***East-West J. of Mathematics, Vol. 25, No. 2*** (2024) pp. 84-120
https://doi.org/10.36853/ewjm0403

# DETERMINATION OF THE FIFTH SINGER ALGEBRAIC TRANSFER IN SOME DEGREES

**Nguyễn Sum**

*Department of Mathematics and Applications, Saigon University,*
*273 An Duong Vuong, District 5, Ho Chi Minh city, Viet Nam*
*e-mail: nguyensum@sgu.edu.vn*

**Abstract**

Let $P_k$ be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$ over the prime field $\mathbb{F}_2$ with two elements and the degree of each variable $x_i$ being 1, and let $GL_k$ be the general linear group over $\mathbb{F}_2$ which acts on $P_k$ as the usual manner. The algebra $P_k$ is considered as a module over the mod-2 Steenrod algebra $\mathcal{A}$. In 1989, Singer [19] defined the $k$-th homological algebraic transfer, which is a homomorphism

$$\varphi_k : \mathrm{Tor}^{\mathcal{A}}_{k,k+d}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)^{GL_k}_d$$

from the homological group of the Steenrod algebra $\mathrm{Tor}^{\mathcal{A}}_{k,k+d}(\mathbb{F}_2, \mathbb{F}_2)$ to the subspace $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)^{GL_k}_d$ of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ consisting of all the $GL_k$-invariant classes of degree $d$.

In this paper, by using the results of the Peterson hit problem we present the proof of the fact that the Singer algebraic transfer of rank five is an isomorphism in the internal degrees $d = 20$ and $d = 30$. Our result refutes the proof for the case of $d = 20$ in Phúc [15].

# 1 Introduction

Let $P_k$ be the polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$ over the field $\mathbb{F}_2$ with two elements, in $k$ variables $x_1, x_2, \ldots, x_k$, each variable of degree 1. It is well-

---

**Key words:** Steenrod squares, Peterson hit problem, polynomial algebra, modular representation.
2010 AMS Classification: Primary 55S10; 55S05; 55T15.

known that this algebra is the mod-2 cohomology of an elementary abelian 2-group $V_k$ of rank $k$. Hence, $P_k$ is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on $P_k$ can be explicitly determined by the elementary properties of the Steenrod operations $Sq^j$ and subject to the Cartan formula $Sq^d(gh) = \sum_{j=0}^{d} Sq^j(g)Sq^{d-j}(h)$, for $g,\ h \in P_k$ (see Steenrod and Epstein [21]).

The *Peterson hit problem* asks for a minimal generating set for $P_k$ regarded as a module over the mod-2 Steenrod algebra. Equivalently, this problem is to find a vector space basis for $QP_k := \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ in each degree $d$. Such a basis can be represented by a list of monomials of degree $d$. This problem is completely computed for $k \leqslant 4$, unknown in general.

Denote by $GL_k$ the general linear group over the field $\mathbb{F}_2$. This group acts naturally on $P_k$ by matrix substitution. The actions of $\mathcal{A}$ and $GL_k$ on $P_k$ commute with each other, so there is an action of $GL_k$ on $QP_k$.

Denote $(P_k)_d$ the vector subspace of $P_k$ consisting of all the homogeneous polynomials of degree $d$ in $P_k$ and by $(QP_k)_d$ the vector subspace of $QP_k$ consisting of all the classes represented by the elements in $(P_k)_d$. In [19], Singer defined the homological algebraic transfer, which is a homomorphism

$$\varphi_k : \mathrm{Tor}^{\mathcal{A}}_{k,k+d}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_k)_d^{GL_k}$$

from the homology of the Steenrod algebra $\mathrm{Tor}^{\mathcal{A}}_{k,k+d}(\mathbb{F}_2, \mathbb{F}_2)$ to the subspace of $(QP_k)_d$ consisting of all the $GL_k$-invariant classes of degree $d$. It can be a useful tool in describing the homology groups of the Steenrod algebra in terms of $GL_k$-invariants in $QP_k$. By passing to the dual we get the cohomological algebraic transfer

$$\varphi_k^* : \mathbb{F}_2 \otimes_{GL_k} \mathcal{P}((P_k)_d^*) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{k,k+d}(\mathbb{F}_2, \mathbb{F}_2),$$

where $(P_k)^*$ is the dual of $P_k$ and $\mathcal{P}((P_k)^*)$ is the primitive subspace consisting of all elements in $(P_k)^*$ that are annihilated by every positive degree Steenrod squares.

The algebraic transfer was studied by Boardman [1], Bruner, Hà and Hưng [2], Chơn and Hà [4, 5, 6], Hưng [8], Hà [7], Hưng and Quỳnh [9], Nam [13], Minami [12], Phúc [14, 15], Quỳnh [18] the present author [23] and others.

Singer proved in [19] that $\varphi_k$ is an isomorphism for $k = 1, 2$. Boardman also showed in [1] that $\varphi_3$ is also an isomorphism. However, $\varphi_k$ is not a monomorphism in infinitely many degrees for any $k \geqslant 4$ (see Singer [19], Bruner, Hà and Hưng [2], Hưng [8].) Singer gave the following conjecture.

**Conjecture 1.1** (Singer [19]). *The algebraic transfer $\varphi_k$ is an epimorphism for any $k \geqslant 0$.*

This conjecture is true for $k \leqslant 3$. Phúc stated in [14] that the conjecture is also true for $k = 4$ but the computations are incomplete and the proof is not explicit. The results in [14] is only a description of the dimension for $QP_4$ in each degree when Singer conjecture is true for $k = 4$. It is not a proof for this conjecture. Recently, we have proved in [28] that this conjecture is not true for $k = 5$ and the internal degree $d = 108$. This result refused a one of Phúc in [15].

In this paper, by using some results of the Peterson hit problem we prove that the Singer algebraic transfer of rank 5 is an isomorphism in the internal degrees $d = 20$ and $d = 30$.

Consider the case $d = 20$, we have the following.

**Theorem 1.2.** *The dimension of the $\mathbb{F}_2$-vector spaces $(QP_5)_{20}^{GL_5}$ is one with a basis element presented by the following polynomial:*

$$\begin{aligned}
p = {} & x_1x_2x_3^3x_4^3x_5^{12} + x_1x_2x_3^3x_4^5x_5^{10} + x_1x_2x_3^3x_4^6x_5^9 + x_1x_2x_3^3x_4^{12}x_5^3 \\
& + x_1x_2x_3^6x_4^3x_5^9 + x_1x_2x_3^6x_4^9x_5^3 + x_1x_2^3x_3x_4^3x_5^{12} + x_1x_2^3x_3x_4^5x_5^{10} \\
& + x_1x_2^3x_3x_4^6x_5^9 + x_1x_2^3x_3x_4^{12}x_5^3 + x_1x_2^3x_3^3x_4^4x_5^9 + x_1x_2^3x_3^3x_4^5x_5^8 \\
& + x_1x_2^3x_3^5x_4^3x_5^8 + x_1x_2^3x_3^5x_4^8x_5^3 + x_1x_2^6x_3x_4^3x_5^9 + x_1x_2^6x_3x_4^9x_5^3 \\
& + x_1^3x_2x_3^4x_4^3x_5^9 + x_1^3x_2x_3^4x_4^9x_5^3 + x_1^3x_2x_3^5x_4^3x_5^8 + x_1^3x_2x_3^5x_4^8x_5^3 \\
& + x_1^3x_2^4x_3x_4^3x_5^9 + x_1^3x_2^4x_3x_4^9x_5^3 + x_1^3x_2^5x_3x_4^3x_5^8 + x_1^3x_2^5x_3x_4^8x_5^3 \\
& + x_1x_2^3x_3^5x_4^5x_5^6 + x_1x_2^3x_3^5x_4^6x_5^5 + x_1x_2^3x_3^6x_4^5x_5^5 + x_1x_2^6x_3^3x_4^5x_5^5 \\
& + x_1^3x_2x_3^5x_4^5x_5^6 + x_1^3x_2x_3^5x_4^6x_5^5 + x_1^3x_2^5x_3x_4^5x_5^6 + x_1^3x_2^5x_3x_4^6x_5^5 \\
& + x_1^3x_2^3x_3^4x_4^5x_5^5 + x_1^3x_2^3x_3^5x_4^4x_5^5 + x_1^3x_2^3x_3^5x_4^5x_5^4 + x_1^3x_2^4x_3^3x_4^5x_5^5 \\
& + x_1^3x_2^5x_3^3x_4^4x_5^5 + x_1^3x_2^5x_3^3x_4^5x_5^4 + x_1^3x_2^5x_3^5x_4^3x_5^4 + x_1^3x_2^5x_3^5x_4^4x_5^3.
\end{aligned}$$

In [15], Phúc also stated that $\dim(QP_5)_{20}^{GL_5} = 1$ but he cannot provide any basis element. Furthermore, the proof of this result in [15] is false.

From Lin [11] and Chen [3], we have $\mathrm{Ext}_{\mathcal{A}}^{5,25}(\mathbb{F}_2, \mathbb{F}_2) = \langle h_0g_1 = h_2e_0 \rangle$, where where $e_0 \in \mathrm{Ext}_{\mathcal{A}}^{4,21}(\mathbb{F}_2, \mathbb{F}_2)$, $g_1 \in \mathrm{Ext}_{\mathcal{A}}^{4,24}(\mathbb{F}_2, \mathbb{F}_2)$ and $h_i$ is the Adams element in $\mathrm{Ext}_{\mathcal{A}}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ for $i \geqslant 0$. Combining the results of Hà [7] and Singer [19], we have $\varphi_5((h_2e_0)^*) = [p]$. Hence, Theorem 1.2 implies that

$$\varphi_5 : \mathrm{Tor}_{5,25}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_5)_{20}^{GL_5}$$

is an isomorphism. So, we get the following.

