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TEMPERATURE OF PERFECT FLUID BLACK HOLES IN VARIOUS COORDINATES AND ENTROPY COMPOSITION OF BLACK HOLE SYSTEMS

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Abstract

The laws of thermodynamics describe the temperature and the entropy of perfect fluid black holes. The temperature can be calculated from the thermal radiation or Hawking radiation using surface gravity, while the entropy is related to the determined temperature. In this paper, we explain in detail the temperature and the entropy of black holes. Finally, we calculate the entropy composition of the perfect fluid black hole systems that have three black holes in the same coordinates. We are interested in two of these coordinates, the Schwarzschild coordinates and the isotropic coordinates, and will also analyze the two types of entropy compositions.

Key words: Temperature, Additive Entropy Composition, Nonadditive Entropy Composition, Perfect Fluid Black Hole
2020 Mathematics Subject Classification: 00A69, 00A79, 34A34, 83C05, 83C20, 83C57

1 Introduction

Perfect fluid black holes are used in the modeling of stars that have sizes smaller than the Schwarzschild radius, which will be transformed into black holes. In the previous research, we were interested in temperature and entropy of perfect fluid black holes in Schwarzschild and isotropic coordinates. There are static spherically symmetric perfect fluid solutions of the Einstein field equations in general relativity such as Wyman ssspf, Buchdahl ssspf, Kuchowicz ssspf, Heintzmann ssspf, Goldman ssspf, Stewart ssspf, and the BVW algorithm [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Ngampitipan et al., has studied and researched about greybody factors, and the temperature and entropy of black holes in dRGT massive gravity [13], while Sansuk has obtained about the temperature and entropy of perfect fluid black holes, and the entropy composition of black hole systems that contain two black holes [14]. The Hawking temperature of the black hole is related to the surface gravity and the important thermodynamic quantities associated with the black hole horizon, which is the entropy. Thus, temperature is correlated with entropy [15, 16]. We use the entropy composition rule from V. G. Czinner, et al[17]. In this paper, we are interested in the entropy composition of black hole systems that contain three black holes, and we will focus on perfect fluid black holes in two coordinates; Schwarzschild and isotropic coordinates. We use perfect fluid black holes classified by Sansuk [14] to calculate the entropy composition of black hole systems. We also analyze the entropy of black hole systems in terms of additive and nonadditive entropy compositions. An additive quantity is comprehensive, and comprehensive quantities can also be nonadditive, and can be used with the Rényi model in entropy of black holes [18, 19, 20].

2 Perfect fluid black hole

Perfect fluid spheres are the simplest model used in the modeling of stars like black holes. There are three properties of the perfect fluid sphere; no viscosity, no heat conductivity, and isotropy [14]. If the radius of the perfect fluid sphere is smaller than the Schwarzschild radius, it will be transformed into a perfect fluid black hole. In this paper, we are interested in perfect fluid black holes in two different coordinates; Schwarzschild and isotropic coordinates, because they are the most commonly used. We are also interested in perfect fluid spheres in two coordinates, namely, Schwarzschild and isotropic coordinates. The Schwarzschild coordinates is estimated to be about 55% and the isotropic coordinates is estimated to be about 35% of the generally used coordinates. Both these coordinates constitute the major proportion of the overall coordinates used [14, 21].

Table 1: Perfect fluid black hole in Schwarzschild coordinates [14]

Black holes	Metrics
Schwarzschild Exterior	$ds^2 = -(1-\frac{2m}{r})dt^2 + (1-\frac{2m}{r})^{-1}dr^2 + r^2 d\Omega^2$
Tolman VI	$ds^2 = -(Ar^{1-n} + Br^{1+n})^2 dt^2 + (2-n^2)dr^2 + r^2 d\Omega^2$
Kuch 68 II	$ds^2 = -(1-\frac{2m}{r})dt^2 + [(1-\frac{2m}{r})(1+C(2r-2m)^2)]^{-1}dr^2 + r^2 d\Omega^2$
M-W III	$ds^2 = -Ar(r-a)dt^2 + \frac{7/4}{1-r^2/a^2}dr^2 + r^2 d\Omega^2$

