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EDGE-ODD GRACEFUL LABELINGS OF SOME PRISMS AND PRISM-LIKE GRAPHS.

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Abstract

A simple graph G with q edges is called an *edge-odd graceful graph*, EOGG, if there is a bijection f from the edge set of the graph to the set $\{1, 3, 5, \dots, 2q-1\}$ such that, when each vertex is assigned the sum of all values of the edges incident to it modulo $2q$, the resulting vertex labels are distinct.

In this paper, we define new graphs called a prism of star S_n, $\text{Prism}(S_n)$, a prism-like graph, $\text{Prism}_3(S_n)$, and a prism of wheel graph W_n, $\text{Prism}(W_n)$. We give necessary conditions on n that force these graphs to be EOGG, namely, (i) if $n \geq 3$, then $\text{Prism}(S_n)$ is an EOGG; (ii) if $n \geq 3$ and $n \equiv 2 \pmod 6$, then $\text{Prism}_3(S_n)$ is an EOGG; (iii) if $n \geq 3$ and $2|n$, then $\text{Prism}(W_n)$ is an EOGG.

Key words: edge-odd graceful labeling, prism, star, wheel.
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1 Introduction

Let G be a simple graph with q edges. In this article, we let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. In 1967, Rosa [3] gave a definition of a *graceful labeling* of G which is an injection f from $V(G)$ to the set $\{0, 1, 2, \dots, q\}$ such that each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. In 1991, Gnanojothi [2] introduced an *odd-graceful labeling* concept for a graph, that is an injection f from $V(G)$ to the set $\{0, 1, 2, \dots, 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edges labels are in the set $\{1, 3, 5, \dots, 2q-1\}$. In 2009, Solairaju and Chithra [6] reversed the concepts of those two previous vertex labelings by defining an *edge-odd graceful labeling* as the following.

Definition 1.1. Let G be a simple graph an *edge-odd graceful labeling* of G is a bijection f from $E(G)$ to the set $\{1, 3, 5, \dots, 2q-1\}$ so that the induced mapping f^+ from $V(G)$ to the set $\{0, 1, 2, \dots, 2q-1\}$ given by $f^+(x) = \sum f(xy) \pmod{2q}$ where the vertex x is adjacent to the vertex y. The edge labels and vertex labels are distinct. A graph that admitted an edge-odd graceful labeling is called an *edge-odd graceful graph* denoted by EOGG.

Solairaju and Chithra [6] showed edge-odd graceful labelings of graphs related to paths. Later, Singhun [4] showed edge-odd graceful labelings of graphs related to cycles, $SF(n, m)$, where n is an odd integer and m is an even integer such that $n \geq 3$ and $n|m$ and a wheel graph W_n, where n is even. In 2013, Boonklurb et. al. [1] showed edge-odd graceful labelings of prism of cycle C_n, where $n \geq 3$ and $\mathrm{Shaft}(n, 1)$, where $n \geq 3$ and $n \equiv 1 \pmod 2$. Note that $\mathrm{Shaft}(n, 1)$ is a graph consists of 2 copies of wheel graphs joining at the middle. Here, we define new graphs from star, S_n, and wheel, W_n. Thus, for ease of reference, we give definitions of them as follow.

Definition 1.2. Let $n \in \mathbb{N}$. A complete bipartite $K_{1,n}$ is called *star*, denoted by S_n.

In this article, we usually let the first partite with one element be $\{u\}$ and the second partite with n elements be $\{u_1, u_2, u_3, \dots, u_n\}$.

Definition 1.3. A *wheel* graph W_n is a graph with $n+1$ vertices obtained by connecting a single vertex u to all vertices of a cycle $u_1u_2u_3 \cdots u_nu_1$. Then, the vertex set of W_n is the set $\{u, u_1, u_2, u_3, \dots, u_n\}$ and the edge set of W_n is the set $\{uu_i | i \in \{1, 2, 3, \dots, n\}\} \cup \{u_iu_{i+1} | i \in \{1, 2, 3, \dots, n-1\}\} \cup \{u_1u_n\}$.

In Section 2, a prism of star, $\mathrm{Prism}(S_n)$, is defined, algorithms for constructing edge-odd graceful labelings are given and we can prove that $\mathrm{Prism}(S_n)$ is an EOGG for $n \geq 3$. In Section 3, we extend the idea of prism to a prism-like graph by defining $\mathrm{Prism}_3(S_n)$. After that, we can show that if $n \geq 3$ and $n \equiv 2 \pmod 6$, then we can be able to label the edge of $\mathrm{Prism}_3(S_n)$ in such a way that $\mathrm{Prism}_3(S_n)$ is an EOGG. In Section 4, we give a definition of a prism of wheel, Prism(W_n). If we impose the conditions: $n \geq 3$ and $n \equiv 0 \pmod 2$, then we can find an algorithm for labeling Prism(W_n) and can prove that this labeling is an edge-odd graceful labeling for Prism(W_n). In the last section, we give a conclusion of our results and some discussion on the on-going research.

2 Prism of star

Definition 2.1. For $n \geq 3$. Let S_n be a star and S'_n be a copy of S_n. Define Prism(S_n), called the *prism* of S_n, by joining u of S_n to the corresponding vertex u' of S'_n and each u_i of S_n to the corresponding vertex u'_i of S'_n for all $i \in \{1, 2, 3, \ldots, n\}$. Thus,

$$E(\text{Prism}(S_n)) = E(S_n) \cup E(S'_n) \cup \{u_i u'_i \mid i \in \{1, 2, 3, \ldots, n\}\} \cup \{uu'\}.$$

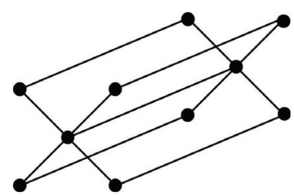

Figure 1: Prism(S_4)

Note that, Prism(S_n) can be called a *book* graph. Solairaju et.al. [5] gave algorithms to construct an edge-odd graceful labeling for book graphs without rigorous proofs. The following algorithms are different from those given in [5] and later we give rigorous proofs to show that the labelings from the following algorithms are edge-odd graceful labelings for Prism(S_n).