**Corollary 1.3.** *Singer's conjecture is true for $k = 5$ and the internal degree* 20.

This result is also stated in [15, Corollary 1.7], but it is derived from a result with a false proof.

Consider the case $d = 30$. By using the explicit results for a basis of $(QP_5)_{30}$ in our work [26], we prove the following.

**Theorem 1.4.** *The dimension of the $\mathbb{F}_2$-vector spaces $(QP_5)_{30}^{GL_5}$ is one with a basis element presented by the following polynomial:*

$$\begin{aligned}
q = {} & x_4^{15}x_5^{15} + x_3^{15}x_5^{15} + x_3^{15}x_4^{15} + x_2^{15}x_5^{15} + x_2^{15}x_4^{15} + x_2^{15}x_3^{15} + x_1^{15}x_5^{15} + x_1^{15}x_4^{15} \\
& + x_1^{15}x_3^{15} + x_1^{15}x_2^{15} + x_3x_4^{14}x_5^{15} + x_3x_4^{15}x_5^{14} + x_3^{15}x_4x_5^{14} + x_2x_4^{14}x_5^{15} \\
& + x_2x_4^{15}x_5^{14} + x_2x_3^{14}x_5^{15} + x_2x_3^{14}x_4^{15} + x_2x_3^{15}x_5^{14} + x_2x_3^{15}x_4^{14} + x_2^{15}x_4x_5^{14} \\
& + x_2^{15}x_3x_5^{14} + x_2^{15}x_3x_4^{14} + x_1x_4^{14}x_5^{15} + x_1x_4^{15}x_5^{14} + x_1x_3^{14}x_5^{15} + x_1x_3^{14}x_4^{15} \\
& + x_1x_3^{15}x_5^{14} + x_1x_3^{15}x_4^{14} + x_1x_2^{14}x_5^{15} + x_1x_2^{14}x_4^{15} + x_1x_2^{14}x_3^{15} + x_1x_2^{15}x_5^{14} \\
& + x_1x_2^{15}x_4^{14} + x_1x_2^{15}x_3^{14} + x_1^{15}x_4x_5^{14} + x_1^{15}x_3x_5^{14} + x_1^{15}x_3x_4^{14} + x_1^{15}x_2x_5^{14} \\
& + x_1^{15}x_2x_4^{14} + x_1^{15}x_2x_3^{14} + x_3^{3}x_4^{13}x_5^{14} + x_2^{3}x_4^{13}x_5^{14} + x_2^{3}x_3^{13}x_5^{14} + x_2^{3}x_3^{13}x_4^{14} \\
& + x_1^{3}x_4^{13}x_5^{14} + x_1^{3}x_3^{13}x_5^{14} + x_1^{3}x_3^{13}x_4^{14} + x_1^{3}x_2^{13}x_5^{14} + x_1^{3}x_2^{13}x_4^{14} + x_1^{3}x_2^{13}x_3^{14} \\
& + x_2x_3x_4^{14}x_5^{14} + x_2x_3^{14}x_4x_5^{14} + x_2^{3}x_3^{5}x_4^{10}x_5^{12} + x_1x_3x_4^{14}x_5^{14} + x_1x_3^{14}x_4x_5^{14} \\
& + x_1x_2x_4^{14}x_5^{14} + x_1x_2x_3^{14}x_5^{14} + x_1x_2x_3^{14}x_4^{14} + x_1x_2^{14}x_4x_5^{14} + x_1x_2^{14}x_3x_5^{14} \\
& + x_1x_2^{14}x_3x_4^{14} + x_1^{3}x_3^{5}x_4^{10}x_5^{12} + x_1^{3}x_2^{5}x_4^{10}x_5^{12} + x_1^{3}x_2^{5}x_3^{10}x_5^{12} + x_1^{3}x_2^{5}x_3^{10}x_4^{12} \\
& + x_2x_3^{2}x_4^{13}x_5^{14} + x_2x_3^{3}x_4^{12}x_5^{14} + x_2x_3^{3}x_4^{14}x_5^{12} + x_2^{3}x_3x_4^{12}x_5^{14} + x_2^{3}x_3x_4^{14}x_5^{12} \\
& + x_2^{3}x_3^{13}x_4^{2}x_5^{12} + x_1x_3^{2}x_4^{13}x_5^{14} + x_1x_3^{3}x_4^{12}x_5^{14} + x_1x_3^{3}x_4^{14}x_5^{12} + x_1x_2^{2}x_4^{13}x_5^{14} \\
& + x_1x_2^{2}x_3^{13}x_5^{14} + x_1x_2^{2}x_3^{13}x_4^{14} + x_1x_2^{3}x_4^{12}x_5^{14} + x_1x_2^{3}x_4^{14}x_5^{12} + x_1x_2^{3}x_3^{12}x_5^{14} \\
& + x_1x_2^{3}x_3^{12}x_4^{14} + x_1x_2^{3}x_3^{14}x_5^{12} + x_1x_2^{3}x_3^{14}x_4^{12} + x_1^{3}x_3x_4^{12}x_5^{14} + x_1^{3}x_3x_4^{14}x_5^{12} \\
& + x_1^{3}x_3^{13}x_4^{2}x_5^{12} + x_1^{3}x_2x_4^{12}x_5^{14} + x_1^{3}x_2x_4^{14}x_5^{12} + x_1^{3}x_2x_3^{12}x_5^{14} + x_1^{3}x_2x_3^{12}x_4^{14} \\
& + x_1^{3}x_2x_3^{14}x_5^{12} + x_1^{3}x_2x_3^{14}x_4^{12} + x_1^{3}x_2^{13}x_4^{2}x_5^{12} + x_1^{3}x_2^{13}x_3^{2}x_5^{12} + x_1^{3}x_2^{13}x_3^{2}x_4^{12} \\
& + x_2x_3^{2}x_4^{12}x_5^{15} + x_2x_3^{2}x_4^{15}x_5^{12} + x_2x_3^{15}x_4^{2}x_5^{12} + x_2^{15}x_3x_4^{2}x_5^{12} + x_1x_3^{2}x_4^{12}x_5^{15} \\
& + x_1x_3^{2}x_4^{15}x_5^{12} + x_1x_3^{15}x_4^{2}x_5^{12} + x_1x_2^{2}x_4^{12}x_5^{15} + x_1x_2^{2}x_4^{15}x_5^{12} + x_1x_2^{2}x_3^{12}x_5^{15} \\
& + x_1x_2^{2}x_3^{12}x_4^{15} + x_1x_2^{2}x_3^{15}x_5^{12} + x_1x_2^{2}x_3^{15}x_4^{12} + x_1x_2^{15}x_4^{2}x_5^{12} + x_1x_2^{15}x_3^{2}x_5^{12} \\
& + x_1x_2^{15}x_3^{2}x_4^{12} + x_1^{15}x_3x_4^{2}x_5^{12} + x_1^{15}x_2x_4^{2}x_5^{12} + x_1^{15}x_2x_3^{2}x_5^{12} + x_1^{15}x_2x_3^{2}x_4^{12} \\
& + x_1x_2x_3^{2}x_4^{14}x_5^{12} + x_1x_2x_3^{6}x_4^{10}x_5^{12} + x_1x_2x_3^{14}x_4^{2}x_5^{12} + x_1x_2^{2}x_3^{3}x_4^{12}x_5^{12} \\
& + x_1x_2^{2}x_3^{4}x_4^{8}x_5^{15} + x_1x_2^{2}x_3^{4}x_4^{9}x_5^{14} + x_1x_2^{2}x_3^{4}x_4^{15}x_5^{8} + x_1x_2^{2}x_3^{5}x_4^{8}x_5^{14} \\
& + x_1x_2^{2}x_3^{5}x_4^{10}x_5^{12} + x_1x_2^{2}x_3^{5}x_4^{14}x_5^{8} + x_1x_2^{2}x_3^{12}x_4x_5^{14} + x_1x_2^{2}x_3^{13}x_4^{2}x_5^{12} \\
& + x_1x_2^{2}x_3^{15}x_4^{4}x_5^{8} + x_1x_2^{3}x_3^{4}x_4^{8}x_5^{14} + x_1x_2^{3}x_3^{4}x_4^{14}x_5^{8} + x_1x_2^{3}x_3^{6}x_4^{12}x_5^{8} \\
& + x_1x_2^{3}x_3^{12}x_4^{2}x_5^{12} + x_1x_2^{3}x_3^{14}x_4^{4}x_5^{8} + x_1x_2^{14}x_3x_4^{2}x_5^{12} + x_1x_2^{15}x_3^{2}x_4^{4}x_5^{8} \\
& + x_1^{3}x_2x_3^{4}x_4^{8}x_5^{14} + x_1^{3}x_2x_3^{4}x_4^{14}x_5^{8} + x_1^{3}x_2x_3^{6}x_4^{12}x_5^{8} + x_1^{3}x_2x_3^{12}x_4^{2}x_5^{12} \\
& + x_1^{3}x_2x_3^{14}x_4^{4}x_5^{8} + x_1^{3}x_2^{5}x_3^{2}x_4^{12}x_5^{8} + x_1^{3}x_2^{5}x_3^{10}x_4^{4}x_5^{8} + x_1^{3}x_2^{13}x_3^{2}x_4^{4}x_5^{8} \\
& + x_1^{15}x_2x_3^{2}x_4^{4}x_5^{8} + x_1^{3}x_2^{5}x_3^{6}x_4^{6}x_5^{10}.
\end{aligned}$$

By using a computer program with an algorithm of the mathematics system

SAGEMATH, Tín proved in [30, Page 1923] that $\dim(QP_5)_{30} = 840$. This dimensional result is presented again in Phúc [16, Appendix] but there are no any details of the computations for a basis of this space. So, it is not a solution of the hit problem because dimensions are not the object of study of the hit problem. When a basis cannot be found, the dimensional results are meaningless. In [26] we present a solution of the hit problem by explicitly computing a basis of $(QP_5)_{30}$. Our results in this paper are completely new and confirm the accuracy of dimensional result for $(QP_5)_{30}$ in Tín's work [30].

