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# BASIC PROPERTY OF GENERALIZED COMPLETELY MULTIPLICATIVE FUNCTIONS

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**Abstract**

An arithmetic function $f$ is said to be completely multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers $m$ and $n$. In this paper, we define that an arithmetic function $f$ is a generalized completely multiplicative function if $f(1) = 1$ and there is a completely multiplicative function $f_b$ such that $f(mn) = f(m)^{f_b(n)}f(n)^{f_b(m)}$ for all positive integers $m$ and $n$. We consider some basic structure properties of these functions. The functions $v(n) = n^n$ and $\mathrm{Exp}D$ are examples of generalized completely multiplicative functions, where $D$ is the arithmetic derivative.

## 1 Introduction

By an arithmetic function we mean a real-valued function defined on the set of positive integers. An arithmetic function $f$ is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$, and it is said to be completely multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers $m$ and $n$. Each completely multiplicative function is multiplicative. Multiplicative functions are totally determined by their values at prime powers,

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and completely multiplicative functions are totally determined by their values at primes.

The works of E. T. Bell and R. Vaidyanathaswamy are prominent in the history of multiplicative type functions, see e.g. [3, 14]. Multiplicative functions were called factorable functions by E. T. Bell, and completely multiplicative functions were called linear functions by R. Vaidyanathaswamy. Modern literature applies the terms multiplicative and completely multiplicative functions. For material on these functions, see [1, 8, 9, 12, 13].

An arithmetic function $f$ is a generalized multiplicative function if $f(1) = 1$ and there is a multiplicative function $f_b$ such that

$$f(mn) = f(m)^{f_b(n)} f(n)^{f_b(m)} \tag{1}$$

for all positive integers $m$ and $n$ with $\gcd(m, n) = 1$ ([5, 15]). If $f(m)$ is zero for some $m$, then $f_b(n)$ has to be nonnegative for all $n$ with $\gcd(m, n) = 1$. For basic properties of generalized multiplicative functions, see [5, 15].

The concept of a generalized multiplicative function suggests we define that an arithmetic function $f$ is *a generalized completely multiplicative function* if $f(1) = 1$ and there is a completely multiplicative function $f_b$ such that (1) holds for all positive integers $m$ and $n$. If $f$ is a generalized completely multiplicative function such that $f_b$ is identically 1, then $f$ is completely multiplicative. Conversely, if $f$ is a completely multiplicative function, then it is a generalized completely multiplicative function such that $f_b$ is identically 1. It should be noted that in the case that $f$ is somewhere zero, $f_b$ need not be unique.

The condition (1) implies that $f(1) = 0$ or $f(1) = 1$, and further $f(1) = 0$ implies that $f$ is identically zero. For the sake of brevity, we assume that $f(1) = 1$, which means that the function identically zero is not a generalized completely multiplicative function.

An arithmetic function $f$ is said to be additive if $f(mn) = f(m) + f(n)$ whenever $\gcd(m, n) = 1$, and it is said to be completely additive if $f(mn) = f(m) + f(n)$ for all positive integers $m$ and $n$. Each completely additive function is additive. For material on additive functions and completely additive functions, see [7, 9, 10, 11].

An arithmetic function $f$ is said to be a generalized additive function if there is a multiplicative function $f_b$ such that

$$f(mn) = f(m)f_b(n) + f(n)f_b(m) \tag{2}$$

for all positive integers $m$ and $n$ with $\gcd(m, n) = 1$ ([5]). If $f$ is a generalized additive function such that $f_b$ is identically 1, then $f$ is additive. An arithmetic function $f$ is said to be Leibniz-additive if there is a completely multiplicative function $f_b$ such that (2) for all positive integers $m$ and $n$ ([6]). These functions could also be termed as generalized completely additive functions. Chawla [4] refers to generalized additive functions and generalized completely additive functions (or Leibniz-additive functions) as distributive and completely

distributive arithmetic functions. For basic properties of generalized additive functions and generalized completely additive functions, see [4, 5, 6].

Let $\Omega(n)$ denote the total number of prime factors of $n$ with $\Omega(1) = 0$, and let $\omega(n)$ denote the number of distinct prime factors of $n$ with $\omega(1) = 0$. Then $\Omega$ is a completely additive function and $\omega$ is an additive function . This implies that the function $f(n) = n\Omega(n)$ for all positive integers $n$ is an example of a generalized completely additive function and the function $f(n) = n\omega(n)$ for all positive integers $n$ is an example of a generalized additive function. The function $f(n) = n\log(n)$ for all positive integers $n$ is another example of a generalized completely additive function.