2.1 Perfect fluid black hole in Schwarzschild coordinates

We start with the spherically symmetric geometry in Schwarzschild (curvature) coordinates [14]

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2. \tag{1}$$

We calculate the radius of perfect fluid spheres in Schwarzschild coordinates by matching it with the Schwarzschild exterior black hole.
Using a perfect fluid constraint [12, 26]

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}, \tag{2}$$

we consider

$$G_{\hat{r}\hat{r}} = \frac{-2rB(r)\zeta(r) + \zeta(r) - \zeta(r)B(r)}{r^2\zeta(r)}. \tag{3}$$

Considering the Einstein field equation

$$G_{\hat{r}\hat{r}} = 8\pi G T_{\hat{r}\hat{r}}, \tag{4}$$

where $G_{\hat{r}\hat{r}}$ is the Einstein tensor, $T_{\hat{r}\hat{r}}$ is the stress energy tensor, G is the gravitational constant, and the pressure inside the perfect fluid sphere is expressed by [14, 26]

$$p = \frac{G_{\hat{r}\hat{r}}}{8\pi G} = \frac{1}{8\pi G}\frac{-2rB(r)\zeta(r) + \zeta(r) - \zeta(r)B(r)}{r^2\zeta(r)}. \tag{5}$$

So, the radius r of a perfect fluid sphere must satisfy

$$p(r) = 0. \tag{6}$$

If the radius is smaller than the Schwarzschild radius, it will be transformed into a black hole.
We will be using these perfect fluid spheres to calculate the entropy composition in additive and nonadditive entropy compositions.

Table 2: Perfect fluid black hole in isotropic coordinates [14].

Black holes	Metrics
Schwarzschild Exterior	$ds^2 = -\frac{(1-\frac{M}{2r})^2}{(1+\frac{M}{2r})^2}dt^2 + (1+\frac{M}{2r})^4\{dr^2 + r^2 d\Omega^2\}$
N-P-V Ia	$ds^2 = -(ar^{1+\frac{x}{2}} + br^{1-\frac{x}{2}})^2(Ar^{1+\frac{n}{2}} + Br^{1-\frac{n}{2}})^{-2}dt^2 +$ $(Ar^{1+\frac{n}{2}} + Br^{1-\frac{n}{2}})^{-2}\{dr^2 + r^2 d\Omega^2\}$
Burl I	$ds^2 = -A(1+r^2)^{\frac{4(a+1)}{2a^2+4a+1}}dt^2 + (1+r^2)^{\frac{4a}{2a^2+4a+1}}\{dr^2 + r^2 d\Omega^2\}$

2.2 Perfect fluid black hole in isotropic coordinates

We start with the spherically symmetric geometry in isotropic coordinates [14]

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{\zeta(r)^2 B(r)^2}\{dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\}. \tag{7}$$

We calculate the radius of perfect fluid sphere in isotropic coordinates by matching it with Schwarzschild exterior black hole. Using a perfect fluid constraint [12, 26]

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}, \tag{8}$$

we use $G_{\hat{r}\hat{r}}$ to calculate the radius of the perfect fluid sphere.
We consider

$$G_{\hat{r}\hat{r}} = (\zeta^{'})^2 B^2 - (B^{'})^2\zeta^2 + 2B^{'}B\frac{\zeta^2}{r}. \tag{9}$$

Considering the Einstein field equation

$$G_{\hat{r}\hat{r}} = 8\pi G T_{\hat{r}\hat{r}}, \tag{10}$$

the pressure inside the perfect fluid sphere is given by [14, 26]

$$p = \frac{G_{\hat{r}\hat{r}}}{8\pi G} = \frac{1}{8\pi G}(\zeta^{'})^2 B^2 - (B^{'})^2\zeta^2 + 2B^{'}B\frac{\zeta^2}{r}. \tag{11}$$

So, the radius r of a perfect fluid sphere must satisfy

$$p(r) = 0. \tag{12}$$

If the radius is smaller than the Schwarzschild radius, it will be transformed into a black hole.