Phúc [16] also stated that by using this dimensional result he proved that $\dim(QP_5)_{30}^{GL_5} = 1$ but this result must be proven by using a basis, it cannot be proven by using a dimensional result. So, his statement is only a prediction.

From Lin [11] and Chen [3], we have $\mathrm{Ext}_{\mathcal{A}}^{5,35}(\mathbb{F}_2,\mathbb{F}_2) = \langle h_0^3h_4^2\rangle$. Hence, using a result of Singer [19], we have $\varphi_5((h_0^3h_4^2)^*) = [q]$. Hence, Theorem 1.4 implies that

$$\varphi_5 : \mathrm{Tor}^{\mathcal{A}}_{5,35}(\mathbb{F}_2,\mathbb{F}_2) \longrightarrow (QP_5)_{30}^{GL_5}$$

is an isomorphism. So, we get the following.

**Corollary 1.5.** *Singer's conjecture is true for $k = 5$ and the internal degree $d = 30$.*

This result is also stated in [16] but it is based on an unconfirmed result.

The paper is organized as follows. In Section 2, we recall some notions and results on the admissible monomials in $P_k$, criterion of Singer on the hit monomials and some needed notations. In Section 3, we present a structure of the space $(QP_5)_{20}$ and prove Theorem 1.2. Theorem 1.4 is proved in Section 4. Finally, in Section 5, we present some needed data for the proofs of the main results.

# 2 Preliminaries on the hit problem

In this the section we present some needed notions and results for studying the hit problem such as the weight vector of a monomial, admissible monomial, criterion of Singer for hit monomial from the works of Kameko [10], Singer [20] and our work [22, 24].

## 2.1 The admissible monomials and Singer's criterion on hit monomials

**Definition 2.1.1.** Let $u = x_1^{c_1}x_2^{c_2}\ldots x_k^{c_k}$ be a monomial in $P_k$. Denote $\nu_j(w) = c_j, 1 \leqslant j \leqslant k$. We define

$$\omega(u) = (\omega_1(u),\ldots,\omega_t(u),\ldots),\quad \sigma(u) = (\nu_1(u),\nu_2(u),\ldots,\nu_k(u)),$$

where $\omega_t(u) = \sum_{1\leqslant j\leqslant k} \alpha_{t-1}(\nu_j(w)),\ t \geqslant 1$. The sequences $\sigma(u)$ is called the exponent vector and $\omega(u)$ is called the weight vector of $u$.

A a sequence of non-negative integers $\omega = (\omega_1, \ldots, \omega_s, \ldots)$ is called a weight vector if $\omega_s = 0$ for $s \gg 0$. We define $\deg \omega = \sum_{s>0} 2^{s-1}\omega_s$, the length $\ell(\omega) = \max\{t : \omega_t > 0\}$. We denote $\omega = (\omega_1, \ldots, \omega_r)$ if $\omega_s = 0$ for $s > r$.

For weight vectors $\omega = (\omega_1, \ldots, \omega_t, \ldots)$ and $\xi = (\xi_1, \ldots, \xi_t, \ldots)$, we define the concatenation of weight vectors $\omega|\xi = (\omega_1, \ldots, \omega_r, \xi_1, \xi_2, \ldots)$ if $\ell(\omega) = r$ and $(b)|^r = (b)|(b)|\ldots|(b)$, ($r$ times of $(b)$'s), where $b$, $r$ are positive integers.

We order the sets of weight vectors and exponent vectors by the left lexicographical order.

For any weight vector, we denote

$$P_k(\omega) = \langle u \in P_k : \deg u = \deg \omega,\ \text{and}\ \omega(u) \leqslant \omega\rangle,$$
$$P_k^-(\omega) = \langle u \in P_k(\omega) : \omega(u) < \omega\rangle.$$

**Definition 2.1.2.** Suppose $\omega$ is a weight vector and $f$, $f_1$, $f_2$ are polynomials of the same degree in $P_k$. We define

i) $f_1 \equiv f_2$ if $f_1 + f_2 \in \mathcal{A}^+P_k$ and $f$ is called hit if $f \equiv 0$.

ii) $f_1 \equiv_\omega f_2$ if $f_1 + f_2 \in \mathcal{A}^+P_k + P_k^-(\omega)$.

It is easy to see that $\equiv$ and $\equiv_\omega$ are equivalence relations. We set

$$QP_k(\omega) = P_k(\omega)/((P_k^-(\omega) + \mathcal{A}^+P_k \cap P_k(\omega))).$$

**Proposition 2.1.3** ([25])**.** *For a weight vector $\omega$, the vector space $QP_k(\omega)$ is an $GL_k$-module.*

**Definition 2.1.4.** Let $u$, $v$ be monomials in $P_k$ with $\deg u = \deg v$. We say that $u < v$ if one of the following conditions satisfies:

i) $\omega(u) < \omega(v)$;

ii) $\omega(u) = \omega(v)$ and $\sigma(u) < \sigma(v)$.

**Definition 2.1.5.** A monomial $w$ in $P_k$ is said to be inadmissible if there are monomials $w_1, w_2, \ldots, w_t$ such that $w_s < w$ for $s = 1, 2, \ldots, t$ and $w + \sum_{s=1}^t w_s \in \mathcal{A}^+P_k$. A monomial $w$ is called admissible if it is not inadmissible.

It is clear that, the set of all admissible monomials of degree $d$ in $P_k$ is a minimal set of generators for $\mathcal{A}$-module $P_k$ in degree $d$.

**Definition 2.1.6.** If $z$ is a monomial in $P_k$ such that $\nu_j(z) = 2^{s_t} - 1$ with $s_t$ a non-negative integer for $t = 1, 2, \ldots, k$, then $z$ is called a spike. If $s_1 > s_2 > \ldots > s_{u-1} \geqslant s_u > 0$ and $s_t = 0$ for $t > u$, then it is called the minimal spike.

**Theorem 2.1.7** (Singer [20])**.** *Suppose $y$ is a monomial of degree $d$ in $P_k$ such that $\mu(d) \leqslant k$ and $z$ is the minimal spike of degree $d$. If $\omega(y) < \omega(z)$, then $y$ is hit.*

### 2.2 Some other tools and notations

We set $P_k^+ = \langle\{y \in P_k : \nu_s(y) > 0, \text{ for all } s\}\rangle$ and $P_k^0 = \langle\{y \in P_k : \nu_s(y) = 0 \text{ for some } s\}\rangle$. The spaces $P_k^0$ and $P_k^+$ are the $\mathcal{A}$-submodules of $P_k$. We denote $QP_k^0 = P_k^0/\mathcal{A}^+P_k^0$ and $QP_k^+ = P_k^+/\mathcal{A}^+P_k^+$. Then we have $QP_k = QP_k^0 \oplus QP_k^+$.

Set $\mathcal{N}_k = \{(i;I) : I = (s_1, s_2, \ldots, s_t), 1 \leqslant i < s_1 < \ldots < s_t \leqslant k,\ 0 \leqslant t < k\}$. For $(i;I) \in \mathcal{N}_k$, we define the $\mathcal{A}$-homomorphism of algebras $p_{(i;I)} : P_k \to P_{k-1}$ by setting

$$p_{(i;I)}(x_s) = \begin{cases} x_s, & \text{if } 1 \leqslant s < i, \\ \sum_{v \in I} x_{v-1}, & \text{if } s = i, \\ x_{s-1}, & \text{if } i < s \leqslant k. \end{cases} \tag{2.1}$$

**Lemma 2.2.1** ([17])**.** *If $y$ is a monomial in $P_k$, then $p_{(i;I)}(y) \in P_{k-1}(\omega(y))$.*

From Lemma 2.2.1, we can see that if $\omega$ is a weight vector and $y \in P_k(\omega)$, then $p_{(i;I)}(y) \in P_{k-1}(\omega)$. Hence, we have the homomorphisms

$$p_{(i;I)}^{(n)} : (QP_k)_n \longrightarrow (QP_{k-1})_n, \quad p_{(i;I)}^{(\omega)} : QP_k(\omega) \longrightarrow QP_{k-1}(\omega),$$

where $n = \deg \omega$. We set

$$(\widetilde{SF}_k)_n = \bigcap_{(i;I)\in\mathcal{N}_k} \operatorname{Ker}(p_{(i;I)}^{(n)}), \quad \widetilde{SF}_k(\omega) = \bigcap_{(i;I)\in\mathcal{N}_k} \operatorname{Ker}(p_{(i;I)}^{(\omega)}),$$
$$\widetilde{QP}_k(\omega) = QP_k(\omega)/\widetilde{SF}_k(\omega),\ (\widetilde{QP}_k)_n = (QP_k)_d/(\widetilde{SF}_k)_n.$$

We note that $(\widetilde{SF}_k)_n$ and $\widetilde{SF}_k(\omega)$ are respectively the subspaces of $(QP_k^+)_n$ and $QP_k^+(\omega)$.