Algorithm 2.1. If $n = 3$, we label each edge as shown in Figure 2.

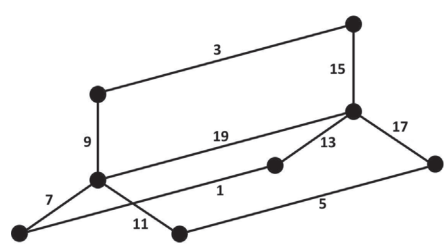

Figure 2: Edge-labelings for Prism(S_3)

Algorithm 2.2. Let $n \geq 4$ and $n \equiv 0 \pmod 2$. Then, $q = |E(\text{Prism}(S_n))| = 3n + 1$. Define $f : E(\text{Prism}((S_n)) \to \{1, 3, 5, \ldots, 6n + 1\}$ by

i $f(u_i u_i') = 2i - 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

ii $f(u_i u) = 2n + 4i - 3$, for $i \in \{1, 2, 3, \ldots, n\}$;

iii $f(u_i' u') = 2n + 4i - 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

iv $f(uu') = 6n + 1$.

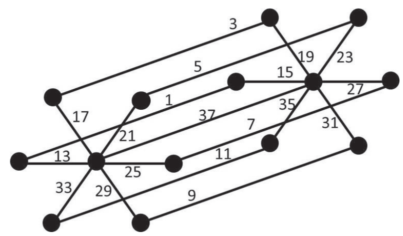

Figure 3: Edge-labelings for Prism(S_6)

For an odd integer n such that $n > 3$, we separate it into 3 cases as follow.

Algorithm 2.3. Let $n > 3$ and $n \equiv 5 \pmod 6$. Then, $q = |E(\text{Prism}(S_n))| = 3n + 1$. Define $f : E(\text{Prism}((S_n)) \to \{1, 3, 5, \ldots, 6n + 1\}$ by

i $f(u_i u_i') = 2n + 2i - 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

ii $f(u_n u) = 1$;

iii $f(u_i u) = 2i + 1$, for $i \in \{1, 2, 3, \ldots, n - 1\}$;

iv $f(u_i' u') = 4n + 2i - 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

v $f(uu') = 6n + 1$.

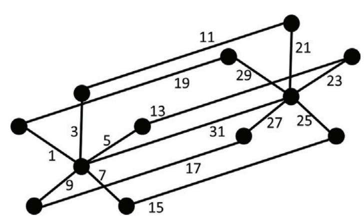

Figure 4: Edge-labelings for Prism(S_5)

Algorithm 2.4. Let $n > 3$ and $n \equiv 1 \pmod 6$. Then, $q = |E(\text{Prism}(S_n))| = 3n+1$. Define $f : E(\text{Prism}((S_n)) \to \{1, 3, 5, \ldots, 6n+1\}$ by

i $f(u_i u_i') = 4n + 2i + 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

ii $f(u_i u) = 2i + 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

iii $f(u_1' u') = 1$;

iv $f(u_i' u') = 2n + 2i + 1$, for $i \in \{2, 3, 4, \ldots, n\}$;

v $f(uu') = 2n + 3$.

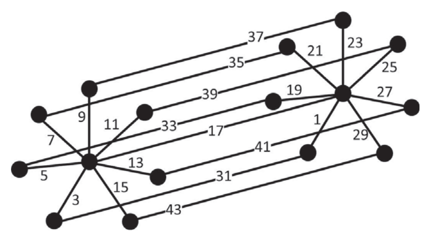

Figure 5: Edge-labelings for Prism(S_7)

Algorithm 2.5. Let $n > 3$ and $n \equiv 3 \pmod 6$. Then, $q = |E(\text{Prism}(S_n))| = 3n+1$. Define $f : E(\text{Prism}((S_n)) \to \{1, 3, 5, \ldots, 6n+1\}$ by

i $f(u_i u_i') = 4n + 2i + 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

ii $f(u_i u) = 2i + 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

iii $f(u_i' u') = 2n + 2i + 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

iv $f(uu') = 1$.

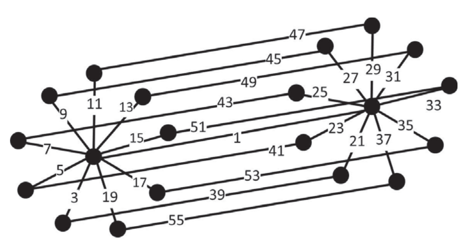

Figure 6: Edge-labelings for Prism(S_9)

Lemma 2.1. *Prism*(S_3) *is an EOGG.*

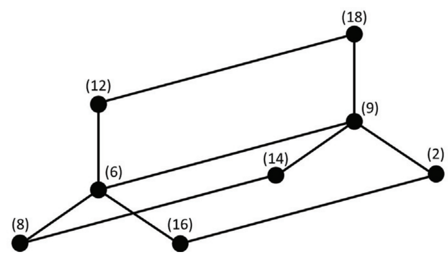

Figure 7: The vertex-labeling for Prism(S_3) induced by Figure 2

Proof. The vertex labeling of Prism(S_3) induced by the edge labeling given in Algorithm 2.1 is shown in Figure 7. Therefore, the function f defined in Algorithm 2.1 is an edge-odd graceful labeling and this implies that Prism(S_3) is an EOGG. □

Lemma 2.2. *If $n \geq 3$ and $n \equiv 0$ (mod 2), then Prism(S_n) is an EOGG.*

Proof. From Algorithm 2.2(i-iv), the edge labels are arranged in the set $\{1, 3, 5, \ldots, 2n-1\}, \{2n+1, 2n+5, 2n+9, \ldots, 6n-3\}, \{2n+3, 2n+7, 2n+11, \ldots, 6n-1\}$, and $\{6n + 1\}$, respectively. Then, f is a bijection from $E(\text{Prism}(S_n))$ to $\{1, 3, 5, \ldots, 6n - 1\}$.