We will be using these perfect fluid spheres to calculate the entropy composition in additive and nonadditive entropy compositions.

3 Temperature and entropy of perfect fluid black hole

The temperature of the Hawking radiation can be calculated in relation with the surface gravity of a black hole, which can be written in this form [13]

$$T = \frac{\kappa}{2\pi}, \tag{13}$$

where $\kappa = (\sqrt{\zeta^2(r)B(r)})'$,
for perfect fluid black holes in Schwarzschild coordinates. For isotropic coordinates, we can obtain κ in this form;

$$\kappa = (\zeta(r)^2 B(r))'. \tag{14}$$

Entropy is also one of the fundamental properties in thermodynamics, which can be expressed in the form [13]

$$S = \int \frac{1}{T} dM. \tag{15}$$

Then, we use this form of entropy to calculate the entropy composition systems of perfect fluid black holes.

3.1 Thermodynamics

The zeroth law of thermodynamics states that when two systems are each in thermal equilibrium with a third system, the first two systems are in thermal equilibrium with each other as well [22].
The first law of thermodynamics, or the law of conservation of energy states that the change in a system's internal energy is equal to the difference between the heat added to the system from its surroundings and the work done by the system on its surroundings [22].

$$dE = Tds + udQ + \Omega dJ, \tag{16}$$

for black holes,

$$dM = \frac{\kappa}{8\pi G} dA + udQ + \Omega dJ. \tag{17}$$

The second law of thermodynamics states that the heat does not flow spontaneously from a colder region to a hotter region, or, equivalently, heat at a given temperature cannot be converted entirely into work [22].

$$\triangle S \geq 0, \tag{18}$$

for black holes, the net area in any process never decreases [22],

$$\triangle A \geq 0. \tag{19}$$

For a black hole, we use the surface area to calculate the temperature. Since temperature is related with entropy, then we can use the temperature to calculate entropy.

3.2 Temperature

Black holes have energy E and entropy S, then it must also have temperature T given by [23]

$$\frac{1}{T} = \frac{dE}{dS}. \tag{20}$$

For Schwarzschild black hole, the area and the entropy scales as $S \sim M^2$. Temperature that scales as M is given by [23]

$$\frac{1}{T} = \frac{dS}{dM} \sim \frac{dM^2}{dM} \sim M. \tag{21}$$

Hawking's calculation showed that the spectrum emitted by black holes is thermal, with temperature [23]

$$T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi GM}. \tag{22}$$

After that, we will explain the black hole entropy formula using the temperature formula.

3.3 Entropy

The formula for the Hawking temperature and the first law of thermodynamics is [24]

$$dM = TdS = \frac{\kappa\hbar}{8\pi G\hbar} dA. \tag{23}$$

The entropy and the area of a black hole is given by [24]

$$S = \frac{Ac^3}{4G\hbar}. \tag{24}$$

4 Entropy composition of black hole systems

We focus on two entropy compositions of black hole systems; additive and nonadditive entropy compositions, while each black hole system has three black holes in the same coordinate. We have extended this topic from Sansuk [14], who obtained the entropy composition of black hole systems that contain only 2 black holes.

Table 3: The additive entropy composition of black hole systems in Schwarzschild coordinates.

Black hole systems	**Metrics**
Schwarzschild Exterior $\oplus$ Tolman VI $\oplus$ Kuch 68 II	$S_{STK} = 2M^2\pi + \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}} + \frac{\pi(d+\ln[M]-\ln[1+d])}{2C}$
Tolman VI $\oplus$ Kuch 68 II $\oplus$ M-W III	$S_{TKM} = \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}} + \frac{\pi(d+\ln[M]-\ln[1+d])}{2C} + \int \frac{dM}{dr}\frac{1}{T}dr$

Table 4: The additive entropy composition of black hole systems in isotropic coordinates.

