFORM HYPERGRAPHS

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## Abstract

With our definition for complete tripartite 3-uniform hypergraphs which contain two types of edges, we show that complete tripartite 3-uniform hypergraphs with partite sets of equal size  $K_{m,m,m}^{(3)}$  is decomposable into (tight) Hamiltonian cycles if and only if  $3 \mid m$ .

## 1 Introduction

A *hypergraph*  $\mathcal{H} = (V, \mathcal{E})$  consists of a nonempty finite set  $V$  of *vertices* with a family  $\mathcal{E}$  of subsets of  $V$ , called (*hyper*)*edges*. If each edge has size  $k$ , we say that  $\mathcal{H}$  is a *k-uniform hypergraph*. A *Hamiltonian decomposition* of a hypergraph is a partition of the set of edges into mutually disjoint Hamiltonian cycles. A (*tight*) *Hamiltonian cycle* in a  $k$ -uniform hypergraph is a cyclic ordering of its vertices such that each consecutive  $k$ -tuple of vertices is an edge. This definition was introduced by Katona and Kierstead [4], and we will use this definition of Hamiltonian cycle for this article. The older definition of a Hamiltonian cycle was given by Berge [2]. The Hamiltonian decomposition of complete 3-uniform

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hypergraphs was completely investigated in 1994 by Verrall [8] using Berge's definition. In 2000s, Bailey and Stevens [1], also Meszka and Rosa [7], Xu and Wang [10], decomposed complete  $k$ -uniform hypergraphs using Katona-Kierstead's definition and this decomposition problem is still not completed and involving the aids of computer programming.

Our motivation comes from the problem of decomposing complete bipartite 3-uniform hypergraphs. This was first introduced by Jirimutu and Wang [3] and was completed later by Xu and Wang [10]. This leads us to extend "bipartite" to "tripartite" and define a complete tripartite 3-uniform hypergraphs as follows.

**Definition 1.** The *complete tripartite 3-uniform hypergraph* has the vertex set  $V$  partitioned into three subsets  $V_0, V_1$  and  $V_2$  and the edge set  $\mathcal{E}$  such that  $\mathcal{E} = \{e : e \subseteq V, |e| = 3 \text{ and } |e \cap V_i| < 3 \text{ for all } i \in \{0, 1, 2\}\}$ , and denoted by  $K_{m,m,m}^{(3)}$  when  $|V_0| = |V_1| = |V_2| = m$ .

For convenience,  $W, \overline{W}$  and  $\overline{\overline{W}}$  are used to denote the vertices of  $K_{m,m,m}^{(3)}$  with

$$\begin{aligned} V_0 &= W = \{0, 1, \dots, m-1\}, \\ V_1 &= \overline{W} = \{\overline{0}, \overline{1}, \dots, \overline{m-1}\}, \\ V_2 &= \overline{\overline{W}} = \{\overline{\overline{0}}, \overline{\overline{1}}, \dots, \overline{\overline{m-1}}\}. \end{aligned}$$

Due to Definition 1, we classify edges of  $K_{m,m,m}^{(3)}$  into two types:

*Type 1* edges are of the form  $\{a, \overline{b}, \overline{\overline{c}}\}$  where  $a, b, c \in \mathbb{Z}_m$ ; and

*Type 2* edges are of the form  $\{x, x', y\}$  in which  $x$  and  $x'$  are in the same partite set, and  $y$  is in a different partite set. Note that there are six forms of  $\{x, x', y\}$ :  $\{a, a', \overline{b}\}$ ,  $\{a, a', \overline{\overline{c}}\}$ ,  $\{\overline{b}, \overline{b'}, a\}$ ,  $\{\overline{b}, \overline{b'}, \overline{\overline{c}}\}$ ,  $\{\overline{\overline{c}}, \overline{\overline{c}'}, a\}$  and  $\{\overline{\overline{c}}, \overline{\overline{c}'}, \overline{b}\}$  where  $a, a', b, b', c, c' \in \mathbb{Z}_m$  and  $a \neq a', b \neq b', c \neq c'$ .

In 2013, Kuhl and Schroeder [5] published their results on Hamiltonian decompositions of complete  $k$ -uniform  $k$ -partite hypergraphs and completely found solutions for  $k = 3$ . A complete 3-uniform 3-partite hypergraph defined in [5] by Kuhl and Schroeder consists of all Type 1 edges but no Type 2 edges, so their hypergraph is a subhypergraph of our  $K_{m,m,m}^{(3)}$  given by Definition 1. In some cases, we can use some of their results, that is  $h(x, y)$  in Section 2, to be a part of our Hamiltonian decompositions of  $K_{m,m,m}^{(3)}$ .

If  $K_{m,m,m}^{(3)}$  has a Hamiltonian decomposition, then the number of edges of  $K_{m,m,m}^{(3)}$  which is equal to  $\binom{3m}{3} - 3\binom{m}{3}$  must be divisible by  $3m$ . Thus, the necessary condition is  $3 \mid m$ . The purpose of this paper is to show that if  $3 \mid m$ , then  $K_{m,m,m}^{(3)}$  has a Hamiltonian decomposition. The proof will be separated into two cases,  $m \equiv 0 \pmod{6}$  and  $m \equiv 3 \pmod{6}$  and a special case  $m = 3$ . In Section 2, we classify four forms of Hamiltonian cycles of  $K_{m,m,m}^{(3)}$ . These forms will be combined and the combination becomes a Hamiltonian decomposition

of  $K_{m,m,m}^{(3)}$  in Section 3. Finally, conclusion and discussion will be given in Section 4.

## 2 Hamiltonian Cycle Constructions

In this section, we provide four forms of a Hamiltonian cycle in  $K_{m,m,m}^{(3)}$  to be used through out this article:  $C(i, j)$ ,  $C'(i, j)$ ,  $(C_M(i)$  and  $C'_M(i))$  and  $h(x, y)$ . First, let us define a useful notation as follows.

**Definition 2.** For  $x, y \in \mathbb{Z}_m$ ,  $\|x - y\| = \min\{(x - y) \pmod{m}, (y - x) \pmod{m}\}$ .

### 2.1 $C(i, j)$

For  $m \equiv 0 \pmod{3}$ , define a Hamiltonian cycle of  $K_{m,m,m}^{(3)}$ ,  $C(i, j)$  by

$$\begin{aligned} C(i, j) = & \overline{(a_0 + i, b_0 + j, c_0 + i + j, c_1 + i + j, a_1 + i, b_1 + j, b_2 + j, c_2 + i + j, a_2 + i,} \\ & \overline{a_3 + i, b_3 + j, c_3 + i + j, c_4 + i + j, a_4 + i, b_4 + j, b_5 + j, c_5 + i + j, a_5 + i,} \\ & \overline{\dots, a_{m-3} + i, b_{m-3} + j, c_{m-3} + i + j, c_{m-2} + i + j, a_{m-2} + i, b_{m-2} + j,} \\ & \overline{b_{m-1} + j, c_{m-1} + i + j, a_{m-1} + i)}, \end{aligned}$$

where  $i, j \in \mathbb{Z}_m$ ,  $\{a_0, a_1, \dots, a_{m-1}\} = \mathbb{Z}_m$ ,  $\{b_0, b_1, \dots, b_{m-1}\} = \mathbb{Z}_m$ , and  $\{c_0, c_1, \dots, c_{m-1}\} = \mathbb{Z}_m$ .