Next, from Algorithm 2.2, we have

$f^+(u_i) = (f(u_i u_i') + f(u_i u)) \pmod{6n + 2} = 2n + 6i - 4 \pmod{6n + 2}$ for $i \in \{1, 2, 3, \ldots, n\}$;

$f^+(u_i') = (f(u_i u_i') + f(u_i' u')) \pmod{6n + 2} = 2n + 6i - 2 \pmod{6n + 2}$ for $i \in \{1, 2, 3, \ldots, n\}$;

$f^+(u) = ((\sum_{i=1}^{n} f(u_i u)) + f(uu')) \pmod{6n+2} = (4n^2 + 5n + 1) \pmod{6n+2}$;

$f^+(u') = ((\sum_{i=1}^{n} f(u_i' u')) + f(uu')) \pmod{6n+2} = (4n^2 + 7n + 1) \pmod{6n+2}$.

Since n is even, we can see from the division algorithm easily that there is an integer t for which $f^+(u) = (4n^2 + 5n + (6n + 2)t) + 1$. Thus, $f^+(u)$ is an odd integer. Similarly, we can use the division algorithm to prove that $f^+(u')$ is an odd integer and $f^+(u_i)$ and $f^+(u_i')$ are even integers. Since the sequences $\{6i - 4\}_{i=1}^{n}$ and $\{6i - 2\}_{i=1}^{n}$ are all distinct, it clearly that $\{f^+(u_i)\}_{i=1}^{n}$ and $\{f^+(u_i')\}_{i=1}^{n}$ are also distinct.

Next, we will show that $f^+(u)$ and $f^+(u')$ are distinct. Suppose in the contrary that $f^+(u) \equiv f^+(u') \pmod{6n + 2}$. Then, $4n^2 + 5n + 1 \equiv 4n^2 + 7n + 1 \pmod{6n + 2}$. This implies that $2n \equiv 0 \pmod{6n + 2}$, which is a contradiction. Thus, for all $i \in \{1, 2, 3, \ldots, n\}$, $f^+(u_i), f^+(u_i'), f^+(u)$ and $f^+(u')$ are all distinct and they are in $\{0, 1, 2, \ldots, 6n + 1\}$. Therefore, the function f defined in Algorithm 2.2 is an edge-odd graceful labeling and this implies that Prism(S_n) is an EOGG. □

Lemma 2.3. *If $n \geq 3$ and $n \equiv 5$ (mod 6), then Prism(S_n) is an EOGG.*

Proof. From Algorithm 2.3(i-v), the edge labels are arranged in the set $\{2n+1, 2n+3, 2n+5, \ldots, 4n-1\}, \{1\}, \{3, 5, 7, \ldots, 2n-1\}, \{4n+1, 4n+3, 4n+5, \ldots, 6n-1\}$ and $\{6n+1\}$, respectively. Then, f is a bijection from $E(\text{Prism}(S_n))$ to $\{1, 3, 5, \ldots, 6n-1\}$.

Next, from Algorithm 2.3, we have
$f^+(u_i) = (f(u_iu_i') + f(u_iu)) \pmod{6n+2} = 2n+4i$, for $i \in \{1, 2, 3, \ldots, n-1\}$;
$f^+(u_n) = 4n$;
$f^+(u_i') = (f(u_iu_i') + f(u_i'u')) \pmod{6n+2} = 4i-4$, for $i \in \{1, 2, 3, \ldots, n\}$;
$f^+(u) = ((\sum_{i=1}^{n} f(u_iu)) + f(uu')) \pmod{6n+2} = (n^2-1) \pmod{6n+2}$;
$f^+(u') = ((\sum_{i=1}^{n} f(u_i'u')) + f(uu')) \pmod{6n+2} = (5n^2-1) \pmod{6n+2}$.

Since $n = 6k-1$ for some $k \in \mathbb{N}$, $f^+(u_i) = 2(6k-1)+4i = 4(3k+i-1)+2$ for all $i \in \{1, 2, 3, \ldots, n-1\}$. Then, $4 \nmid f^+(u_i)$. However, $4 | f^+(u_i')$ for all $i \in \{1, 2, 3, \ldots, n\}$, $4|f^+(u_n)$, and $\max_{1 \le i \le n}\{f^+(u_i')\} = 4n-4$. We can conclude that $\{f^+(u_i) | i \in \{1, 2, 3, \ldots, n\}\}$ and $\{f^+(u_i') | i \in \{1, 2, 3, \ldots, n\}\}$ are disjoint. Since $n = 6k-1$ for some $k \in \mathbb{N}$, we can use the division algorithm to prove that $4|f^+(u)$ and $4|f^+(u')$. Then, to complete the prove, we will show that the values of $f^+(u)$ and $f^+(u')$ under the integer modulo $6n+2$ are greater than $4n$.

Let k be an integer such that $n = 6k-1$, then $6n+2 = 36k-4, n^2-1 = 36k^2-12k$ and $5n^2-1 = 180k^2-60k+4$. Thus, $f^+(u) = (36k^2-12k) \pmod{36k-4} = 28k-4 > 24k-4 = 4n$, and $f^+(u') = (5n^2-1) \pmod{6n+2} = 32k-4 > 24k-4 = 4n$. Hence, for all $i \in \{1, 2, 3, \ldots, n\}$, $f^+(u_i), f^+(u_i'), f^+(u)$ and $f^+(u')$ are distinct and they are in $\{0, 1, 2, \ldots, 6n+1\}$. Therefore, the function f defined in Algorithm 2.3 is an edge-odd graceful labeling and this implies that $\text{Prism}(S_n)$ is an EOGG. □

Lemma 2.4. *If $n \ge 3$ and $n \equiv 1 \pmod 6$, then Prism(S_n) is an EOGG.*

Proof. From Algorithm 2.4(i-iv), the edge labels are arranged in the set $\{4n+3, 4n+5, 4n+7, \ldots, 6n+1\}, \{3, 5, 7, \ldots, 2n+1\}, \{1\}, \{2n+5, 2n+7, 2n+9, \ldots, 4n+1\}$ and $\{2n+3\}$, respectively. Then, f is a bijection from $E(\text{Prism}(S_n))$ to $\{1, 3, 5, \ldots, 6n-1\}$.