**Notation 2.2.2.** We denote $[g]$ the class in $QP_k$ represented by $g \in P_k$. If $g \in P_k(\omega)$, we denote $[g]_\omega$ the class in $QP_k(\omega)$ represented by $g$. If $R$ is a subset of $P_k$, then we denote $[R] = \{[g] : g \in R\}$. If $R \subset P_k(\omega)$, then we set $[R]_\omega = \{[g]_\omega : g \in L\}$. We denote $|R|$ the cardinal of a set $R$.

Denote by $B_k(n)$ the set of all admissible monomials of degree $n$ in $P_k$ for any nonnegative integer $n$. Set $B_k^+(n) = B_k(n) \cap P_k^+,\ B_k^0(n) = B_k(n) \cap P_k^0$. If $\omega$ is a weight vector of degree $n$, we denote $B_k(\omega) = B_k(n) \cap P_k(\omega),\ B_k^+(\omega) = B_k^+(n) \cap P_k(\omega),\ QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$.

For any subgroup $G \subset GL_k$ and $S \subset P_k(\omega)$, we denote

$$[G(S)]_\omega = \langle [gs]_\omega : g \in G, s \in S\rangle \subset QP_k(\omega).$$

Obviously, $[G(S)]_\omega$ is an $G$-submodule of $QP_k(\omega)$. If $\omega$ is a minimal weight vector, then we denote $[G(S)]_\omega = [G(S)] \subset QP_k$.

If $\mathbb{J}$ is an index set and $\gamma_j \in \mathbb{F}_2$ with $j \in \mathbb{J}$, then we denote $\gamma_{\mathbb{J}} = \sum_{j\in\mathbb{J}} \gamma_j$.

For a sequence $T = (t_1, t_2, \ldots, t_r)$, $1 \leqslant t_1 < \ldots < t_r \leqslant k$, we define a monomorphism $\theta_T : P_r \to P_k$ of algebras by setting

$$\theta_T(x_i) = x_{t_i} \quad \text{for} \quad 1 \leqslant i \leqslant r. \tag{2.2}$$

Obviously, $\theta_T$ is an $\mathcal{A}$-homomorphism. If $\omega$ is a weight vector of degree $n$, then

$$Q\theta_T(P_r^+)(\omega) \cong QP_r^+(\omega) \text{ and } (Q\theta_T(P_r^+))_n \cong (QP_r^+)_n$$

for $1 \leqslant r \leqslant k$. Here, $Q\theta_T(P_r^+) = \theta_T(P_r^+)/\mathcal{A}^+\theta_T(P_r^+)$. Then we have

$$B_k(\omega) = \bigcup_{\substack{\mu(n)\leqslant r\leqslant k,\\ \ell(T)=r}} \theta_T(B_r^+(\omega)),\ B_k(n) = \bigcup_{\substack{\mu(n)\leqslant r\leqslant k,\\ \ell(T)=r}} \theta_T(B_r^+(n)). \tag{2.3}$$

# 3 A structure of $(QP_5)_{20}$ and a proof of Theorem 1.2

By using a computer program with an algorithm of the mathematics system SAGEMATH, Tín [29] proved that $\dim(QP_5)_{20} = 641$. This result is also repeated in Phúc [15] but the computation for a basis of $(QP_5)_{20}$ is incomplete and the proof of this result in [15] is false.

Firstly, we recall the results in our work [27] on the admissible monomials of degree 20 in $P_5$. Recall from [27] and [15] that if $x$ is an admissible monomial of degree 20 in $P_5$, then

$$\omega(x) = (4)|(2)|(1)|^2 \text{ or } \omega(x) = (4)|(2)|(3) \text{ or } \omega(x) = (4)|^2|(2).$$

From this result we get

$$(QP_5)_{20} \cong QP_5((4)|(2)|(1)|^2) \bigoplus QP_5((4)|(2)|(3)) \bigoplus QP_5((4)|^2|(2)).$$

In [15], the author says that

$$\begin{aligned}&\dim QP_5((4)|(2)|(1)|^2) = 450,\\ &\dim QP_5((4)|(2)|(3)) = 70,\\ &\dim QP_5((4)|^2|(2)) = 121.\end{aligned}$$

However, the detailed computation is only presented for the space $QP_5((4)|(2)|(3))$ but the proof of it is false.

The set of all admissible monomials of the degree 20 in $P_5$ is explicitly presented in Subsection 5.1.

Consider the spaces $(\widetilde{SF}_k)_n$ and $\widetilde{SF}_k(\omega)$ as defined in Section 2.

### 3.1 Computation of $(\widetilde{SF}_5)_{20}$

We determined $\widetilde{SF}_5((4)|(2)|(1)|^2)$. The following is announced [27] with no the proof.

**Proposition 3.1.1.** *We have*

$$\widetilde{SF}_5((4)|(2)|(1)|^2) = \langle [g_u] : 1 \leqslant u \leqslant 10\rangle,$$

*where the polynomials $g_u$ are explicitly determined in Subsection 5.2.*

*Proof.* By a direct computation, it is not difficult to verify that $p_{(i;I)}(g_u) \equiv 0$ for all $(i;I) \in \mathcal{N}_5$. Hence, $[g_u] \in \widetilde{SF}_5((4)|(2)|(1)|^2)$ for $1 \leqslant u \leqslant 10$. The set $\{[a_t] : 1 \leqslant t \leqslant 225\}$ is a basis of $QP_5^+((4)|(2)|(1)|^2)$ (see Subsection 5.1), hence if $[\theta] \in \widetilde{SF}_5((4)|(2)|(1)|^2) \subset QP_5^+((4)|(2)|(1)|^2)$, then we have $\theta \equiv \sum_{1\leqslant t\leqslant 225} \bar\gamma_t a_t$, where $\bar\gamma_t \in \mathbb{F}_2$. The leading monomial of $g_u$ is $a_{(u+215)}$, $1 \leqslant u \leqslant 10$. Hence, we get

$$\widetilde{\theta} := \theta + \sum_{1\leqslant u\leqslant 10} \bar\gamma_{(u+215)} g_u \equiv \sum_{1\leqslant t\leqslant 215} \gamma_t a_t, \tag{3.1}$$

where $\gamma_t \in \mathbb{F}_2$, $1 \leqslant t \leqslant 215$, and $p_{(i;I)}(\widetilde{\theta}) \equiv 0$ for all $(i;I) \in \mathcal{N}_5$.

Let $u_s$, $1 \leqslant s \leqslant 45$, be as in Subsection 5.1. Apply $p_{(1;j)}$, $2 \leqslant j \leqslant 5$, to (3.1) to obtain

$$\begin{aligned}
p_{(1;2)}(\widetilde{\theta}) \equiv{}& \gamma_{10}v_{16} + \gamma_{46}v_{17} + \gamma_{47}v_{18} + \gamma_{48}v_{19} + \gamma_{49}v_{20} + \gamma_{11}v_{21} + \gamma_{50}v_{22}\\
&+ \gamma_{51}v_{23} + \gamma_{52}v_{24} + \gamma_{53}v_{25} + \gamma_{54}v_{26} + \gamma_{55}v_{27} + \gamma_{56}v_{28} + \gamma_{57}v_{29}\\
&+ \gamma_{58}v_{30} + \gamma_{59}v_{31} + \gamma_{60}v_{32} + \gamma_{12}v_{33} + \gamma_{\{97,169\}}v_{34} + \gamma_{\{98,170\}}v_{35}\\
&+ \gamma_{99}v_{36} + \gamma_{\{100,171\}}v_{37} + \gamma_{\{101,172\}}v_{38} + \gamma_{\{102,173\}}v_{39} + \gamma_{103}v_{40}\\
&+ \gamma_{104}v_{41} + \gamma_{\{105,174\}}v_{42} + \gamma_{121}v_{43} + \gamma_{122}v_{44} + \gamma_{123}v_{45} \equiv 0,\\
p_{(1;3)}(\widetilde{\theta}) \equiv{}& \gamma_{5}v_{4} + \gamma_{20}v_{5} + \gamma_{21}v_{6} + \gamma_{22}v_{7} + \gamma_{23}v_{8} + \gamma_{6}v_{9} + \gamma_{\{34,141\}}v_{10}\\
&+ \gamma_{\{35,142\}}v_{11} + \gamma_{\{36,143\}}v_{12} + \gamma_{\{37,144\}}v_{13} + \gamma_{\{44,155,206\}}v_{14}\\
&+ \gamma_{\{45,156,207\}}v_{15} + \gamma_{71}v_{22} + \gamma_{72}v_{23} + \gamma_{73}v_{24} + \gamma_{78}v_{25} + \gamma_{79}v_{26}\\
&+ \gamma_{80}v_{27} + \gamma_{81}v_{28} + \gamma_{\{88,164,185\}}v_{29} + \gamma_{\{89,165,186\}}v_{30} + \gamma_{\{92,185\}}v_{31}\\
&+ \gamma_{\{93,186\}}v_{32} + \gamma_{\{96,168,213\}}v_{33} + \gamma_{112}v_{38} + \gamma_{113}v_{39} + \gamma_{116}v_{40}\\
&+ \gamma_{117}v_{41} + \gamma_{\{120,192\}}v_{42} + \gamma_{9}v_{45} \equiv 0,\\
p_{(1;4)}(\widetilde{\theta}) \equiv{}& \gamma_{2}v_{1} + \gamma_{\{17,127\}}v_{2} + \gamma_{\{19,131,199\}}v_{3} + \gamma_{25}v_{5} + \gamma_{27}v_{6} + \gamma_{\{29,138,148\}}v_{7}\\
&+ \gamma_{\{31,148\}}v_{8} + \gamma_{\{33,140,205\}}v_{9} + \gamma_{39}v_{11} + \gamma_{41}v_{12} + \gamma_{\{43,154\}}v_{13}\\
&+ \gamma_{4}v_{15} + \gamma_{62}v_{17} + \gamma_{64}v_{18} + \gamma_{\{66,161,178\}}v_{19} + \gamma_{\{68,178\}}v_{20}\\
&+ \gamma_{\{70,163,212\}}v_{21} + \gamma_{75}v_{23} + \gamma_{\{77,184\}}v_{24} + \gamma_{83}v_{26} + \gamma_{85}v_{27}\\
&+ \gamma_{\{87,167,184\}}v_{28} + \gamma_{91}v_{30} + \gamma_{95}v_{32} + \gamma_{107}v_{35} + \gamma_{109}v_{36}
\end{aligned}$$