**Lemma 1.** *Let  $m \equiv 0 \pmod{3}$ . Suppose  $C(0, 0)$  has properties that  $c_k - b_k = c_{k'} - b_{k'}$  for all  $k, k' \in \mathbb{Z}_m$  with  $k \neq k'$ , and  $\|a_{3k-1} - a_{3k}\| \neq \|a_{3k'-1} - a_{3k'}\|$ ,  $\|b_{3k+1} - b_{3k+2}\| \neq \|b_{3k'+1} - b_{3k'+2}\|$ ,  $\|c_{3k} - c_{3k+1}\| \neq \|c_{3k'} - c_{3k'+1}\|$  for all  $k, k' \in \{0, 1, \dots, \frac{m}{3} - 1\}$  with  $k \neq k'$ . Then  $\{C(i, j) : i, j \in \mathbb{Z}_m\}$  is a set of  $m^2$  disjoint Hamiltonian cycles of  $K_{m,m,m}^{(3)}$ .*

*Proof.* For edges of the form  $\{a, \overline{b, c}\}$ , we will show that if  $\{a_k + i, \overline{b_k + j, c_k + i + j}\} = \{a_{k'} + i', \overline{b_{k'} + j', c_{k'} + i' + j'}\}$ , then  $i = i'$ ,  $j = j'$  and  $k = k'$ .

Suppose that  $\{a_k + i, \overline{b_k + j, c_k + i + j}\} = \{a_{k'} + i', \overline{b_{k'} + j', c_{k'} + i' + j'}\}$  for some  $i, i', j, j', k, k' \in \mathbb{Z}_m$ . Then

$$\begin{aligned} a_k + i &\equiv a_{k'} + i' \pmod{m}, \\ b_k + j &\equiv b_{k'} + j' \pmod{m}, \\ c_k + i + j &\equiv c_{k'} + i' + j' \pmod{m}. \end{aligned}$$

Since  $c_k - b_k = c_{k'} - b_{k'}$ , we get  $i = i'$  and then  $a_k = a_{k'}$ . Then  $j = j'$ . Hence,  $i = i'$ ,  $j = j'$  and  $k = k'$ .

For edges of the form  $\{a, a', \bar{b}\}$ , we will show that if  $\{a_{3k-1} + i, a_{3k} + i, \overline{b_{3k} + j}\} = \{a_{3k'-1} + i', a_{3k'} + i', \overline{b_{3k'} + j'}\}$ , then  $i = i'$ ,  $j = j'$  and  $k = k'$ .

Suppose that  $\{a_{3k-1} + i, a_{3k} + i, \overline{b_{3k} + j}\} = \{a_{3k'-1} + i', a_{3k'} + i', \overline{b_{3k'} + j'}\}$  for some  $i, i', j, j' \in \mathbb{Z}_m$  and  $k, k' \in \{0, 1, \dots, \frac{m}{3} - 1\}$ . Then

$$\begin{aligned} a_{3k-1} + i &\equiv a_{3k'-1} + i' \pmod{m}, \\ a_{3k} + i &\equiv a_{3k'} + i' \pmod{m}, \\ b_{3k} + j &\equiv b_{3k'} + j' \pmod{m}, \end{aligned}$$

or

$$\begin{aligned} a_{3k-1} + i &\equiv a_{3k'} + i' \pmod{m}, \\ a_{3k} + i &\equiv a_{3k'-1} + i' \pmod{m}, \\ b_{3k} + j &\equiv b_{3k'} + j' \pmod{m}. \end{aligned}$$

Since  $\|a_{3k-1} - a_{3k}\| \neq \|a_{3k'-1} - a_{3k'}\|$  for all  $k \neq k'$  but  $a_{3k-1} - a_{3k} \equiv a_{3k'-1} - a_{3k'} \pmod{m}$  or  $a_{3k-1} - a_{3k} \equiv a_{3k'} - a_{3k'-1} \pmod{m}$ , we have  $k = k'$ . Then  $i = i'$  and  $j = j'$ .

For other edge-forms:  $\{a, a', \bar{c}\}$ ,  $\{\bar{b}, \bar{b}', \bar{c}\}$ ,  $\{\bar{b}, \bar{b}', a\}$ ,  $\{\bar{c}, \bar{c}', a\}$ ,  $\{\bar{c}, \bar{c}', \bar{b}\}$ , we can prove the same result in a similar manner. Thus, all  $3m \times m^2$  edges of  $\{C(i, j) : i, j \in \mathbb{Z}_m\}$  are distinct and  $\{C(i, j) : i, j \in \mathbb{Z}_m\}$  is a set of  $m^2$  disjoint Hamiltonian cycles of  $K_{m,m,m}^{(3)}$ .  $\square$

**Lemma 2.** *Let  $m \equiv 0 \pmod{3}$ . Let  $c_i = b_i = x_i$  and  $a_i = x_{i+1}$  for all  $i \in \mathbb{Z}_m$ , where*

$$\begin{aligned} x_{3k} &= \begin{cases} 3k/2 & \text{if } k \text{ is even,} \\ (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases} \\ x_{3k+1} &= 3k+1, \\ x_{3k+2} &= \begin{cases} \lceil m/2 \rceil + 3k/2 & \text{if } k \text{ is even,} \\ \lceil m/2 \rceil + (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases} \end{aligned}$$

and  $k \in \{0, 1, \dots, \frac{m}{3} - 1\}$ . Then  $C(0, 0)$  has properties as in Lemma 1. Moreover,  $\|x - x'\| \equiv 1$  or  $2 \pmod{3}$  for all Type 2 edges of the form  $\{x, x', y\}$  in  $C(0, 0)$ .