Next, from Algorithm 2.4, we have
$f^+(u_i) = (f(u_iu_i') + f(u_iu)) \pmod{6n+2} = (4n+4i+2) \pmod{6n+2}$,
for $i \in \{1, 2, 3, \ldots, n\}$;
$f^+(u) = ((\sum_{i=1}^{n} f(u_iu)) + f(uu')) \pmod{6n+2} = (n^2+4n+3) \pmod{6n+2}$;
$f^+(u_1') = 4n+4$;
$f^+(u_i') = (f(u_iu_i') + f(u_i'u')) \pmod{6n+2} = 4i$, for $i \in \{2, 3, 4, \ldots, n\}$;
$f^+(u') = ((\sum_{i=1}^{n} f(u_i'u')) + f(uu')) \pmod{6n+2} = (3n^2+2n+1) \pmod{6n+2}$.

Since $n = 6k+1$ for some $k \in \mathbb{N}$,
$\{f^+(u_i) | i \in \{1, 2, 3, \ldots, \frac{n-1}{2}\}\} \cup \{f^+(u_i) | i \in \{\frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n\}\}$
$= \{4n+6, 4n+10, 4n+14, \ldots, 6n\} \cup \{2, 6, 10, \ldots, 2n\}$
$= \{24k+10, 24k+14, 24k+18, \ldots, 36k+6\} \cup \{2, 6, 10, \ldots, 12k+2\}$ and
$\{f^+(u_i') | i \in \{1, 2, 3, \ldots, n\}\}$

$= \{8, 12, 16, \ldots, 4n, 4n+4\} = \{8, 12, 16, \ldots, 24k+4, 24k+8\}$.
Since $n = 6k+1$ for some $k \in \mathbb{N}$, we have $6n+2 = 36k+8, n^2+4n+3 = 36k^2+36k+8 \equiv 28k+8 \pmod{36k+8}$ and $3n^2+2n+1 = 108k^2+48k+6 \equiv 24k+6 \pmod{36k+8}$. Then, $f^+(u) = 28k+8$ and $f^+(u') = 24k+6$. Hence, for all $i \in \{1, 2, 3, \ldots, n\}$, $f^+(u_i), f^+(u_i'), f^+(u)$ and $f^+(u')$ are distinct and they are in $\{0, 1, 2, \ldots, 6n+1\}$. Therefore, the function f defined in Algorithm 2.4 is an edge-odd graceful labeling and this implies that Prism(S_n) is an EOGG. □

Lemma 2.5. *If $n \geq 3$ and $n \equiv 3 \pmod 6$, then Prism(S_n) is an EOGG.*

Proof. From Algorithm 2.5, the edge labels are arranged in the set $\{4n+3, 4n+5, 4n+7, \ldots, 6n+1\} \cup \{3, 5, 7, \ldots, 2n+1\} \cup \{2n+3, 2n+5, 2n+7, \ldots, 4n+1\} \cup \{1\}$, respectively. Then f is a bijection from $E(\text{Prism}(S_n))$ to $\{1, 3, 5, \ldots, 6n-1\}$.

Next, from Algorithm 2.5, we have
$f^+(u_i) = (f(u_iu_i') + f(u_iu)) \pmod{6n+2} = (4n+4i+2) \pmod{6n+2}$, for $i \in \{1, 2, 3, \ldots, n\}$;
$f^+(u) = ((\sum_{i=1}^{n} f(u_iu)) + f(uu')) \pmod{6n+2} = (n^2+2n+1) \pmod{6n+2}$;
$f^+(u_i') = (f(u_iu_i') + f(u_i'u')) \pmod{6n+2} = 4i$, for $i \in \{1, 2, 3, \ldots, n\}$;
$f^+(u') = ((\sum_{i=1}^{n} f(u_i'u')) + f(uu')) \pmod{6n+2} = (3n^2+2n+1) \pmod{6n+2}$.

Since $n = 6k+3$ for some $k \in \mathbb{N}$, the similar argument as in Lemma 2.4 can show that $f^+(u_i), f^+(u_i'), f^+(u)$ and $f^+(u')$ are distinct and they are subsets of $\{0, 1, 2, \ldots, 6n+1\}$. Therefore, the function f defined in Algorithm 2.5 is an edge-odd graceful labeling and this implies that Prism(S_n) is an EOGG. □

The results from those of Lemma 2.1-2.5 can be concluded as the following theorem.