The property (2) may be considered *a generalized Leibniz rule*. The arithmetic derivative $D(n) = n'$ is defined as the arithmetic function satisfying the property $D(p) = 1$ for all primes $p$ and the usual Leibniz rule

$$D(mn) = D(m)n + D(n)m$$

for all positive integers $m$ and $n$. Thus it satisfies the generalized Leibniz rule (2) for all positive integers $m$ and $n$ with $D_b(n) = n$. The function $D_b(n) = n$ is completely multiplicative. This implies that the arithmetic derivative $D(n) = n'$ is a Leibniz-additive function. For example, it possesses the property $D(p^k) = kp^{k-1}$ for all primes $p$ and integers $k \geq 0$. See [2, 6].

The property (1) may be considered *a multiplicative analogue of generalized Leibniz rule* and the property

$$f(mn) = f(m)^n f(n)^m \tag{3}$$

may be considered *a multiplicative analogue of Leibniz rule*.

In this paper we consider basic structure properties of generalized completely multiplicative functions, that is, the functions satisfying the multiplicative analogue of generalized Leibniz rule. As an example we show that the function $v(n) = n^{u(n)}$ is a generalized completely multiplicative function with $v_b(n) = u(n)$, where $u$ is a completely multiplicative function. The function Exp$D$ is another example of a generalized completely multiplicative function.

## 2 Properties

Given arithmetic functions $g$ and $h$, we define the power function $g^h$ elementwise as $(g^h)(n) = g(n)^{h(n)}$. If $g(n) = 0$ for some positive integer $n$, then we have to assume that $h(n) \geq 0$.

**Theorem 1.** *If $f$ is a generalized completely multiplicative function such that $f_b(n) \neq 0$ for all positive integers $n$, then the function $g$ defined as $g(n) = f(n)^{1/f_b(n)}$ for all positive integers $n$ is completely multiplicative.*

*Proof.* Clearly, $g(1) = f(1)^{1/f_b(1)} = 1^{1/1} = 1$. From (1) we obtain

$$\begin{aligned} g(mn) &= f(mn)^{\frac{1}{f_b(mn)}} = [f(m)^{f_b(n)} f(n)^{f_b(m)}]^{\frac{1}{f_b(m)f_b(n)}} \\ &= f(m)^{\frac{1}{f_b(m)}} f(n)^{\frac{1}{f_b(n)}} = g(m)g(n) \end{aligned}$$

for all positive integers $m$ and $n$. Thus $g = f^{1/f_b}$ is a completely multiplicative function. □

**Theorem 2.** *If $f$ is a generalized completely multiplicative function such that $f_b(n) \neq 0$ for all positive integers $n$, then $f$ can be written in the form*

$$f = g^{f_b},$$

*where $g$ is a completely multiplicative function.*

*Conversely, assume that $f$ is an arithmetic function of the form*

$$f = g^h, \tag{4}$$

*where $g$ and $h$ are completely multiplicative functions such that if $g(n) = 0$ for some positive integer $n$, then $h$ is always nonnegative. Then $f$ is a generalized completely multiplicative function with $f_b = h$.*

*Proof.* Let $n = q_1 q_2 \cdots q_r$, where $q_1, q_2, \ldots, q_r$ are prime numbers (not necessarily distinct). Then

$$\begin{aligned} f(n) &= f(q_1)^{f_b(q_2 \cdots q_r)} f(q_2 \cdots q_r)^{f_b(q_1)} = f(q_1)^{\frac{f_b(n)}{f_b(q_1)}} f(q_2 \cdots q_r)^{f_b(q_1)} = \cdots \\ &= f(q_1)^{\frac{f_b(n)}{f_b(q_1)}} \cdots f(q_r)^{\frac{f_b(n)}{f_b(q_r)}} = [f(q_1)^{\frac{1}{f_b(q_1)}} \cdots f(q_r)^{\frac{1}{f_b(q_r)}}]^{f_b(n)}. \end{aligned}$$

Defining $g$ as

$$g(n) = f(q_1)^{\frac{1}{f_b(q_1)}} \cdots f(q_r)^{\frac{1}{f_b(q_r)}}$$

we see that $f(n) = g(n)^{f_b(n)}$ for all positive integers $n$, that is, $f = g^{f_b}$, where $g$ is completely multiplicative.