*Proof.* By this setting, we have  $c_k - b_k = 0 = c_{k'} - b_{k'}$  for all  $k, k' \in \mathbb{Z}_m$  with  $k \neq k'$ . For  $k \in \{0, 1, \dots, \frac{m}{3} - 1\}$ ,

$$\begin{aligned} \|a_{3k-1} - a_{3k}\| &= \|x_{3k} - x_{3k+1}\| = \begin{cases} (3k+2)/2 & \text{if } k \text{ is even,} \\ (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases} \\ \|b_{3k+1} - b_{3k+2}\| &= \|x_{3k+1} - x_{3k+2}\| = \begin{cases} \lceil m/2 \rceil - (3k+2)/2 & \text{if } k \text{ is even,} \\ \lceil m/2 \rceil - (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases} \\ \|c_{3k} - c_{3k+1}\| &= \|x_{3k} - x_{3k+1}\| = \begin{cases} (3k+2)/2 & \text{if } k \text{ is even,} \\ (3k+1)/2 & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

Thus,  $\|a_{3k-1} - a_{3k}\| \neq \|a_{3k'-1} - a_{3k'}\|$ ,  $\|b_{3k+1} - b_{3k+2}\| \neq \|b_{3k'+1} - b_{3k'+2}\|$ ,  $\|c_{3k} - c_{3k+1}\| \neq \|c_{3k'} - c_{3k'+1}\|$  for all  $k, k' \in \{0, 1, \dots, \frac{m}{3} - 1\}$  with  $k \neq k'$  and  $\|x - x'\| \equiv 1$  or  $2 \pmod{3}$  for all Type 2 edges of the form  $\{x, x', y\}$ .  $\square$

**Example 1.** Let  $m = 6$ . The cycle  $C(0, 0)$  in Lemma 2 is

$$C(0, 0) = (1, \bar{0}, \bar{0}, \bar{1}, 3, \bar{1}, \bar{3}, \bar{3}, 2, 4, \bar{2}, \bar{2}, \bar{4}, 5, \bar{4}, \bar{5}, \bar{5}, 0).$$

## 2.2 $C'(i, j)$

For odd integer  $m$ , define a Hamiltonian cycle of  $K_{m,m,m}^{(3)}$ ,  $C'(i, j)$  by

$$\begin{aligned} C'(i, j) = & (a_0 + j, a_1 + j, \overline{b_0 + i + j}, \overline{b_1 + i + j}, \overline{c_0 + 2i + j}, \overline{c_1 + 2i + j}, \\ & a_2 + j, a_3 + j, \overline{b_2 + i + j}, \overline{b_3 + i + j}, \overline{c_2 + 2i + j}, \overline{c_3 + 2i + j}, \dots, \\ & a_{m-3} + j, a_{m-2} + j, \overline{b_{m-3} + i + j}, \overline{b_{m-2} + i + j}, \overline{c_{m-3} + 2i + j}, \overline{c_{m-2} + 2i + j}, \\ & a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{c_{m-1} + 2i + j}), \end{aligned}$$

where  $i, j \in \mathbb{Z}_m$ ,  $\{a_0, a_1, \dots, a_{m-1}\} = \mathbb{Z}_m$ ,  $\{b_0, b_1, \dots, b_{m-1}\} = \mathbb{Z}_m$ , and  $\{c_0, c_1, \dots, c_{m-1}\} = \mathbb{Z}_m$ .

A similar argument as in the proof of Lemma 1 can be used to prove Lemma 3.

**Lemma 3.** For odd integer  $m$ , suppose  $C'(0, 0)$  has properties that  $a_0 + c_{m-1} \neq a_{m-1} + c_{m-2} \pmod{m}$  and  $\|a_{2k+1} - a_{2k}\| \neq \|a_{2k'+1} - a_{2k'}\|$ ,  $\|b_{2k+1} - b_{2k}\| \neq \|b_{2k'+1} - b_{2k'}\|$ ,  $\|c_{2k+1} - c_{2k}\| \neq \|c_{2k'+1} - c_{2k'}\|$  for all  $k, k' \in \{0, 1, \dots, \frac{m-1}{2} - 1\}$  with  $k \neq k'$ . Then  $\{C'(i, j) : i, j \in \mathbb{Z}_m\}$  is a set of  $m^2$  disjoint Hamiltonian cycles of  $K_{m,m,m}^{(3)}$ .

**Lemma 4.** For odd integer  $m$ , let  $a_i = b_i = x_i$  for all  $i \in \mathbb{Z}_m$ ,  $c_{m-3} = x_0$ ,  $c_{m-2} = x_1$ ,  $c_{m-1} = x_{m-1}$  and  $c_i = x_{i+2}$  for all  $i \in \{0, 1, \dots, m-4\}$ , where  $x_{m-1} = 1$ ,  $x_{2k} = m - k$ ,  $x_{2k+1} = k + 2$ , and  $k \in \{0, 1, \dots, \frac{m-1}{2} - 1\}$ . Then  $C'(0, 0)$  has properties as in Lemma 3. Moreover,  $b_{m-1} - a_{m-1} = 0$ ,  $b_{m-1} - a_0 = 1$ ,  $c_{m-1} - b_{m-1} = 0$ ,  $c_{m-2} - b_{m-1} = 1$ .

*Proof.* By this setting, we have  $a_0 + c_{m-1} = 1$  and  $a_{m-1} + c_{m-2} = 3$ . For  $k \in \{0, 1, \dots, \frac{m-1}{2} - 1\}$ ,

$$\begin{aligned} \|a_{2k+1} - a_{2k}\| &= \|b_{2k+1} - b_{2k}\| = \|x_{2k+1} - x_{2k}\| \\ &= \min\{2k + 2, m - (2k + 2)\}. \end{aligned}$$

For  $k \in \{0, 1, \dots, \frac{m-1}{2} - 2\}$ ,

$$\begin{aligned} \|c_{2k+1} - c_{2k}\| &= \|x_{2k+3} - x_{2k+2}\| \\ &= \min\{2k + 4, m - (2k + 4)\} \end{aligned}$$

and  $c_{m-2} - c_{m-3} = x_1 - x_0 = 2$ .

Since  $m$  is odd,  $\{\|x_{2k+1} - x_{2k}\| : k \in \{0, 1, \dots, \frac{m-1}{2} - 1\}\} = \{1, 2, \dots, \frac{m-1}{2}\}$ . Thus,  $a_0 + c_{m-1} \not\equiv a_{m-1} + c_{m-2} \pmod{m}$  and  $\|a_{2k+1} - a_{2k}\| \neq \|a_{2k'+1} - a_{2k'}\|$ ,  $\|b_{2k+1} - b_{2k}\| \neq \|b_{2k'+1} - b_{2k'}\|$ ,  $\|c_{2k+1} - c_{2k}\| \neq \|c_{2k'+1} - c_{2k'}\|$  for all  $k, k' \in \{0, 1, \dots, \frac{m-1}{2} - 1\}$  with  $k \neq k'$ .  $\square$

**Example 2.** Let  $m = 9$ . The cycle  $C'(0, 0)$  in Lemma 4 is

$$C'(0, 0) = (0, 2, \bar{0}, \bar{2}, \bar{\bar{8}}, \bar{\bar{3}}, 8, 3, \bar{8}, \bar{3}, \bar{\bar{7}}, \bar{\bar{4}}, 7, 4, \bar{7}, \bar{4}, \bar{\bar{6}}, \bar{\bar{5}}, 6, 5, \bar{6}, \bar{5}, \bar{\bar{0}}, \bar{\bar{2}}, 1, \bar{1}, \bar{\bar{1}}).$$

### 2.3 $C_M(i)$ and $C'_M(i)$

First, consider the case where  $m$  is even. We introduce a technique different from those of 2.1 and 2.2 to construct a family of Hamiltonian cycles which contain no edges of the form  $\{a, \bar{b}, \bar{\bar{c}}\}$ . This technique requires the knowledge of 1-factors and orthogonal quasigroups.