$$
\begin{aligned}
&+ \gamma_{\{111,191\}}v_{37} + \gamma_{115}v_{39} + \gamma_{119}v_{41} + \gamma_{8}v_{44} \equiv 0,\\
p_{(1;5)}(\widetilde{\theta}) \equiv\ &\gamma_{\{16,126,198\}}v_1 + \gamma_{\{18,130\}}v_2 + \gamma_1 v_3 + \gamma_{\{24,137,204\}}v_4 + \gamma_{\{26,147\}}v_5\\
&+ \gamma_{\{28,139,147\}}v_6 + \gamma_{30}v_7 + \gamma_{32}v_8 + \gamma_{\{38,153\}}v_{10} + \gamma_{40}v_{11} + \gamma_{42}v_{12}\\
&+ \gamma_3 v_{14} + \gamma_{\{61,160,211\}}v_{16} + \gamma_{\{63,177\}}v_{17} + \gamma_{\{65,162,177\}}v_{18} + \gamma_{67}v_{19}\\
&+ \gamma_{69}v_{20} + \gamma_{\{74,183\}}v_{22} + \gamma_{76}v_{23} + \gamma_{\{82,166,183\}}v_{25} + \gamma_{84}v_{26}\\
&+ \gamma_{86}v_{27} + \gamma_{90}v_{29} + \gamma_{94}v_{31} + \gamma_{\{106,190\}}v_{34} + \gamma_{108}v_{35} + \gamma_{110}v_{36}\\
&+ \gamma_{114}v_{38} + \gamma_{118}v_{40} + \gamma_7 v_{43} \equiv 0.
\end{aligned}
$$

By computing from above relations we have

$$\gamma_t = 0 \text{ for } t \in \mathbb{L}_1,\ \gamma_u = \gamma_u \text{ for } (u,v) \in \mathbb{M}_1, \tag{3.2}$$

where $\mathbb{L}_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 20, 21, 22, 23, 25, 27, 30, 32, 39, 40, 41, 42, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 62, 64, 67, 69, 71, 72, 73, 75, 76, 78, 79, 80, 81, 83, 84, 85, 86, 90, 91, 94, 95, 99, 103, 104, 107, 108, 109, 110, 112, 113, 114, 115, 116, 117, 118, 119, 121, 122, 123\}$ and $\mathbb{M}_1 = \{(17,127), (18,130), (26,147), (31,148), (34,141), (35,142), (36,143), (37,144), (38,153), (43,154), (63,177), (68,178), (74,183), (77,184), (92,185), (93,186), (97,169), (98,170), (100,171), (101,172), (102,173), (105,174), (106,190), (111,191), (120,192)\}$.

By applying the homomorphisms $p_{(2;u)}, u = 3, 4, 5$ to (3.1) and using (3.2), we obtain

$$
\begin{aligned}
p_{(2;3)}(\widetilde{\theta}) \equiv\ &\gamma_{\{34,97\}}v_{10} + \gamma_{\{35,98\}}v_{11} + \gamma_{36}v_{12} + \gamma_{\{37,100\}}v_{13} + \gamma_{\{44,92\}}v_{14}\\
&+ \gamma_{\{45,93\}}v_{15} + \gamma_{134}v_{22} + \gamma_{135}v_{23} + \gamma_{136}v_{24} + \gamma_{\{34,97\}}v_{25}\\
&+ \gamma_{\{35,98\}}v_{26} + \gamma_{36}v_{27} + \gamma_{\{37,100\}}v_{28} + \gamma_{\{101,151,164,181\}}v_{29}\\
&+ \gamma_{\{102,152,165,182\}}v_{30} + \gamma_{\{92,155\}}v_{31} + \gamma_{\{93,156\}}v_{32}\\
&+ \gamma_{\{105,120,159,168,189,195\}}v_{33} + \gamma_{202}v_{38} + \gamma_{203}v_{39} + \gamma_{206}v_{40}\\
&+ \gamma_{207}v_{41} + \gamma_{\{210,213\}}v_{42} + \gamma_{15}v_{45} \equiv 0,\\
p_{(2;4)}(\widetilde{\theta}) \equiv\ &\gamma_{\{17,97\}}v_2 + \gamma_{\{19,68\}}v_3 + \gamma_{\{29,101\}}v_7 + \gamma_{31}v_8 + \gamma_{\{33,77,102\}}v_9\\
&+ \gamma_{\{43,105\}}v_{13} + \gamma_{125}v_{17} + \gamma_{\{17,97\}}v_{18} + \gamma_{\{98,129,161,176\}}v_{19}\\
&+ \gamma_{\{68,131\}}v_{20} + \gamma_{\{100,111,133,163,180,194\}}v_{21} + \gamma_{\{101,138\}}v_{23}\\
&+ \gamma_{\{77,102,140\}}v_{24} + \gamma_{146}v_{26} + \gamma_{31}v_{27} + \gamma_{\{150,167,188\}}v_{28}\\
&+ \gamma_{\{43,105\}}v_{30} + \gamma_{158}v_{32} + \gamma_{197}v_{35} + \gamma_{199}v_{36} + \gamma_{\{201,212,215\}}v_{37}\\
&+ \gamma_{205}v_{39} + \gamma_{209}v_{41} + \gamma_{14}v_{44} \equiv 0,\\
p_{(2;5)}(\widetilde{\theta}) \equiv\ &\gamma_{\{16,63,98\}}v_1 + \gamma_{\{18,100\}}v_2 + \gamma_{\{24,74,101\}}v_4 + \gamma_{26}v_5 + \gamma_{\{28,102\}}v_6\\
&+ \gamma_{\{38,105\}}v_{10} + \gamma_{\{97,106,124,160,175,193\}}v_{16} + \gamma_{\{63,98,126\}}v_{17}\\
&+ \gamma_{\{128,162,179\}}v_{18} + \gamma_{\{18,100\}}v_{19} + \gamma_{132}v_{20} + \gamma_{\{74,101,137\}}v_{22}
\end{aligned}
$$

$$+ \gamma_{\{102,139\}}v_{23} + \gamma_{\{145,166,187\}}v_{25} + \gamma_{26}v_{26} + \gamma_{149}v_{27}$$
$$+ \gamma_{\{38,105\}}v_{29} + \gamma_{157}v_{31} + \gamma_{\{196,211,214\}}v_{34} + \gamma_{198}v_{35} + \gamma_{200}v_{36}$$
$$+ \gamma_{204}v_{38} + \gamma_{208}v_{40} + \gamma_{13}v_{43} \equiv 0.$$

Computing from the above equalities gives

$$\gamma_t = 0 \text{ for } t \in \mathbb{L}_2,\ \gamma_u = \gamma_v \text{ for } (u,v) \in \mathbb{M}_2, \tag{3.3}$$

where $\mathbb{L}_2 = \{13,\ 14,\ 15,\ 26,\ 31,\ 36,\ 125,\ 132,\ 134,\ 135,\ 136,\ 143,\ 146,\ 147,\ 148,\ 149,\ 157,\ 158,\ 197,\ 198,\ 199,\ 200,\ 202,\ 203,\ 204,\ 205,\ 206,\ 207,\ 208,\ 209\}$ and $\mathbb{M}_2 = \{(16{,}126),\ (17{,}34),\ (17{,}97),\ (18{,}37),\ (18{,}100),\ (19{,}68),\ (19{,}131),\ (24{,}137),\ (28{,}102),\ (28{,}139),\ (29{,}101),\ (29{,}138),\ (33{,}140),\ (35{,}98),\ (38{,}43),\ (38{,}105),\ (44{,}92),\ (44{,}155),\ (45{,}93),\ (45{,}156),\ (210{,}213)\}$.