Theorem 2.1. *The Prism(S_n) is an EOGG for every $n \geq 3$.*

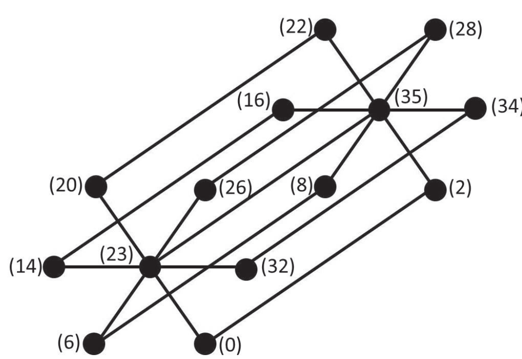

Figure 8: The vertex-labeling for Prism(S_6) induced by Algorithm 2.2

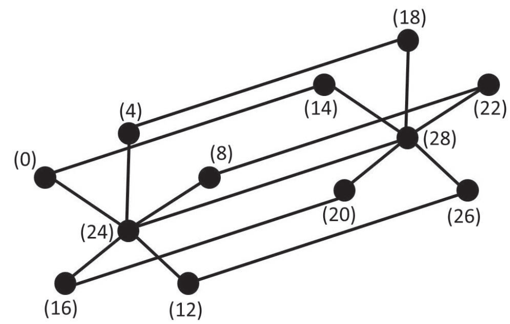

Figure 9: The vertex-labeling for Prism(S_5) induced by Algorithm 2.3

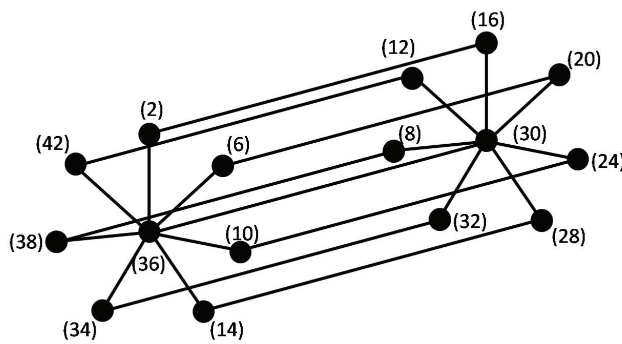

Figure 10: The vertex-labeling for Prism(S_7) induced by Algorithm 2.4

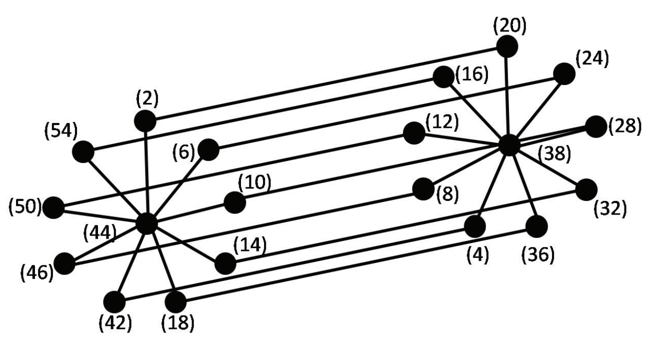

Figure 11: The vertex-labeling for Prism(S_9) induced by Algorithm 2.5

3 Prism-like graph

In this section, we give a definition of Prism$_3(S_n)$ and an algorithm of each edge labeling of Prism$_3(S_n)$ for $n \geq 3$ and $n \equiv 2 \pmod 6$. After that we prove that the labeling given in this algorithm is an edge-odd graceful labeling.

Definition 3.1. For $n \geq 3$, let $S_n^{(1)}$ be a star and $S_n^{(2)}$ and $S_n^{(3)}$ be copies of $S_n^{(1)}$. Define $\text{Prism}_3(S_n)$ by joining $u^{(1)}$ of $S_n^{(1)}$ to the corresponding vertex $u^{(2)}$ of $S_n^{(2)}$, $u^{(2)}$ of $S_n^{(2)}$ to the corresponding vertex $u^{(3)}$ of $S_n^{(3)}$, each $u_i^{(1)}$ of $S_n^{(1)}$ to the corresponding vertex $u_i^{(2)}$ of $S_n^{(2)}$, and each $u_i^{(2)}$ of $S_n^{(2)}$ to the corresponding vertex $u_i^{(3)}$ of $S_n^{(3)}$ for all $i \in \{1, 2, 3, \ldots, n\}$. Thus,

$$E(Prism_3(S_n)) = E(S_n^{(1)}) \cup E(S_n^{(2)}) \cup E(S_n^{(3)}) \cup \{u_i^{(1)} u_i^{(2)} \mid i \in \{1, 2, 3, \ldots, n\}\}.$$

$$\cup \{u_i^{(2)} u_i^{(3)} \mid i \in \{1, 2, 3, \ldots, n\}\} \cup \{u^{(1)} u^{(2)}\} \cup \{u^{(2)} u^{(3)}\}.$$

Algorithm 3.1. Let $n \geq 3$ be an integer and $n \equiv 2 \pmod 6$. We can easily obtain that $q = |E(\text{Prism}_3(S_n))| = 5n + 2$. Define $f : E(\text{Prism}_3(S_n)) \to \{1, 3, 5, \ldots, 10n + 3\}$ by

i $f(u^{(1)} u_i^{(1)}) = 4n + 2i - 1$ for $i \in \{1, 2, 3, \ldots, n\}$;

ii $f(u^{(2)} u_i^{(2)}) = 2i - 1$ for $i \in \{1, 2, 3, \ldots, n\}$;

iii $f(u^{(3)} u_i^{(3)}) = 6n + 2i - 1$ for $i \in \{1, 2, 3, \ldots, n\}$;

iv $f(u_i^{(1)} u_i^{(2)}) = 8n + 2i - 1$ for $i \in \{1, 2, 3, \ldots, n\}$;

v $f(u_i^{(2)} u_i^{(3)}) = 2n + 2i - 1$ for $i \in \{1, 2, 3, \ldots, n\}$;

vi $f(u^{(1)} u^{(2)}) = 10n + 3$;

vii $f(u^{(2)} u^{(3)}) = 10n + 1$.

Theorem 3.1. *If $n \geq 3$ and $n \equiv 2 \ (mod\ 6)$, then $Prism_3(S_n)$ is an EOGG.*

Proof. From Algorithm 3.1(i-vii), the edge labels are arranged in the set $\{4n+1, 4n+3, 4n+5, \ldots, 6n-1\}, \{1, 3, 5, \ldots, 2n-1\}, \{6n+1, 6n+3, 6n+5, \ldots 8n-1\}, \{8n+1, 8n+3, 8n+5, \ldots, 10n-1\}, \{2n+1, 2n+3, 2n+5, \ldots, 4n-1\}, \{10n+3\}$, and $\{10n+1\}$, respectively. Then, f is a bijection from $E(\text{Prism}_3(S_n))$ to $\{1, 3, 5, \ldots, 10n + 4\}$.