**Definition 3.** Let  $G$  be a graph. A *1-factor* of  $G$  is a subgraph of  $G$  in which every vertex has degree 1. A *1-factorization* of  $G$  is a partition of an edge set of  $G$  into 1-factors.

**Definition 4.**  $(\mathbb{Z}_n, \circ)$  is a *quasigroup* if

- (1)  $i \circ j \in \mathbb{Z}_n$  for all  $i, j \in \mathbb{Z}_n$ , and
- (2)  $i \circ j \neq i \circ j'$  and  $i \circ j \neq i' \circ j$  for all  $i, j \in \mathbb{Z}_n$  with  $i \neq i', j \neq j'$ .

Note that the multiplication table of  $(\mathbb{Z}_n, \circ)$  is a *Latin square*.

**Definition 5.**  $(\mathbb{Z}_n, \circ_1)$  and  $(\mathbb{Z}_n, \circ_2)$  are *orthogonal* if for  $(i, j) \neq (i', j') \in \mathbb{Z}_n^2$ ,  $i \circ_1 j = i' \circ_1 j'$  implies  $i \circ_2 j \neq i' \circ_2 j'$ .

**Lemma 5 ([6]).** *There exists a pair of mutually orthogonal Latin squares of order  $n$  for every  $n \neq 2$  or  $6$ .*

For even integer  $m$ , let  $M = \{x_0x_1, x_2x_3, x_4x_5, \dots, x_{m-2}x_{m-1}\}$  be a 1-factor of a graph with  $\mathbb{Z}_m$  as a vertex set. By Lemma 5, there exists a pair of orthogonal quasigroups,  $(\mathbb{Z}_{m/2}, \circ_1)$  and  $(\mathbb{Z}_{m/2}, \circ_2)$  for  $m \neq 4$  or  $12$ . For  $i \in \mathbb{Z}_{m/2}$ , define Hamiltonian cycles of  $K_{m,m,m}^{(3)}$ ,  $C_M(i)$  and  $C'_M(i)$ , by

$$C_M(i) = (x_0, x_1, \overline{x_{2(i\circ_1 0)}}, \overline{x_{2(i\circ_1 0)+1}}, \overline{\overline{x_{2(i\circ_2 0)}}}, \overline{\overline{x_{2(i\circ_2 0)+1}}}, \\ x_2, x_3, \overline{x_{2(i\circ_1 1)}}, \overline{x_{2(i\circ_1 1)+1}}, \overline{\overline{x_{2(i\circ_2 1)}}}, \overline{\overline{x_{2(i\circ_2 1)+1}}}, \dots, \\ x_{m-2}, x_{m-1}, \overline{x_{2(i\circ_1 \frac{m-2}{2})}}, \overline{x_{2(i\circ_1 \frac{m-2}{2})+1}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})}}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})+1}}})$$

and

$$C'_M(i) = (x_1, x_0, \overline{x_{2(i\circ_1 0)+1}}, \overline{x_{2(i\circ_1 0)}}, \overline{\overline{x_{2(i\circ_2 0)+1}}}, \overline{\overline{x_{2(i\circ_2 0)}}}, \\ x_3, x_2, \overline{x_{2(i\circ_1 1)+1}}, \overline{x_{2(i\circ_1 1)}}, \overline{\overline{x_{2(i\circ_2 1)+1}}}, \overline{\overline{x_{2(i\circ_2 1)}}}, \dots, \\ x_{m-1}, x_{m-2}, \overline{x_{2(i\circ_1 \frac{m-2}{2})+1}}, \overline{x_{2(i\circ_1 \frac{m-2}{2})}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})+1}}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})}}}).$$

**Example 3.** Let  $m = 6$ . The multiplication tables of orthogonal quasigroups  $(\mathbb{Z}_3, \circ_1)$  and  $(\mathbb{Z}_3, \circ_2)$  are as follows.

$\circ_1$	0	1	2	$\circ_2$	0	1	2
0	0	1	2	0	0	1	2
1	1	2	0	1	2	0	1
2	2	0	1	2	1	2	0

Let  $M = \{x_0x_1, x_2x_3, x_4x_5\} = \{03, 14, 25\}$ . Then

$$\begin{aligned}
C_M(0) &= (0, 3, \bar{0}, \bar{3}, \bar{\bar{0}}, \bar{\bar{3}}, 1, 4, \bar{1}, \bar{4}, \bar{\bar{1}}, \bar{\bar{4}}, 2, 5, \bar{2}, \bar{5}, \bar{\bar{2}}, \bar{\bar{5}}), \\
C_M(1) &= (0, 3, \bar{1}, \bar{4}, \bar{\bar{2}}, \bar{\bar{5}}, 1, 4, \bar{2}, \bar{5}, \bar{\bar{0}}, \bar{\bar{3}}, 2, 5, \bar{0}, \bar{3}, \bar{\bar{1}}, \bar{\bar{4}}), \\
C_M(2) &= (0, 3, \bar{2}, \bar{5}, \bar{\bar{1}}, \bar{\bar{4}}, 1, 4, \bar{0}, \bar{3}, \bar{\bar{2}}, \bar{\bar{5}}, 2, 5, \bar{1}, \bar{4}, \bar{\bar{0}}, \bar{\bar{3}}), \\
C'_M(0) &= (3, 0, \bar{3}, \bar{0}, \bar{\bar{3}}, \bar{\bar{0}}, 4, 1, \bar{4}, \bar{1}, \bar{\bar{4}}, \bar{\bar{1}}, 5, 2, \bar{5}, \bar{2}, \bar{\bar{5}}, \bar{\bar{2}}), \\
C'_M(1) &= (3, 0, \bar{4}, \bar{1}, \bar{\bar{5}}, \bar{\bar{2}}, 4, 1, \bar{5}, \bar{2}, \bar{\bar{3}}, \bar{\bar{0}}, 5, 2, \bar{3}, \bar{0}, \bar{\bar{4}}, \bar{\bar{1}}), \\
C'_M(2) &= (3, 0, \bar{5}, \bar{2}, \bar{\bar{4}}, \bar{\bar{1}}, 4, 1, \bar{3}, \bar{0}, \bar{\bar{5}}, \bar{\bar{2}}, 5, 2, \bar{4}, \bar{1}, \bar{\bar{3}}, \bar{\bar{0}}).
\end{aligned}$$

For  $m = 4$  and  $12$ , there are no orthogonal quasigroups  $(\mathbb{Z}_{m/2}, \circ_1)$  and  $(\mathbb{Z}_{m/2}, \circ_2)$ , therefore,  $C_M(i)$  and  $C'_M(i)$  will be constructed by the following way.