Applying the homomorphisms $p_{(i;j)}$, $3 \leqslant i < j \leqslant 5$ to (3.1) and using the relations (3.2), (3.3) give

$$\begin{aligned}
p_{(3;4)}(\widetilde{\theta}) \equiv\ & \gamma_{\{19,44\}}v_3 + \gamma_{\{66,88\}}v_7 + \gamma_{\{19,44\}}v_8 + \gamma_{\{45,70,77,87,89,96\}}v_9\\
&+ \gamma_{\{111,120\}}v_{13} + \gamma_{\{29,35,129,151\}}v_{19} + \gamma_{\{19,44\}}v_{20}\\
&+ \gamma_{\{18,33,38,45,133,150,152,159\}}v_{21} + \gamma_{\{161,164\}}v_{23} + \gamma_{\{163,165,167,168\}}v_{24}\\
&+ \gamma_{\{176,181\}}v_{26} + \gamma_{\{19,44\}}v_{27} + \gamma_{\{45,77,180,182,188,189\}}v_{28} + \gamma_{\{111,120\}}v_{30}\\
&+ \gamma_{\{194,195\}}v_{32} + \gamma_{\{201,210\}}v_{37} + \gamma_{\{210,212\}}v_{39} + \gamma_{215}v_{41} \equiv 0,\\
p_{(3;5)}(\widetilde{\theta}) \equiv\ & \gamma_{\{16,35,45\}}v_1 + \gamma_{\{44,61,74,82,88,96\}}v_4 + \gamma_{\{45,63\}}v_5 + \gamma_{\{65,89\}}v_6\\
&+ \gamma_{\{106,120\}}v_{10} + \gamma_{\{17,24,38,44,124,145,151,159\}}v_{16} + \gamma_{\{16,35,45\}}v_{17}\\
&+ \gamma_{\{28,128,152\}}v_{18} + \gamma_{\{160,164,166,168\}}v_{22} + \gamma_{\{162,165\}}v_{23}\\
&+ \gamma_{\{44,74,175,181,187,189\}}v_{25} + \gamma_{\{45,63\}}v_{26} + \gamma_{\{179,182\}}v_{27} + \gamma_{\{106,120\}}v_{29}\\
&+ \gamma_{\{193,195\}}v_{31} + \gamma_{\{196,210\}}v_{34} + \gamma_{\{210,211\}}v_{38} + \gamma_{214}v_{40} \equiv 0,\\
p_{(4;5)}(\widetilde{\theta}) \equiv\ & \gamma_{\{24,28,29,33\}}v_1 + \gamma_{\{19,61,63,65,66,70\}}v_4 + \gamma_{\{74,77\}}v_5 + \gamma_{\{82,87\}}v_6\\
&+ \gamma_{\{106,111\}}v_{10} + \gamma_{\{16,17,18,19,124,128,129,133\}}v_{16} + \gamma_{\{24,28,29,33\}}v_{17}\\
&+ \gamma_{\{145,150\}}v_{18} + \gamma_{\{160,161,162,163\}}v_{22} + \gamma_{\{166,167\}}v_{23}\\
&+ \gamma_{\{19,63,175,176,179,180\}}v_{25} + \gamma_{\{74,77\}}v_{26} + \gamma_{\{187,188\}}v_{27} + \gamma_{\{106,111\}}v_{29}\\
&+ \gamma_{\{193,194\}}v_{31} + \gamma_{\{196,201\}}v_{34} + \gamma_{\{211,212\}}v_{38} + \gamma_{\{214,215\}}v_{40} \equiv 0.
\end{aligned}$$

From these equalities it implies

$$\gamma_{214} = \gamma_{215} = 0 \text{ and } \gamma_u = \gamma_v \text{ for } (u,v) \in \mathbb{M}_3, \tag{3.4}$$

where $\mathbb{M}_3 = \{(19{,}44),\ (45{,}63),\ (65{,}89),\ (66{,}88),\ (74{,}77),\ (82{,}87),\ (106{,}111),\ (106{,}120),\ (145{,}150),\ (161{,}164),\ (162{,}165),\ (166{,}167),\ (176{,}181),\ (179{,}182),\ (187{,}188),\ (193{,}194),\ (193{,}195),\ (196{,}201),\ (196{,}210),\ (196{,}211),\ (196{,}212)\}$.

Apply the homomorphisms $p_{(1;(2,3))}$, $p_{(1;(2,4))}$ and $p_{(1;(3,4))}$ to (3.1) and using (3.2), (3.3), (3.4) to get

$$\begin{aligned}
p_{(1;(2,3))}(\widetilde{\theta}) \equiv\ & \gamma_{17}v_{22} + \gamma_{35}v_{23} + \gamma_{18}v_{24} + \gamma_{\{19,66,151,161\}}v_{29} + \gamma_{\{45,65,152,162\}}v_{30}\\
&+ \gamma_{19}v_{31} + \gamma_{45}v_{32} + \gamma_{\{96,159,168\}}v_{33} + \gamma_{176}v_{38} + \gamma_{179}v_{39} + \gamma_{19}v_{40}\\
&+ \gamma_{45}v_{41} + \gamma_{189}v_{42} + \gamma_{193}v_{45} \equiv 0,\\
p_{(1;(2,4))}(\widetilde{\theta}) \equiv\ & \gamma_{17}v_{17} + \gamma_{\{19,66,129,161\}}v_{19} + \gamma_{19}v_{20} + \gamma_{\{70,133,163\}}v_{21}\\
&+ \gamma_{\{28,33\}}v_{24} + \gamma_{29}v_{26} + \gamma_{\{28,74,82,145,166\}}v_{28} + \gamma_{38}v_{32} + \gamma_{176}v_{35}\\
&+ \gamma_{19}v_{36} + \gamma_{180}v_{37} + \gamma_{74}v_{39} + \gamma_{187}v_{41} + \gamma_{193}v_{44} \equiv 0,\\
p_{(1;(3,4))}(\widetilde{\theta}) \equiv\ & \gamma_{17}v_{5} + \gamma_{129}v_{7} + \gamma_{19}v_{8} + \gamma_{\{133,196\}}v_{9} + \gamma_{\{19,151\}}v_{11} + \gamma_{19}v_{12}\\
&+ \gamma_{\{145,152\}}v_{13} + \gamma_{\{159,196\}}v_{15} + \gamma_{\{19,66,161\}}v_{19} + \gamma_{\{70,163,196\}}v_{21}\\
&+ \gamma_{\{19,161\}}v_{23} + \gamma_{\{163,179,180,196\}}v_{24} + \gamma_{\{19,161\}}v_{26}\\
&+ \gamma_{\{74,82,162,166,179,180\}}v_{28} + \gamma_{\{19,66,161\}}v_{29}\\
&+ \gamma_{\{45,65,162,166,187,189\}}v_{30} + \gamma_{\{168,187,189,196\}}v_{32}\\
&+ \gamma_{\{96,168,196\}}v_{33} + \gamma_{106}v_{39} + \gamma_{106}v_{41} \equiv 0.
\end{aligned}$$

By a direct computation using the above equalities we obtain

$$\gamma_j = 0 \text{ for } j \in \mathbb{L}_4,\ \gamma_j = \gamma_{124} \text{ for } j \in \mathbb{L}_5, \tag{3.5}$$

where $\mathbb{L}_4 = \{16,\ 17,\ 18,\ 19,\ 24,\ 28,\ 29,\ 33,\ 34,\ 35,\ 37,\ 38,\ 43,\ 44,\ 45,\ 61,\ 63,\ 65,\ 66,\ 68,\ 70,\ 74,\ 77,\ 82,\ 87,\ 88,\ 89,\ 92,\ 93,\ 96,\ 97,\ 98,\ 100,\ 101,\ 102,\ 105,\ 106,\ 111,\ 120,\ 126,\ 127,\ 128,\ 129,\ 130,\ 131,\ 137,\ 138,\ 139,\ 140,\ 141,\ 142,\ 144,\ 145,\ 150,\ 151,\ 152,\ 153,\ 154,\ 155,\ 156,\ 161,\ 162,\ 164,\ 165,\ 166,\ 167,\ 169,\ 170,\ 171,\ 172,\ 173,\ 174,\ 175,\ 176,\ 177,\ 178,\ 179,\ 180,\ 181,\ 182,\ 183,\ 184,\ 185,\ 186,\ 187,\ 188,\ 189,\ 190,\ 191,\ 192,\ 193,\ 194,\ 195\}$ and $\mathbb{L}_5 = \{133,\ 159,\ 160,\ 163,\ 168,\ 196,\ 201,\ 210,\ 211,\ 212,\ 213\}$.