Next, from Algorithm 3.1, we have
$f^+(u^{(1)}) = ((\sum_{i=1}^{n} f(u_i^{(1)} u^{(1)})) + f(u^{(1)} u^{(2)})) \pmod{10n + 4}$
$= (5n^2 - 1) \pmod{10n + 4}$;
$f^+(u^{(2)}) = ((\sum_{i=1}^{n} f(u_i^{(2)} u^{(2)})) + f(u^{(1)} u^{(2)}) + f(u^{(2)} u^{(3)})) \pmod{10n + 4}$
$= (n^2 - 4) \pmod{10n + 4}$;
$f^+(u^{(3)}) = ((\sum_{i=1}^{n} f(u_i^{(3)} u^{(3)})) + f(u^{(2)} u^{(3)})) \pmod{10n + 4}$
$= (7n^2 - 3) \pmod{10n + 4}$;
$f^+(u_1^{(2)}) = (f(u_1^{(1)} u_1^{(2)}) + f(u_1^{(2)} u_1^{(3)}) + f(u_1^{(2)} u^{(2)})) \pmod{10n + 4} = 10n + 3$;

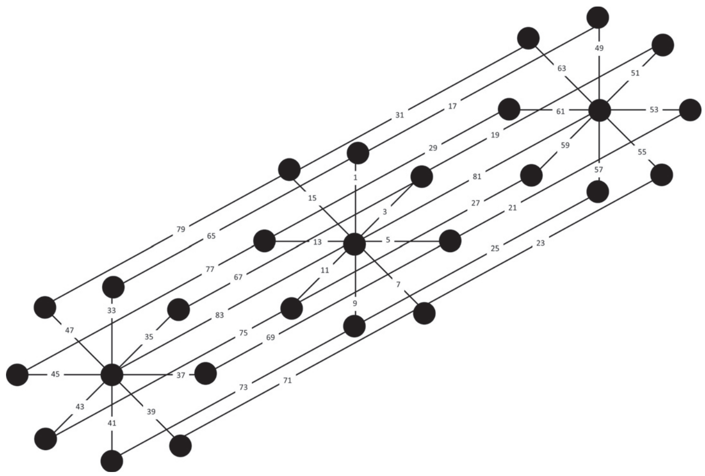

Figure 12: Edge-labelings for $\mathrm{Prism}_3(S_8)$

$f^+(u_i^{(2)}) = (f(u_i^{(1)}u_i^{(2)}) + f(u_i^{(2)}u_i^{(3)}) + f(u_i^{(2)}u^{(2)}))\ (\mathrm{mod}\ 10n+4) = 6i-7$ for $i \in \{2,3,4,\ldots,n\}$;
$f^+(u_i^{(1)}) = (f(u_i^{(1)}u_i^{(2)}) + f(u_i^{(1)}u^{(1)}))\ (\mathrm{mod}\ 10n+4) = 2n+4i-6$;
$f^+(u_i^{(3)}) = (f(u_i^{(2)}u_i^{(3)}) + f(u_i^{(3)}u^{(3)}))\ (\mathrm{mod}\ 10n+4)$
$= (8n+4i-2)\ (\mathrm{mod}\ 10n+4)$.

Since $n \equiv 2\ (\mathrm{mod}\ 6)$, we can see from the division algorithm that for all $i \in \{1,2,3,\ldots,n\}$, $f^+(u^{(2)}), f^+(u_i^{(1)})$, and $f^+(u_i^{(3)})$ are even integers and $f^+(u^{(1)}), f^+(u^{(3)})$, and $f^+(u_i^{(2)})$ are odd integers. First, we claim that $f^+(u^{(2)})$, $f^+(u_i^{(1)})$, and $f^+(u_i^{(3)})$ are distinct for all $i \in \{1,2,3,\ldots,n\}$. Since $n = 6k+2$ for some $k \in \mathbb{N}$, we have

$f^+(u^{(2)}) = (36k^2+24k)\ (\mathrm{mod}\ 60k+24)$;
$f^+(u_i^{(1)}) = (12k+4i-2)\ (\mathrm{mod}\ 60k+24)$;
$f^+(u_i^{(3)}) = (48k+4i+14)\ (\mathrm{mod}\ 60k+24)$.

By the division algorithm, $4|f^+(u^{(2)})$, $4 \nmid f^+(u_i^{(1)})$ and $4 \nmid f^+(u_i^{(3)})$. Suppose that $f^+(u_i^{(1)}) \equiv f^+(u_j^{(3)})\ (\mathrm{mod}\ 10n+4)$ for some $i,j \in \{1,2,3,\ldots,n\}$ and $i \neq j$. Then, $2n+4i-6 \equiv 8n+4j-2\ (\mathrm{mod}\ 10n+4)$. This implies that $4(i-j-1) = 6n+(10n+4)t$ for some integer t. Since $i,j \in \{1,2,3,\ldots,n\}$ and $i \neq j$, we have $-4 \leq 4(i-j-1) \leq 4n-8$. However, if $t<0$, then $4(i-j-1) = 6n+(10n+4)t < -4n-4$ and if $t \geq 0$, then $4(i-j-1) = 6n+(10n+4)t \geq 6n$. This is a contradiction.