For  $m = 4$ , let  $M = \{x_0x_1, x_2x_3\}$ . Then

$$\begin{aligned}
C_M(0) &= (x_0, x_1, \bar{x}_0, \bar{x}_1, \bar{\bar{x}_0}, \bar{\bar{x}_1}, x_2, x_3, \bar{x}_2, \bar{x}_3, \bar{\bar{x}_2}, \bar{\bar{x}_3}), \\
C_M(1) &= (x_0, x_1, \bar{x}_3, \bar{x}_2, \bar{\bar{x}_3}, \bar{\bar{x}_2}, x_2, x_3, \bar{x}_1, \bar{x}_0, \bar{\bar{x}_1}, \bar{\bar{x}_0}), \\
C'_M(0) &= (x_1, x_0, \bar{x}_1, \bar{x}_0, \bar{\bar{x}_2}, \bar{\bar{x}_3}, x_3, x_2, \bar{x}_3, \bar{x}_2, \bar{\bar{x}_0}, \bar{\bar{x}_1}), \\
C'_M(1) &= (x_1, x_0, \bar{x}_2, \bar{x}_3, \bar{\bar{x}_1}, \bar{\bar{x}_0}, x_3, x_2, \bar{x}_0, \bar{x}_1, \bar{\bar{x}_3}, \bar{\bar{x}_2}).
\end{aligned}$$

For  $m = 12$ , let  $(\mathbb{Z}_{m/4}, \circ_3)$  and  $(\mathbb{Z}_{m/4}, \circ_4)$  be orthogonal quasigroups. For  $i \in \mathbb{Z}_{m/2}$ , define  $C_M(i)$  and  $C'_M(i)$  by

$$\begin{aligned}
C_M(i) &= (x_0, x_1, \bar{b}_0(i), \bar{b}_1(i), \bar{\bar{c}}_0(i), \bar{\bar{c}}_1(i), \\
&\quad x_2, x_3, \bar{b}_2(i), \bar{b}_3(i), \bar{\bar{c}}_2(i), \bar{\bar{c}}_3(i), \dots, \\
&\quad x_{m-2}, x_{m-1}, \bar{b}_{m-2}(i), \bar{b}_{m-1}(i), \bar{\bar{c}}_{m-2}(i), \bar{\bar{c}}_{m-1}(i))
\end{aligned}$$

and

$$\begin{aligned}
C'_M(i) &= (x_1, x_0, \bar{b}'_0(i), \bar{b}'_1(i), \bar{\bar{c}}'_0(i), \bar{\bar{c}}'_1(i), \\
&\quad x_3, x_2, \bar{b}'_2(i), \bar{b}'_3(i), \bar{\bar{c}}'_2(i), \bar{\bar{c}}'_3(i), \dots, \\
&\quad x_{m-1}, x_{m-2}, \bar{b}'_{m-2}(i), \bar{b}'_{m-1}(i), \bar{\bar{c}}'_{m-2}(i), \bar{\bar{c}}'_{m-1}(i)),
\end{aligned}$$

where for  $j, k \in \{0, 1, \dots, \frac{m}{4} - 1\}$ ,

$$\begin{aligned}
b_{2k}(j) &= b_{\frac{m}{2}+2k+1}(j + \frac{m}{4}) = b'_{2k+1}(j) = b'_{\frac{m}{2}+2k}(j + \frac{m}{4}) = x_{2(j \circ_3 k)}, \\
c_{2k}(j) &= c_{\frac{m}{2}+2k+1}(j + \frac{m}{4}) = c'_{2k+1}(j + \frac{m}{4}) = c'_{\frac{m}{2}+2k}(j) = x_{2(j \circ_4 k)}, \\
b_{2k+1}(j) &= b_{\frac{m}{2}+2k}(j + \frac{m}{4}) = b'_{2k}(j) = b'_{\frac{m}{2}+2k+1}(j + \frac{m}{4}) = x_{2(j \circ_3 k)+1}, \\
c_{2k+1}(j) &= c_{\frac{m}{2}+2k}(j + \frac{m}{4}) = c'_{2k}(j + \frac{m}{4}) = c'_{\frac{m}{2}+2k+1}(j) = x_{2(j \circ_4 k)+1}, \\
b_{2k+1}(j + \frac{m}{4}) &= b_{\frac{m}{2}+2k}(j) = b'_{2k}(j + \frac{m}{4}) = b'_{\frac{m}{2}+2k+1}(j) = x_{\frac{m}{2}+2(j \circ_3 k)}, \\
c_{2k+1}(j + \frac{m}{4}) &= c_{\frac{m}{2}+2k}(j) = c'_{2k}(j) = c'_{\frac{m}{2}+2k+1}(j + \frac{m}{4}) = x_{\frac{m}{2}+2(j \circ_4 k)}, \\
b_{2k}(j + \frac{m}{4}) &= b_{\frac{m}{2}+2k+1}(j) = b'_{2k+1}(j + \frac{m}{4}) = b'_{\frac{m}{2}+2k}(j) = x_{\frac{m}{2}+2(j \circ_3 k)+1}, \\
c_{2k}(j + \frac{m}{4}) &= c_{\frac{m}{2}+2k+1}(j) = c'_{2k+1}(j) = c'_{\frac{m}{2}+2k}(j + \frac{m}{4}) = x_{\frac{m}{2}+2(j \circ_4 k)+1}.
\end{aligned}$$

**Lemma 6.** For even integer  $m$ , given a 1-factor  $M$  of a graph with  $\mathbb{Z}_m$  as a vertex set,  $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$  is a set of  $m$  disjoint Hamiltonian cycles of  $K_{m,m,m}^{(3)}$ .

*Proof.* Let  $M = \{x_0x_1, x_2x_3, \dots, x_{m-2}x_{m-1}\}$ . Consider the case where  $m \notin \{4, 12\}$ . For edges of the form  $\{a, a', \bar{b}\}$ , we will show that if  $\{x_{2k}, x_{2k+1}, \overline{x_{2(i \circ_1 k)+j}}\} = \{x_{2k'}, x_{2k'+1}, \overline{x_{2(i' \circ_1 k')+j'}}\}$ , then  $i = i'$ ,  $j = j'$  and  $k = k'$ .