Then, $\widetilde{\theta} \equiv \gamma_{124}\sum_{j\in\mathbb{J}_5} w_j$. By a simple computation we get

$$p_{(2;(3,4))}(\widetilde{\theta}) \equiv \gamma_{124}(v_{24} + v_{32} + v_{39} + v_{41}) \equiv 0.$$

This equality implies that $\gamma_{124} = 0$ and $\gamma_t = 0$ for $1 \leqslant t \leqslant 215$. Hence, we get $\theta \equiv \sum_{1\leqslant u\leqslant 10} \bar{\gamma}_{(u+215)} g_u$. The proposition is proved. □

**Proposition 3.1.2.** *Set* $\omega = (4)|(2)|(3)$*, we have*

$$\widetilde{SF}_5(\omega) = \langle [g_{11}]_\omega \rangle,$$

*where the polynomials* $g_{11}$ *is explicitly determined in Subsection 5.2.*

*Proof.* The proposition is proved by an argument analogous to the one in the proof of Proposition 3.1.1. By a direct computation, we can easily verify that

$p_{(i;I)}(g_{11}) \equiv_\omega 0$ for all $(i;I) \in \mathcal{N}_5$. Hence, $[g_{11}] \in \widetilde{SF}_5(\omega)$. The set $\{[b_t]_\omega : 1 \leqslant t \leqslant 50\}$ is a basis of $QP_5^+(\omega)$ (see Subsection 5.1), hence if $\theta^* \in \widetilde{SF}_5(\omega)$, then we have $\theta^* \equiv_\omega \sum_{1\leqslant t\leqslant 50} \gamma_t^* b_t$, where $\gamma_t^* \in \mathbb{F}_2$. The leading monomial of $g_{11}$ is $b_{50}$. Hence, we get

$$\overline{\theta} := \theta^* + \gamma_{50}^* g_{11} \equiv_\omega \sum_{1\leqslant t\leqslant 49} \gamma_t b_t, \tag{3.6}$$

where $\gamma_j \in \mathbb{F}_2$, $1 \leqslant j \leqslant 49$, and $p_{i;I}(\widetilde{\theta}) \equiv_\omega 0$ for all $(i;I) \in \mathcal{N}_5$.

Let $v_s$, $46 \leqslant s \leqslant 49$, be as in Subsection 5.1. Applying $p_{(u;v)}$, $1 \leqslant u < v \leqslant 5$, to (3.6) and using Theorem 2.1.7 give

$$\begin{aligned}
p_{(1;2)}(\overline{\theta}) &\equiv_\omega \gamma_1 v_{46} + \gamma_2 v_{47} + \gamma_3 v_{48} + \gamma_{\{34,46\}} v_{49} \equiv_\omega 0,\\
p_{(1;3)}(\overline{\theta}) &\equiv_\omega \gamma_6 v_{46} + \gamma_7 v_{47} + \gamma_{\{33,43\}} v_{48} + \gamma_4 v_{49} \equiv_\omega 0,\\
p_{(1;4)}(\overline{\theta}) &\equiv_\omega \gamma_8 v_{46} + \gamma_{\{32,44,47\}} v_{47} + \gamma_{10} v_{48} + \gamma_{12} v_{49} \equiv_\omega 0,\\
p_{(1;5)}(\overline{\theta}) &\equiv_\omega \gamma_{\{31,45,48,49\}} v_{46} + \gamma_9 v_{47} + \gamma_{11} v_{48} + \gamma_{13} v_{49} \equiv_\omega 0,\\
p_{(2;3)}(\overline{\theta}) &\equiv_\omega \gamma_{14} v_{46} + \gamma_{15} v_{47} + \gamma_{\{37,41,43,46\}} v_{48} + \gamma_5 v_{49} \equiv_\omega 0,\\
p_{(2;4)}(\overline{\theta}) &\equiv_\omega \gamma_{16} v_{46} + \gamma_{\{36,44\}} v_{47} + \gamma_{18} v_{48} + \gamma_{26} v_{49} \equiv_\omega 0,\\
p_{(2;5)}(\overline{\theta}) &\equiv_\omega \gamma_{\{35,42,45\}} v_{46} + \gamma_{17} v_{47} + \gamma_{19} v_{48} + \gamma_{27} v_{49} \equiv_\omega 0,\\
p_{(3;4)}(\overline{\theta}) &\equiv_\omega \gamma_{20} v_{46} + \gamma_{\{39,41,47\}} v_{47} + \gamma_{23} v_{48} + \gamma_{28} v_{49} \equiv_\omega 0,\\
p_{(3;5)}(\overline{\theta}) &\equiv_\omega \gamma_{\{38,41,42,48\}} v_{46} + \gamma_{21} v_{47} + \gamma_{24} v_{48} + \gamma_{29} v_{49} \equiv_\omega 0,\\
p_{(4;5)}(\overline{\theta}) &\equiv_\omega \gamma_{\{40,42,49\}} v_{46} + \gamma_{22} v_{47} + \gamma_{25} v_{48} + \gamma_{30} v_{49} \equiv_\omega 0.
\end{aligned}$$

From the above relations we get

$$\gamma_j = 0,\ 1 \leqslant t \leqslant 30,\ \gamma_{33} = \gamma_{43},\ \gamma_{34} = \gamma_{46},\ \gamma_{36} = \gamma_{44}. \tag{3.7}$$

By applying the homomorphisms $p_{(1;(u,v))}$, $1 < u < v < 5$, to (3.6) and using (3.7), we get

$$\begin{aligned}
p_{(1;(2,3))}(\overline{\theta}) &\equiv_\omega \gamma_{37} v_{48} + \gamma_{41} v_{49} \equiv_\omega 0,\\
p_{(1;(2,4))}(\overline{\theta}) &\equiv_\omega \gamma_{\{32,34,47\}} v_{47} + \gamma_{47} v_{49} \equiv_\omega 0,\\
p_{(1;(3,4))}(\overline{\theta}) &\equiv_\omega \gamma_{\{32,33,36,39,41,47\}} v_{47} + \gamma_{36} v_{48} \equiv_\omega 0,
\end{aligned}$$

By a direct computation using these equalities we get

$$\gamma_j = 0 \text{ for } j \in \{32,\ 33,\ 34,\ 36,\ 37,\ 39,\ 41,\ 43,\ 44,\ 46,\ 47\}, \tag{3.8}$$

Apply the homomorphisms $p_{(2;(3,5))}$, $p_{(2;(4,5))}$ to (3.6) using (3.7), (3.8) to obtain

$$p_{(2;(3,5))}(\overline{\theta}) \equiv_\omega \gamma_{\{35,38,42,45\}} v_{46} + \gamma_{\{45,48\}} v_{48} \equiv_\omega 0,$$

$$p_{(2;(4,5))}(\overline{\theta}) \equiv_\omega \gamma_{\{35,40,42,45\}}v_{46} + \gamma_{\{45,49\}}v_{47} \equiv_\omega 0.$$

From the above relations we obtain

$$\gamma_{45} = \gamma_{48} = \gamma_{49}. \tag{3.9}$$

Apply the homomorphisms $p_{(1;(2,5))}$, $p_{(1;(3,5))}$, $p_{(1;(4,5))}$ to (3.6) using (3.7), (3.8), (3.9) to get

$$\begin{aligned} p_{(1;(2,5))}(\overline{\theta}) &\equiv_\omega \gamma_{35}v_{46} + \gamma_{42}v_{49} \equiv_\omega 0,\\ p_{(1;(3,5))}(\overline{\theta}) &\equiv_\omega \gamma_{38}v_{46} + \gamma_{42}v_{48} \equiv_\omega 0,\\ p_{(1;(4,5))}(\overline{\theta}) &\equiv_\omega \gamma_{40}v_{46} + \gamma_{42}v_{47} \equiv_\omega 0, \end{aligned}$$

These equalities imply that $\gamma_t = 0$ for $1 \leqslant t \leqslant 49$. Hence, we get $\theta^* \equiv \gamma_{50}^* g_{11}$. The proposition is proved. □

**Remark 3.1.3.**

i) Consider the relation (3.13) at the end of Pages 15 of Phúc [15]:

$$\eta := \sum_{501 \leqslant j \leqslant 550} \mathrm{adm}_j \equiv_\omega 0$$

with $\omega = (4,2,3)$. Here, we use the notations as defined in [15]. From the proof of Proposition 3.1.2, we see that $p_{(i;I)}(\eta) \equiv_\omega 0$ for all $(i;I) \in \mathcal{N}_5$ if and only if $\gamma_j = \gamma_{531}$ for $531 \leqslant j \leqslant 550$, $j \neq 535, 540, 541, 542$ and $\gamma_j = 0$ for $j \in \{501, 502, \ldots, 530, 535, 540, 541, 542\}$. However, in [15], the author says that from $p_{(u;v)}(\eta) \equiv_\omega 0$, $1 \leqslant u < v \leqslant 5$, $p_{(1;(2,4))}(\eta) \equiv_\omega 0$ and $p_{(1;(2,5))}(\eta) \equiv_\omega 0$ he obtains $\gamma_j = 0$ for all $501 \leqslant j \leqslant 550$. Thus, the computations for $p_{(i;I)}(\eta)$ on Page 16 of [15] are false.

ii) It is not difficult to show that $[g_{11}]_\omega$ is an $GL_5$-invariant, hence we obtain $\dim QP_5(\omega)^{GL_5} \geqslant 1$. However, in [15], the author stated that $\dim QP_5(\omega)^{GL_5} = 0$. Thus, the computations for $QP_5(\omega)^{GL_5}$ on Pages 16, 17 of [15] are false.

By similar computations as presented on the proof of Proposition 3.1.1 we get the following.

**Proposition 3.1.4.** *We have* $\widetilde{SF}_5((4)|^2|(2)) = 0$.

## 3.2 Computation of $QP_5((4)|(2)|(1)|^2)^{GL_5}$

To prove the main results, we need to use the following homomorphism.

**Definition 3.2.1.** Recall that $V_k \cong \langle x_1, x_2, \ldots, x_k\rangle \subset P_k$. For $1 \leqslant j \leqslant k$. Define the $\mathbb{F}_2$-linear map $\rho_j : V_k \to V_k$, by $\rho_k(x_1) = x_1 + x_2$, $\rho_k(x_t) = x_t$ for $t > 1$ and $\rho_j(x_j) = x_{j+1}, \rho_j(x_{j+1}) = x_j$, $\rho_j(x_t) = x_t$ for $t \neq j, j+1,\ 1 \leqslant j < k$. The group $GL_k \cong GL(V_k)$ is generated by $\rho_j,\ 1 \leqslant j \leqslant k$, and the subgroup $\Sigma_k$ is generated by $\rho_j,\ 1 \leqslant j < k$.