Next, we claim that $f^+(u^{(1)}), f^+(u^{(3)})$, and $f^+(u_i^{(2)})$ are distinct. Since $n = 6k+2$ for some $k \in \mathbb{N}$, $f^+(u_1^{(2)}) = 60k+23$, $f^+(u^{(1)}) = (180k^2 + 120k+19)\ (\mathrm{mod}\ 60k+24) = 48k+19$ and $f^+(u^{(3)}) = (252k^2+168k+25)\ (\mathrm{mod}\ 60k+24) = (12k^2+12k+1)\ (\mathrm{mod}\ 60k+24)$. Since $f^+(u^{(3)}) =$

$(12k^2+12k+1) \pmod{60k+24}$, by the division algorithm, there exist unique integers q and r such that $12k^2+12k+1 = (60k+24)q+r$. Then, $r = 6(2k^2+8k-4)q+1$. Notice that $f^+(u_1^{(2)}) = 6(10k+3)+5$, $f^+(u_i^{(2)}) = 6(i-2)+5$ for all $i \in \{2,3,4,\ldots,n\}$, and $f^+(u^{(1)}) = 6(8k+3)+1$. Thus, $\{f^+(u_i^{(2)})|i \in \{1,2,3,\ldots,n\}\}$ and $\{f^+(u^{(1)}), f^+(u^{(3)})\}$ are distinct. Moreover, since $\max_{2\le i\le n}\{6i-7\}$=$6n-7 < 10n+3$=$f^+(u_1^{(2)})$, It's clearly that $f^+(u_1^{(2)})$ and $f^+(u_i^{(2)})$ are distinct.

Finally, to obtain the second claim we have to show that $f^+(u^{(1)})$ and $f^+(u^{(3)})$ are distinct. Suppose that $f^+(u^{(1)}) \equiv f^+(u^{(3)}) \pmod{10n+4}$. Then, $5n^2-1 \equiv 7n^2-3 \pmod{10n+4}$. This implies that $n^2-1 \equiv 0 \pmod{5n+2}$ that is there exist $m \in \mathbb{Z}$ such that $n^2-1 = (5n+2)m$. Since n is even, n^2-1 is odd and $5n+2$ is even integer. This is a contradiction. Hence, $f^+(u^{(1)})$ and $f^+(u^{(3)})$ are distinct.

Therefore, the function f defined in Algorithm 3.1 is an edge-odd graceful labeling and this implies that $\text{Prism}_3(S_n)$ is an EOGG. □

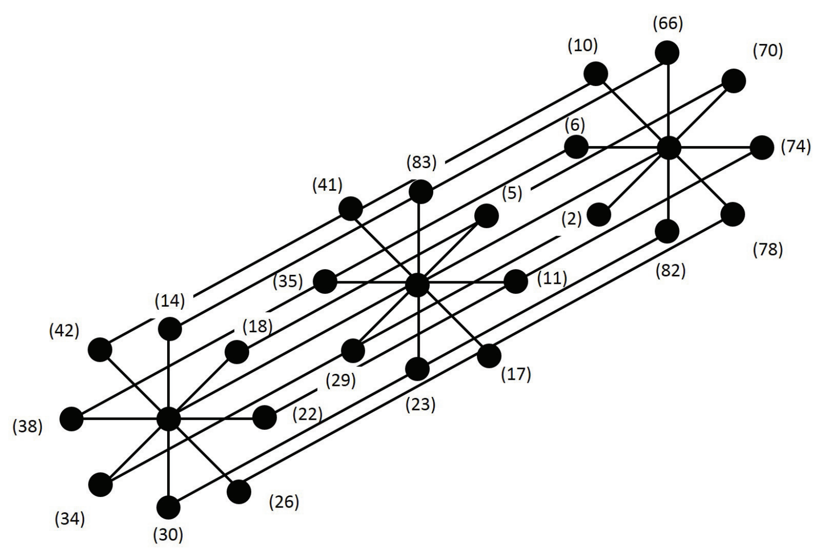

Figure 13: The vertex-labeling for $\text{Prism}_3(S_8)$ induced by Algorithm 3.1

4 Prism of wheel

In this section, we give a definition of $\text{Prism}(W_n)$ and an algorithm to label each edge of $\text{Prism}(W_n)$. Then, we prove that the $\text{Prism}(W_n)$ is an edge-odd graceful graph if $2|n$.

Definition 4.1. For $n \geq 3$, let W_n be a wheel graph and W'_n be a copy of W_n. Define Prism(W_n), called the *prism* of W_n, by joining u of W_n to the corresponding vertex u' of W'_n and each u_i of W_n to the corresponding vertex u'_i of W'_n for all $i \in \{1, 2, 3, \ldots, n\}$. Thus,

$$E(Prism(W_n)) = E(W_n) \cup E(W'_n) \cup \{u_i u'_i \mid i \in \{1, 2, 3, \ldots, n\}\} \cup \{uu'\}.$$

Algorithm 4.1. Let $n \geq 3$ and $2|n$. We can easily to obtain that $q = |E(Prism(W_n))| = 5n + 1$. Define $f : E(G) \to \{1, 3, 5, \ldots, 10n + 1\}$ by

i $f(u_i u'_i) = 2i - 1$ for $i \in \{1, 2, 3, \ldots, n\}$;

ii $f(u_1 u_n) = 2n + 1$;

iii $f(u_i u_{i+1}) = 4n - 2i + 1$ for $i \in \{1, 2, 3, \ldots, n - 1\}$;

iv $f(u'_1 u'_n) = 6n + 1$;

v $f(u'_i u'_{i+1}) = 8n - 2i + 1$ for $i \in \{1, 2, 3, \ldots, n - 1\}$;

vi $f(u_1 u) = 4n + 1$;

vii $f(u_i u) = 6n - 2i + 3$ for $i \in \{2, 3, 4, \ldots, n\}$;

viii $f(u'_1 u') = 8n + 1$;

ix $f(u'_i u') = 10n - 2i + 3$ for $i \in \{2, 3, 4, \ldots, n\}$;

x $f(uu') = 10n + 1$.