Suppose that  $\{x_{2k}, x_{2k+1}, \overline{x_{2(i \circ_1 k)+j}}\} = \{x_{2k'}, x_{2k'+1}, \overline{x_{2(i' \circ_1 k')+j'}}\}$  for some  $i, i', k, k' \in \mathbb{Z}_{m/2}$  and  $j, j' \in \{0, 1\}$ . Then

$$\begin{aligned}
2k &= 2k', \\
2(i \circ_1 k) + j &= 2(i' \circ_1 k') + j'.
\end{aligned}$$

That is  $k = k'$ ,  $j = j'$  and  $i \circ_1 k = i' \circ_1 k$ . Since  $(\mathbb{Z}_{m/2}, \circ_1)$  is a quasigroup,  $i = i'$ .

The proof for edges of the form  $\{a, a', \bar{c}\}$  can be done in the same way.

For edges of the form  $\{\bar{b}, \bar{b}', \bar{c}\}$ , we will show that if  $\{\overline{x_{2(i \circ_1 k)}}, \overline{x_{2(i \circ_1 k)+1}}, \overline{x_{2(i \circ_2 k)+j}}\} = \{\overline{x_{2(i' \circ_1 k')}}}, \overline{x_{2(i' \circ_1 k')+1}}, \overline{x_{2(i' \circ_2 k')+j'}}\}$ , then  $i = i'$ ,  $j = j'$  and  $k = k'$ .

Suppose that  $\{\overline{x_{2(i \circ_1 k)}}, \overline{x_{2(i \circ_1 k)+1}}, \overline{x_{2(i \circ_2 k)+j}}\} = \{\overline{x_{2(i' \circ_1 k')}}}, \overline{x_{2(i' \circ_1 k')+1}}, \overline{x_{2(i' \circ_2 k')+j'}}\}$  for some  $i, i', k, k' \in \mathbb{Z}_{m/2}$  and  $j, j' \in \{0, 1\}$ . Then

$$\begin{aligned}
i \circ_1 k &= i' \circ_1 k', \\
i \circ_2 k &= i' \circ_2 k', \\
j &= j'.
\end{aligned}$$

Since  $(\mathbb{Z}_{m/2}, \circ_1)$  and  $(\mathbb{Z}_{m/2}, \circ_2)$  are orthogonal quasigroups, we have  $i = i'$  and  $k = k'$ .

The proof for edges of the forms  $\{\bar{b}, \bar{b}', a\}$ ,  $\{\bar{c}, \bar{c}', a\}$  and  $\{\bar{c}, \bar{c}', \bar{b}\}$  can also be done in the same way. Thus, all  $3m \times m$  edges of  $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$



are distinct and  $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$  is a set of  $m$  disjoint Hamiltonian cycles of  $K_{m,m,m}^{(3)}$ .

For  $m = 4$ , it is easy to see that  $C_M(0)$ ,  $C_M(1)$ ,  $C'_M(0)$  and  $C'_M(1)$  are mutually disjoint Hamiltonian cycles of  $K_{m,m,m}^{(3)}$ .

For  $m = 12$ , consider edges of the form  $\{a, a', \bar{b}\}$ :  $e_1 = \{x_{2k}, x_{2k+1}, \bar{x}_i\}$  and  $e_2 = \{x_{\frac{m}{2}+2k}, x_{\frac{m}{2}+2k+1}, \bar{x}_i\}$ , where  $k \in \mathbb{Z}_{m/4}$  and  $i \in \mathbb{Z}_m$ . Note that  $\{2(j \circ_3 k), 2(j \circ_3 k) + 1, \frac{m}{2} + 2(j \circ_3 k), \frac{m}{2} + 2(j \circ_3 k) + 1 : j, k \in \mathbb{Z}_{m/4}\} = \mathbb{Z}_m$  by means of a quasigroup.

If  $i = 2(j \circ_3 k)$ , then  $e_1 \in C_M(j)$  and  $e_2 \in C'_M(j + \frac{m}{4})$ .

If  $i = 2(j \circ_3 k) + 1$ , then  $e_1 \in C'_M(j)$  and  $e_2 \in C_M(j + \frac{m}{4})$ .

If  $i = \frac{m}{2} + 2(j \circ_3 k)$ , then  $e_1 \in C'_M(j + \frac{m}{4})$  and  $e_2 \in C_M(j)$ .

If  $i = \frac{m}{2} + 2(j \circ_3 k) + 1$ , then  $e_1 \in C_M(j + \frac{m}{4})$  and  $e_2 \in C'_M(j)$ .

Thus, each edge of the form  $\{a, a', \bar{b}\}$  is in a unique Hamiltonian cycle. Also use this way to show the same result for edges of the form  $\{a, a', \bar{c}\}$ .

For edges of the form  $\{\bar{b}, \bar{b}', \bar{c}\}$ :  $\{\overline{x_{2(j \circ_3 k)}}, \overline{x_{2(j \circ_3 k) + 1}}, \overline{x_i}\}$  (or  $\{\overline{x_{\frac{m}{2} + 2(j \circ_3 k)}}, \overline{x_{\frac{m}{2} + 2(j \circ_3 k) + 1}}, \overline{x_i}\}$ ), we will show that if  $\{\overline{x_{2(j \circ_3 k)}}, \overline{x_{2(j \circ_3 k) + 1}}, \overline{x_i}\} = \{\overline{x_{2(j' \circ_3 k')}}, \overline{x_{2(j' \circ_3 k') + 1}}, \overline{x_{i'}}\}$ , then  $i = i'$ ,  $j = j'$  and  $k = k'$ .

Suppose that  $\{\overline{x_{2(j \circ_3 k)}}, \overline{x_{2(j \circ_3 k) + 1}}, \overline{x_i}\} = \{\overline{x_{2(j' \circ_3 k')}}, \overline{x_{2(j' \circ_3 k') + 1}}, \overline{x_{i'}}\}$  for some  $j, j', k, k' \in \mathbb{Z}_{m/4}$  and  $i \in \mathbb{Z}_m$ . Then

$$\begin{aligned} j \circ_3 k &= j' \circ_3 k', \\ i &= i'. \end{aligned}$$

There are four possibilities for  $i$ :  $2(j \circ_4 k)$ ,  $2(j \circ_4 k) + 1$ ,  $\frac{m}{2} + 2(j \circ_4 k)$  or  $\frac{m}{2} + 2(j \circ_4 k) + 1$  (also for  $i'$ :  $2(j' \circ_4 k')$ ,  $2(j' \circ_4 k') + 1$ ,  $\frac{m}{2} + 2(j' \circ_4 k')$  or  $\frac{m}{2} + 2(j' \circ_4 k') + 1$ ). Since  $i = i'$ , in any cases, we have  $j \circ_4 k = j' \circ_4 k'$ . The orthogonality of  $(\mathbb{Z}_{m/4}, \circ_3)$  and  $(\mathbb{Z}_{m/4}, \circ_4)$  implies  $j = j'$  and  $k = k'$ .