Black hole systems	**Metrics**
Schwarzschild Exterior (isotropic) $\oplus$ N-P-V Ia $\oplus$ Burl I	$S_{SNB} = \frac{125M}{24\pi}\pi - \frac{4(r(4b+3ar)+3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{3a\pi\sqrt{-r^2(b+ar)^3}} + - \frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g-(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}}$

4.1 Additive Entropy Composition

The additive entropy composition is given by [17]

$$S_{123} = S_1 + S_2 + S_3. \tag{25}$$

where S_1, S_2, and S_3 are the first black hole, second black hole, and third black hole, respectively.

Tables 3 and 4 show the additive entropy composition of black hole systems in Schwarzschild and isotropic coordinates, and also that the three black holes in each system are not correlated.

4.2 Nonadditive Entropy Composition

The nonadditive entropy composition rule can be obtained using the Abe's equation [17]

$$S_{123} = S_1 + S_2 + S_3 + \lambda(S_1 + S_2 + \lambda S_1 S_2)S_3, \tag{26}$$

where $0 < \lambda < 1$.

Table 5: The nonadditive entropy composition of black hole systems in Schwarzschild coordinates.

Black hole systems	Metrics
Schwarzschild Exterior ⊕ Tolman VI ⊕ Kuch 68 II	$S_{STK} = 2M^2\pi + \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}} + \frac{\pi(d+\ln[M]-\ln[1+d])}{2C} + \lambda\left(2M^2\pi + \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}}\right)\frac{\pi(d+\ln[M]-\ln[1+d])}{2C} + \lambda^2(2M^2\pi)\left(\frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}}\right)\left(\frac{\pi(d+\ln[M]-\ln[1+d])}{2C}\right)$
Tolman VI ⊕ Kuch 68 II ⊕ M-W III	$S_{TKM} = \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}} + \frac{\pi(d+\ln[M]-\ln[1+d])}{2C} + \int \frac{dM}{dr}\frac{1}{T}dr + \lambda\left(\frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}} + \frac{\pi(d+\ln[M]-\ln[1+d])}{2C}\right)\int \frac{dM}{dr}\frac{1}{T}dr + \lambda^2\left(\frac{15\pi(A+Br^8)\tan[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}}\right)\left(\frac{\pi(d+\ln[M]-\ln[1+d])}{2C}\right)\left(\int \frac{dM}{dr}\frac{1}{T}dr\right)$

Table 6: The nonadditive entropy composition of black hole systems in isotropic coordinates.

Black hole systems	Metrics
Schwarzschild Exterior (isotropic) ⊕ N-P-V Ia ⊕ Burl I	$S_{SNB} = \frac{125M}{24\pi}\pi - \frac{4(r(4b+3ar)+3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{3a\pi\sqrt{-r^2(b+ar)^3}} - \frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g-(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}} + \left[\lambda\left(\frac{125M}{24\pi}\pi\right) - \frac{4(r(4b+3ar)+3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{3a\pi\sqrt{-r^2(b+ar)^3}} + \lambda^2\left(\frac{125M}{24\pi}\pi\right)\left(\frac{4(r(4b+3ar)+3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{3a\pi\sqrt{-r^2(b+ar)^3}}\right)\right]\frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g-(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}}$

Tables 5 and 6 show the non-additive entropy composition of black hole systems in Schwarzschild and isotropic coordinates, and that the three black holes in each system are correlated through λ.

5 Conclusion

We are interested in perfect fluid black holes that have a radius smaller than the Schwarzschild radius. We then used these perfect fluid black holes to calculate the two entropy compositions; additive and nonadditive entropy compositions of the black hole systems in two coordinates, with each system having three black holes in the same coordinates. The results show that the three black holes in the additive entropy composition of black hole systems in Schwarzschild and isotropic coordinates are not correlated, while the nonadditive entropy composition of black hole systems in Schwarzschild and isotropic coordinates are correlated through λ.

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