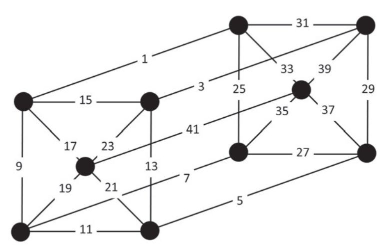

Figure 14: Edge-labelings for Prism(W_4)

Theorem 4.1. *If $n \geq 3$ and $2|n$, then Prism(W_n) is an EOGG.*

Proof. From Algorithm 4.1(i-x), the edge labels are arranged in the set $\{1, 3, 5, \ldots, 2n-1\}$, $\{2n+1\}$, $\{4n-1, 4n-3, 4n-5, \ldots, 2n+5, 2n+3\}$, $\{6n+1\}$, $\{8n-1, 8n-3, 8n-5, \ldots, 6n+3\}$, $\{4n+1\}$, $\{6n-1, 6n-3, 6n-5, \ldots, 4n+3\}$, $\{8n+1\}$, $\{10n-1, 10n-3, 10n-5, \ldots, 8n+3\}$, and $\{10n+1\}$, respectively. Then, f is a bijection from $E(\text{Prism}(W_n))$ to $\{1, 3, 5, \ldots, 10n+1\}$.

Next, from Algorithm 4.1, we have
$f^+(u_1) = (f(u_1u_n) + f(u_1u_2) + f(u_1u) + f(u_1u_1')) \pmod{10n+2} = 0$;
$f^+(u_i) = (f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iu) + f(u_iu_i')) \pmod{10n+2} = 4n-4i+4$ for $i \in \{2,3,4,\ldots,n\}$;
$f^+(u_1') = (f(u_1'u_n') + f(u_1'u_2') + f(u_1'u') + f(u_1u_1')) \pmod{10n+2} = 2n-2$;
$f^+(u_i') = (f(u_{i-1}'u_i') + f(u_i'u_{i+1}') + f(u_i'u') + f(u_iu_i')) \pmod{10n+2} = 6n-4i+2$ for $i \in \{2,3,4,\ldots,n\}$;
$f^+(u) = ((\sum_{i=1}^{n} f(u_iu) + f(uu')) \pmod{10n+2} = (5n^2-1) \pmod{10n+2}$;
$f^+(u') = ((\sum_{i=1}^{n} f(u_i'u') + f(uu')) \pmod{10n+2} = (9n^2-1) \pmod{10n+2}$;
It's clearly that for all $i \in \{1,2,3,\ldots,n\}$, $f^+(u_i)$ and $f^+(u_i')$ are even integers and by the division algorithm, we have $f^+(u)$ and $f^+(u')$ are odd integers. First, we show that $f^+(u_i)$ and $f^+(u_i')$ are distinct. Since the sequences $\{4n-4i+4\}_{i=2}^{n}$,$\{0\}$,$\{6n-4i+2\}_{i=2}^{n}$, and $\{2n-2\}$ are all distinct, it clearly that $\{f^+(u_i)\}_{i=2}^{n}$,$\{f^+(u_1)\}$,$\{f^+(u_i')\}_{i=2}^{n}$ and $\{f^+(u_1')\}$ are also distinct. Next, we claim that $f^+(u)$ and $f^+(u')$ are distinct. Suppose in the contrary that $f^+(u) \equiv f^+(u') \pmod{10n+2}$. Then, $5n^2-1 \equiv 9n^2-1 \pmod{10n+2}$. This implies that $2n^2 \equiv 0 \pmod{5n+1}$
For $n \in \{4,6,8\}$, $2n^2 \pmod{5n+1}$ is $11, 10$ and 5, respectively.
If $n = 10k$ for some $k \in \mathbb{N}$, $2n^2 \pmod{5n+1} = 200k^2 \pmod{50k+1} = 46k+1$.
Similarly,
If $n = 10k+2$ for some $k \in \mathbb{N}$, $2n^2 \pmod{5n+1} = 36k+8$.
If $n = 10k+4$ for some $k \in \mathbb{N}$, $2n^2 \pmod{5n+1} = 26k+11$.
If $n = 10k+6$ for some $k \in \mathbb{N}$, $2n^2 \pmod{5n+1} = 16k+10$.
If $n = 10k+8$ for some $k \in \mathbb{N}$, $2n^2 \pmod{5n+1} = 6k+5$.
Thus, $2n^2 \not\equiv 0 \pmod{5n+1}$. Hence, $f^+(u)$ and $f^+(u')$ are distinct.

Therefore, the function f defined in Algorithm 4.1 is an edge-odd graceful labeling and this implies that Prism(W_n) is an EOGG.

□

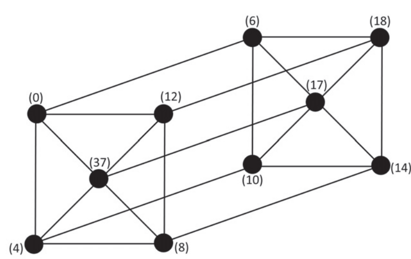

Figure 15: The vertex-labeling for Prism(W_4) induced by Algorithm 4.1

5 Conclusion and Discussion

As we can see in this article that $\text{Prism}(S_n)$ is an EOGG for all $n \in \mathbb{N}$. However, we can only prove that if $n \equiv 2 \pmod 6$, then $\text{Prism}_3(S_n)$ is an EOGG and if $2|n$, then $\text{Prism}(W_n)$ is an EOGG. One may try to find algorithms for edge labeling of $\text{Prism}_3(S_n)$ and $\text{Prism}(W_n)$ for other cases in such away that force $\text{Prism}_3(S_n)$ and $\text{Prism}(W_n)$ to be EOGG. Another way is extend the definition of $\text{Prism}_3(S_n)$ to be $\text{Prism}_m(S_n)$ for $m \in \mathbb{N}$ and investigate whether we can find the way to have the edge-odd graceful labeling for this type of Prism-like graph or not.

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