Edges of the forms  $\{\bar{b}, \bar{b}', a\}$ ,  $\{\bar{c}, \bar{c}', a\}$  and  $\{\bar{c}, \bar{c}', \bar{b}\}$  can be showed in a similar manner. This completes the proof.  $\square$

## 2.4 $h(x, y)$

For  $(x, y) \in \mathbb{Z}_m^2$ , define a Hamiltonian cycle of  $K_{m,m,m}^{(3)}$ ,  $h(x, y)$  by

$$h(x, y) = (0, \overline{x}, \overline{x+y}, m-1, \overline{m-1+x}, \overline{m-1+x+y}, \dots, 1, \overline{1+x}, \overline{1+x+y}).$$

Kuhl and Schroeder [5] define a *difference type* of each edge of the form  $\{a, \bar{b}, \bar{c}\}$  to be  $(b-a, c-b)$  in modulo  $m$ . There are  $m$  edges with a specific difference type.  $h(x, y)$  has  $3m$  edges and contains all edges of difference types  $(x, y)$ ,  $(x+1, y)$  and  $(x, y+1)$ .

Note that all  $m^3$  edges in  $K_{m,m,m}^{(3)}$  are classified into  $m^2$  distinct difference types.

**Lemma 7 ([5]).** *Let  $m \equiv 0 \pmod{3}$  and  $\mathcal{A}_0 = \{(x, y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod{3}\}$ . Then  $\{h(x, y) : (x, y) \in \mathcal{A}_0\}$  is a set of  $m^2/3$  disjoint Hamiltonian cycles of  $K_{m,m,m}^{(3)}$ .*

### 3 Main Results

Let  $H$  be a subhypergraph of  $K_{m,m,m}^{(3)}$ . Let  $n_1(H)$  and  $n_2(H)$  denote the number of Type 1 and Type 2 edges in  $H$ , respectively. Each Hamiltonian cycle in Section 2,  $C(i, j)$ ,  $C'(i, j)$ ,  $(C_M(i)$  and  $C'_M(i))$  and  $h(x, y)$  can be regarded as a subhypergraph of  $K_{m,m,m}^{(3)}$ . We count the number of Type 1 edges and Type 2 edges for each of the four forms of Hamiltonian cycle in Section 2 and overall edges in  $K_{m,m,m}^{(3)}$  as shown in the following table.

$H$	$n_1(H)$	$n_2(H)$	condition
$K_{m,m,m}^{(3)}$	$m^3$	$3m^3 - 3m^2$	–
$C(i, j)$	$m$	$2m$	$m \equiv 0 \pmod{3}$
$C'(i, j)$	$3$	$3m - 3$	$m$ is odd
$C_M(i), C'_M(i)$	$0$	$3m$	$m$ is even
$h(x, y)$	$3m$	$0$	–

Let  $C(0, 0)$  be a Hamiltonian cycle in Lemma 2 and  $C'(0, 0)$  be a Hamiltonian cycle in Lemma 4. We obtain several results as follows.

#### 3.1 $m = 3$

For  $m = 3$ ,  $n_1(K_{3,3,3}^{(3)}) = 27$  and  $n_2(K_{3,3,3}^{(3)}) = 54$ . The sets of Hamiltonian cycles  $\mathcal{C}_1 = \{C(i, j) : i, j \in \mathbb{Z}_m\}$  and  $\mathcal{C}_2 = \{C'(i, j) : i, j \in \mathbb{Z}_m\}$  both have  $m^2$  Hamiltonian cycles. We calculate the number of edges in  $\mathcal{C}_1$ ,  $\sum_{H \in \mathcal{C}_1} n_1(H) = m^3 = 27$  and  $\sum_{H \in \mathcal{C}_1} n_2(H) = 2m^3 = 54$ , and the number of edges in  $\mathcal{C}_2$ ,  $\sum_{H \in \mathcal{C}_2} n_1(H) = 3m^2 = 27$  and  $\sum_{H \in \mathcal{C}_2} n_2(H) = 3m^3 - 3m^2 = 54$ . By Lemma 1 and Lemma 3, we can conclude that  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are both Hamiltonian decompositions of  $K_{3,3,3}^{(3)}$ .

**Example 4.** Let  $C(0, 0) = (0, \bar{0}, \bar{0}, \bar{1}, 1, \bar{1}, \bar{2}, \bar{2}, 2)$ . Then the Hamiltonian decomposition  $\mathcal{C}_1$  of  $K_{3,3,3}^{(3)}$  obtained from Section 2.1 is shown below.

$$C(0, 0) = (0, \bar{0}, \bar{0}, \bar{1}, 1, \bar{1}, \bar{2}, \bar{2}, 2),$$

$$C(0, 1) = (0, \bar{1}, \bar{1}, \bar{2}, 1, \bar{2}, \bar{0}, \bar{0}, 2),$$

$$C(0, 2) = (0, \bar{2}, \bar{2}, \bar{0}, 1, \bar{0}, \bar{1}, \bar{1}, 2),$$

$$\begin{aligned}
C(1, 0) &= (1, \bar{0}, \bar{1}, \bar{2}, 2, \bar{1}, \bar{2}, \bar{0}, 0), \\
C(1, 1) &= (1, \bar{1}, \bar{2}, \bar{0}, 2, \bar{2}, \bar{0}, \bar{1}, 0), \\
C(1, 2) &= (1, \bar{2}, \bar{0}, \bar{1}, 2, \bar{0}, \bar{1}, \bar{2}, 0), \\
C(2, 0) &= (2, \bar{0}, \bar{2}, \bar{0}, 0, \bar{1}, \bar{2}, \bar{1}, 1), \\
C(2, 1) &= (2, \bar{1}, \bar{0}, \bar{1}, 0, \bar{2}, \bar{0}, \bar{2}, 1), \\
C(2, 2) &= (2, \bar{2}, \bar{1}, \bar{2}, 0, \bar{0}, \bar{1}, \bar{0}, 1).
\end{aligned}$$

### 3.2 $m \equiv 0 \pmod{6}$

For  $m \equiv 0 \pmod{6}$ , we have two families  $\mathcal{H}_1$  and  $\mathcal{H}_2$  of Hamiltonian cycles forming two Hamiltonian decompositions of  $K_{m,m,m}^{(3)}$ . First,

$$\mathcal{H}_1 = \{C(i, j) : i, j \in \mathbb{Z}_m\} \cup \{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_1\},$$

where  $\mathcal{F}_1$  is a 1-factorization of a graph  $G$  with  $V(G) = \mathbb{Z}_m = [\bar{0}] \cup [\bar{1}] \cup [\bar{2}]$  and  $E(G) = \{uv : u, v \in \mathbb{Z}_m, \|u - v\| \equiv 0 \pmod{3}\}$ .  $G$  is isomorphic to  $3K_{m/3}$ , three copies of  $K_{m/3}$ . Each component consists of vertices in the same class of modulo 3. Next,

$$\mathcal{H}_2 = \{h(x, y) : (x, y) \in \mathcal{A}_0\} \cup \{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_2\},$$

where  $\mathcal{F}_2$  is a 1-factorization of  $K_m$  with  $\mathbb{Z}_m$  as a vertex set and  $\mathcal{A}_0 = \{(x, y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod{3}\}$ .