Conversely, assume that $f$ is of the form (4). Then, for all positive integers $m$ and $n$,

$$\begin{aligned} f(mn) &= [g(mn)]^{h(mn)} = [g(m)g(n)]^{h(m)h(n)} \\ &= [g(m)^{h(m)}]^{h(n)} [g(n)^{h(n)}]^{h(m)} = f(m)^{h(n)} f(n)^{h(m)}. \end{aligned}$$

This completes the proof. □

Theorem 2 shows that each generalized completely multiplicative function $f$ such that $f_b$ is always nonzero can be represented as a pair of two completely multiplicative functions $g$ and $h$; we write $f = (g, h) = (f_a, f_b)$. This means that $f = g^h = f_a^{f_b}$.

**Example 1.** *All functions $f$ satisfying the multiplicative Leibniz property*

$$f(mn) = f(m)^n f(n)^m$$

*are of the form $f(n) = g(n)^n$, where $g$ is a completely multiplicative function. Thus, $f = (g, N)$, where $N(n) = n$ for all positive integers $n$.*

**Theorem 3.** *If $f$ is a generalized completely multiplicative function such that $f(p) \neq 0, \pm 1$ and $f_b(p) \neq 0$ for all primes $p$, then the representation $f = (g, h) = (f_a, f_b)$ is unique.*

*Proof.* Let $f = (g_1, h_1) = (g_2, h_2)$. If $p$ is a prime, then

$$f(p) = g_1(p)^{h_1(p)} = g_2(p)^{h_2(p)}, \quad f(p^2) = f(p)^{2h_1(p)} = f(p)^{2h_2(p)},$$

and thus $g_1(p) = g_2(p)$ and $h_1(p) = h_2(p)$. This completes the proof. □

A completely multiplicative function $f$ is totally determined by its values at primes. Next theorem shows that a generalized completely multiplicative function $f$ is totally determined by the values of $f$ and $f_b$ at primes.

**Theorem 4.** *A generalized completely multiplicative function $f$ is totally determined by the values of $f$ and $f_b$ at primes. In fact, if $n = q_1 q_2 \cdots q_r$, where $q_1, q_2, \ldots, q_r$ are prime numbers, then*

$$f(n) = \prod_{i=1}^{r} f(q_i)^{f_b(q_1)\cdots f_b(q_{i-1}) f_b(q_{i+1})\cdots f_b(q_r)}.$$

*If $f_b$ is always nonzero, then*

$$f(n) = \left[ \prod_{i=1}^{r} f(q_i)^{\frac{1}{f_b(q_i)}} \right]^{f_b(n)}.$$

*If $n = p_1^{n_1} \cdots p_s^{n_s}$, where $p_1, \ldots, p_s$ are distinct primes, and $f_b$ is always nonzero, then*

$$f(n) = \left[ \prod_{i=1}^{s} f(p_i)^{\frac{n_i}{f_b(p_i)}} \right]^{f_b(n)}.$$

*Proof.* Similar to the proof of Theorem 2. □

**Theorem 5.** *Assume that $f$ is a generalized completely multiplicative function and $u$ is a completely multiplicative function such that if $f(n) = 0$ for some positive integer $n$, then $u$ is always nonnegative. Then the function $f^u$ defined by $f^u(n) = f(n)^{u(n)}$ is a generalized completely multiplicative function with $(f^u)_b = f_b u$.*

*Proof.* For all positive integers $m$ and $n$,

$$\begin{aligned}(f^u)(mn) &= f(mn)^{u(mn)} = [f(m)^{f_b(n)} f(n)^{f_b(m)}]^{u(m)u(n)} \\ &= (f^u)(m)^{(f_b u)(n)} (f^u)(n)^{(f_b u)(m)}.\end{aligned}$$

Thus $f^u$ is a generalized completely multiplicative function with $(f^u)_b = f_b u$. □

**Example 2.** *Assume that $f$ and $u$ are completely multiplicative functions such that if $f(n) = 0$ for some positive integer $n$, then $u$ is always nonnegative. Then the function $f^u$ is not necessarily a completely multiplicative function but according to Theorem 5 it is a generalized completely multiplicative function. For example, the function $v(n) = n^{u(n)}$ is a generalized completely multiplicative function with $v_b(n) = u(n)$, where $u$ is a completely multiplicative function. In particular, the function $v(n) = n^n$ is not a completely multiplicative function but it is a generalized completely multiplicative function with $v_b(n) = n$. Also the function $v(n) = n^{\lambda(n)}$ is not a completely multiplicative function but it is a generalized completely multiplicative function with $v_b(n) = \lambda(n)$, where $\lambda$ is Liouville's function.*

If $f$ is a completely multiplicative function and $u$ is a completely multiplicative function with positive integer values, then their composite function $f \circ u$ is completely multiplicative. In fact, for all positive integers $m$ and $n$,

$$(f \circ u)(mn) = f(u(m)u(n)) = f(u(m))f(u(n)) = (f \circ u)(m)(f \circ u)(n).$$

In next theorem we apply this result to provide a generalization with respect to $f$.

**Theorem 6.** *If $f$ is a generalized completely multiplicative function and $u$ is a completely multiplicative function with positive integer values, then their composite function $f \circ u$ is generalized completely multiplicative function with $(f \circ u)_b = f_b \circ u$.*

*Proof.* For all positive integers $m$ and $n$,

$$\begin{aligned}(f \circ u)(mn) &= f(u(mn)) = f(u(m)u(n)) = f(u(m))^{f_b(u(n))} f(u(n))^{f_b(u(m))} \\ &= (f \circ u)(m)^{(f_b \circ u)(n)} (f \circ u)(n)^{(f_b \circ u)(m)}.\end{aligned}$$

Now, the result follows, since $f_b \circ u$ is completely multiplicative. This completes the proof. □

If $f$ is a completely multiplicative function with positive values, then the function $\mathrm{Log}f$ defined as $(\mathrm{Log}f)(n) = \log(f(n))$ is completely additive, where log is the natural logarithm of a real number. Conversely, if $f$ is a completely additive function, then the function $\mathrm{Exp}f$ defined as $(\mathrm{Exp}f)(n) = \exp(f(n))$ is completely multiplicative, where exp is the real exponential function. See [11]. Next theorem generalizes this result.

**Theorem 7.** *If $f$ is a generalized completely multiplicative function with positive values, then the function* $\mathrm{Log}f$ *is a generalized completely additive function with* $(\mathrm{Log}f)_b = f_b$.

*Conversely, if $f$ is a generalized completely additive function, then the function* $\mathrm{Exp}f$ *is a generalized completely multiplicative function with* $(\mathrm{Exp}f)_b = f_b$.

*Proof.* If $f$ is a generalized completely multiplicative function with positive values, then

$$f(mn) = f(m)^{f_b(n)} f(n)^{f_b(m)}$$

for all positive integers $m$ and $n$, and thus

$$\log(f(mn)) = f_b(n)\log(f(m)) + f_b(m)\log(f(n))$$

or

$$(\mathrm{Log}f)(mn) = f_b(n)(\mathrm{Log}f)(m) + f_b(m)(\mathrm{Log}f)(n).$$

This proves the first part of the theorem.

Conversely, if $f$ is a generalized completely additive function, then

$$f(mn) = f_b(n)f(m) + f_b(m)f(n)$$

for all positive integers $m$ and $n$, and thus

$$\exp(f(mn)) = (\exp(f(m)))^{f_b(n)}(\exp(f(n)))^{f_b(m)}$$

or

$$(\mathrm{Exp}f)(mn) = ((\mathrm{Exp}f)(m))^{f_b(n)}((\mathrm{Exp}f)(n))^{f_b(m)}.$$

This proves the second part of the theorem. □

**Example 3.** *If $D$ is the arithmetic derivative, then the function* $\mathrm{Exp}D$ *is the generalized completely multiplicative function such that* $g(p) = (\mathrm{Exp}D)_a(p) = \exp(1/p)$ *and* $h(p) = (\mathrm{Exp}D)_b(p) = p$ *in Theorem 2. The function* $\mathrm{Exp}D$ *may be considered a multiplicative analogue of the arithmetic derivative. For example,* $(\mathrm{Exp}D)(p^k) = \exp(kp^{k-1}) = (\exp(p^{k-1}))^k$. *According to Theorem 7,* $(\mathrm{Exp}D)_b(n) = n$, *and thus* $\mathrm{Exp}D$ *satisfies the multiplicative analogue of Leibniz rule* (3).

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