Since  $K_{2n}$  is factorizable into  $2n-1$  1-factors [9], we have  $|\mathcal{F}_1| = m/3-1$  and  $|\mathcal{F}_2| = m-1$ . Then we calculate the number of edges in  $\mathcal{H}_1$ ,  $\sum_{H \in \mathcal{H}_1} n_1(H) = m^2 \times m = m^3$  and  $\sum_{H \in \mathcal{H}_1} n_2(H) = m^2 \times 2m + m(m/3-1) \times 3m = 3m^3 - 3m^2$  and the number of edges in  $\mathcal{H}_2$ ,  $\sum_{H \in \mathcal{H}_2} n_1(H) = m^2/3 \times 3m = m^3$  and  $\sum_{H \in \mathcal{H}_2} n_2(H) = m(m-1) \times 3m = 3m^3 - 3m^2$ .

We make some observations.

1. For any two 1-factors  $M$  and  $M'$  in  $K_m$ , we see that if  $M$  and  $M'$  are disjoint, then  $C_M(i)$  and  $C_{M'}(i)$  are also disjoint for all  $i \in \mathbb{Z}_{m/2}$ .
2. For all Type 2 edges  $\{x, x', y\}$  in  $\{C(i, j) : i, j \in \mathbb{Z}_m\}$ ,  $\|x - x'\| \equiv 1$  or  $2 \pmod{3}$ .
3. For all Type 2 edges  $\{x, x', y\}$  in  $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_1\}$ ,  $\|x - x'\| \equiv 0 \pmod{3}$ .
4.  $\{h(x, y) : (x, y) \in \mathcal{A}_0\}$  contains only Type 1 edges.
5.  $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_2\}$  contains only Type 2 edges.

By these observations, Lemma 1, Lemma 6 and Lemma 7, we see that  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are both Hamiltonian decompositions of  $K_{m,m,m}^{(3)}$ , where  $m \equiv 0 \pmod{6}$ .

**Example 5.** For  $m = 6$ , the Hamiltonian decomposition  $\mathcal{H}_1$  consists of  $C(i, j)$ , where  $i, j \in \mathbb{Z}_6$  with  $C(0, 0)$  in Example 1 and  $C_M(i), C'_M(i)$  where  $i \in \mathbb{Z}_3$  and  $M = \{03, 14, 25\}$  in Example 3. The Hamiltonian decomposition  $\mathcal{H}_2$  consists of  $\{h(x, y) : (x, y) \in \mathcal{A}_0\}$  and  $C_M(i), C'_M(i)$  where  $i \in \mathbb{Z}_3$  and  $M \in \{\{01, 25, 34\}, \{02, 31, 45\}, \{03, 42, 51\}, \{04, 53, 12\}, \{05, 14, 23\}\}$ .

### 3.3 $m \equiv 3 \pmod{6}$

For  $m \equiv 3 \pmod{6}$ , let

$$\mathcal{H}_3 = \{C'(i, j) : i, j \in \mathbb{Z}_m\} \cup \{h(x, y) : (x, y) \in \mathcal{A}_0, x \neq y\}$$

where  $\mathcal{A}_0 = \{(x, y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod{3}\}$ . We calculate the number of edges in  $\mathcal{H}_3$ ,  $\sum_{H \in \mathcal{H}_3} n_1(H) = m^2 \times 3 + (m^2/3 - m) \times 3m = m^3$  and  $\sum_{H \in \mathcal{H}_3} n_2(H) = m^2 \times (3m - 3) = 3m^3 - 3m^2$ .

To show that  $\mathcal{H}_3$  is a Hamiltonian decomposition of  $K_{m,m,m}^{(3)}$ , we must show that  $\{C'(i, j) : i, j \in \mathbb{Z}_m\}$  contains all edges of difference types  $(x, y)$ ,  $(x+1, y)$ , and  $(x, y+1)$  for all  $x, y \in \mathbb{Z}_m$  with  $x \neq y$ .

Three edges of the form  $\{a, \bar{b}, \bar{c}\}$  in  $C'(i, j)$  for each  $i, j \in \mathbb{Z}_m$  have difference types

$$\begin{aligned} (b_{m-1} - a_{m-1} + i, c_{m-2} - b_{m-1} + i) &= (i, i+1) \\ \text{for the edge } \{a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{c_{m-2} + 2i + j}\}, \\ (b_{m-1} - a_{m-1} + i, c_{m-1} - b_{m-1} + i) &= (i, i) \\ \text{for the edge } \{a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{c_{m-1} + 2i + j}\}, \\ (b_{m-1} - a_0 + i, c_{m-1} - b_{m-1} + i) &= (i+1, i) \\ \text{for the edge } \{a_0 + j, \overline{b_{m-1} + i + j}, \overline{c_{m-1} + 2i + j}\}. \end{aligned}$$

Since each  $i, j$  corresponds to  $m$  different values of  $\mathbb{Z}_m$ ,  $\{C'(i, j) : i, j \in \mathbb{Z}_m\}$  contains all edges of difference type  $(i, i)$ ,  $(i+1, i)$  and  $(i, i+1)$  as desired.

Thus,  $\mathcal{H}_3$  is a Hamiltonian decompositions of  $K_{m,m,m}^{(3)}$ , where  $m \equiv 3 \pmod{6}$ . The following theorem concludes all the results.

**Theorem 1.**  $K_{m,m,m}^{(3)}$  is decomposable into Hamiltonian cycles if and only if  $3 \mid m$ .

## 4 Discussion

If  $3 \nmid m$ , it is reasonable to consider a Hamiltonian decomposition of  $K_{m,m,m}^{(3)} - I$  where  $I$  is a 1-factor of  $K_{m,m,m}^{(3)}$ . When  $m$  is even,  $m \neq 4$  and  $3 \nmid m$ ,  $K_{m,m,m}^{(3)} - I$  has a Hamiltonian decomposition by a combination of Hamiltonian cycles  $h(x, y)$  retrieved from [5] and  $C_M(i), C'_M(i)$ . Thus, the case of  $m$  is

odd with  $3 \nmid m$  and the case of  $m = 4$  are still open for investigating the existence of Hamiltonian decomposition of  $K_{m,m,m}^{(3)} - I$  where  $K_{m,m,m}^{(3)}$  is given by Definition 1.

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