The linear map $\rho_j$ induces an $\mathcal{A}$-homomorphism of algebras $\rho_j : P_k \to P_k$. Hence, a class $[g]_\omega \in QP_k(\omega)$ is an element of $QP_k^{GL_k}$ if and only if $\rho_j(g) \equiv_\omega g$ for $1 \leqslant j \leqslant k$. It is an element of $QP_k(\omega)^{\Sigma_k}$ if and only if $\rho_j(g) \equiv_\omega g$ for $1 \leqslant j < k$.

**Proposition 3.2.2.** *We have* $QP_5((4)|(2)|(1)|^2)^{GL_5} = 0$.

To compute $QP_5((4)|(2)|(1)|^2)^{GL_5}$ we need the following.

**Lemma 3.2.3.** *We have*

$$QP_5((4)|(2)|(1)|^2)^{\Sigma_5} = \langle [h_s] : 1 \leqslant s \leqslant 7\rangle,$$

*where the polynomials $h_s$ are determined as in Subsection 5.3.*

*Proof.* We have $B_5((4)|(2)|(1)|^2) = B_5^0((4)|(2)|(1)|^2) \cup \{a_t : 1 \leqslant t \leqslant 225\}$, where the monomials $a_t$ are determined as in Subsection 5.1. By using this set we see that there is a direct summand decompositions of the $\Sigma_5$-modules

$$QP_5((4)|(2)|(1)|^2)[\Sigma_5(v_1)] \bigoplus [\Sigma_5(v_2)] \bigoplus [\Sigma_5(v_5, v_6, v_7)] \bigoplus [\Sigma_5(v_{23})] \bigoplus [\Sigma_5(a_1)] \bigoplus \mathcal{M},$$

where $v_s$, $a_t$ are defined as in Subsection 5.1 and $\mathcal{M} = \langle [a_t] : 16 \leqslant t \leqslant 225\rangle$. By a direct computation, we have

$$[\Sigma_5(v_1)]^{\Sigma_5} = \langle [h_1]\rangle, \quad [\Sigma_5(v_2)]^{\Sigma_5} = \langle [h_2]\rangle,$$
$$[\Sigma_5(v_5, v_6, v_7)]^{\Sigma_5} = \langle [h_3]\rangle, \quad [\Sigma_5(w_{23})]^{\Sigma_5} = \langle [h_4]\rangle,$$
$$[\Sigma_5(a_1)]^{\Sigma_5} = 0, \quad \mathcal{M}^{\Sigma_5} = \langle [h_5], [h_6], [h_7]\rangle.$$

To illustrate the proofs, we present a proof for the relation $[\Sigma_5(a_1)]^{\Sigma_5} = 0$. The proofs of other cases are carried out by a similar computation.

Note that $[\Sigma_5(a_1)] = \langle [a_t] : 1 \leqslant t \leqslant 15\rangle$. Suppose $h \in P_5((4)|(2)|(1)|^2)$ and $[h] \in [\Sigma_5(a_1)]^{\Sigma_5}$. Then we have $h \equiv \sum_{t=1}^{15} \gamma_t a_t$. Cnssider the homomorphisms $\rho_j : P_5 \to P_5$ as defined in Section 2 with $1 \leqslant j \leqslant 5$. A direct computation shows

$$\begin{aligned}\rho_1(h) + h &\equiv \gamma_7 a_1 + \gamma_8 a_2 + \gamma_7 a_3 + \gamma_8 a_4 + \gamma_9 a_5 + \gamma_9 a_6 + \gamma_{\{10,13\}} a_{10}\\ &\quad + \gamma_{\{11,14\}} a_{11} + \gamma_{\{12,15\}} a_{12} + \gamma_{\{10,13\}} a_{13} + \gamma_{\{11,14\}} a_{14}\\ &\quad + \gamma_{\{12,15\}} a_{15} \equiv 0,\\ \rho_2(h) + h &\equiv \gamma_{\{3,7\}} a_3 + \gamma_{\{4,8\}} a_4 + \gamma_{\{5,10\}} a_5 + \gamma_{\{6,11\}} a_6 + \gamma_{\{3,7\}} a_7\\ &\quad + \gamma_{\{4,8\}} a_8 + \gamma_{\{9,12\}} a_9 + \gamma_{\{5,10\}} a_{10} + \gamma_{\{6,11\}} a_{11} + \gamma_{\{9,12\}} a_{12}\\ &\quad + \gamma_{15} a_{13} + \gamma_{15} a_{14} \equiv 0,\\ \rho_3(h) + h &\equiv \gamma_{\{1,3\}} a_1 + \gamma_{\{2,5\}} a_2 + \gamma_{\{1,3\}} a_3 + \gamma_{\{4,6\}} a_4 + \gamma_{\{2,5\}} a_5\end{aligned}$$

$$\begin{aligned}
&+ \gamma_{\{4,6\}}a_6 + \gamma_{\{8,9\}}a_8 + \gamma_{\{8,9\}}a_9 + \gamma_{\{11,12\}}a_{11} + \gamma_{\{11,12\}}a_{12}\\
&+ \gamma_{\{14,15\}}a_{14} + \gamma_{\{14,15\}}a_{15} \equiv 0,\\
\rho_4(h) + h \equiv\ &\gamma_{\{1,2\}}a_1 + \gamma_{\{1,2\}}a_2 + \gamma_{\{3,4\}}a_3 + \gamma_{\{3,4\}}a_4 + \gamma_{\{5,6\}}a_5\\
&+ \gamma_{\{5,6\}}a_6 + \gamma_{\{7,8\}}a_7 + \gamma_{\{7,8\}}a_8 + \gamma_{\{10,11\}}a_{10} + \gamma_{\{10,11\}}a_{11}\\
&+ \gamma_{\{13,14\}}a_{13} + \gamma_{\{13,14\}}a_{14} \equiv 0.
\end{aligned}$$

By computing from the above relations, we get $\gamma_t = 0$ for $1 \leqslant t \leqslant 15$. Hence, $h \equiv 0$ and $[\Sigma_5(a_1)]^{\Sigma_5} = 0$. □

*Proof of Proposition 3.2.2.* Suppose $f \in QP_5((4)|(2)|(1)|^2)$ such that the class $[f] \in P_5((4)|(2)|(1)|^2)^{GL_5}$. Then, $[f] \in P_5((4)|(2)|(1)|^2)^{\Sigma_5}$. By Lemma 3.2.3, we have

$$f \equiv \sum_{1\leqslant s\leqslant 7} \gamma_s h_s,$$

where $\gamma_s \in \mathbb{F}_2$. By computing $\rho_5(f) + f$ in terms of the admissible monomials, we get

$$\begin{aligned}
\rho_5(f) + f \equiv\ &\gamma_1 a_1 + \gamma_2 a_{125} + \gamma_3 a_{16} + \gamma_4 a_{26} + \gamma_5 a_{121}\\
&+ (\gamma_2 + \gamma_6)a_{61} + \gamma_7 a_{63} + \text{ other terms } \equiv 0.
\end{aligned}$$

This equality implies $\gamma_s = 0$ for $1 \leqslant s \leqslant 7$. Hence, $[f] = 0$. The proposition is proved. □

## 3.3 Computation of $QP_5((4)|(2)|(3))^{GL_5}$

In this subsection we prove the following.

**Proposition 3.3.1.** *Set $\overline{\omega} = (4)|(2)|(3)$. Then, we have*

$$QP_5(\overline{\omega})^{GL_5} = \langle [h_{10}]_{\overline{\omega}} \rangle.$$

Note that $h_{10} = g_{11}$. We need the following for the proof of Proposition 3.3.1.

**Lemma 3.3.2.** *We have*

$$QP_5((4)|(2)|(1)|^2)^{\Sigma_5} = \langle [h_8]_{\overline{\omega}}, [h_9]_{\overline{\omega}}, [h_{10}]_{\overline{\omega}} \rangle.$$

*Proof.* We have $B_5(\overline{\omega}) = B_5^0(\overline{\omega}) \cup \{b_t : 1 \leqslant t \leqslant 50\}$, where the monomials $b_t$ are determined as in Subsection 5.1. By using this set we see that there is a direct summand decompositions of the $\Sigma_5$-modules

$$QP_5(\overline{\omega}) = [\Sigma_5(v_{46})]_{\overline{\omega}} \bigoplus [\Sigma_5(b_1)]_{\overline{\omega}} \bigoplus [\Sigma_5(b_{31})]_{\overline{\omega}},$$

where $v_s$, $b_t$ are defined as in Subsection 5.1. By a direct computation, we have

$$[\Sigma_5(v_{46})]^{\Sigma_5} = \langle[h_8]_{\overline{\omega}}\rangle, \quad [\Sigma_5(b_1)]^{\Sigma_5} = \langle[h_9]_{\overline{\omega}}\rangle, \quad [\Sigma_5(b_{31})]^{\Sigma_5} = \langle[h_{10}]_{\overline{\omega}}\rangle.$$

We present a detailed proof for the case $[\Sigma_5(b_{31})]^{\Sigma_5} = \langle[h_{10}]_{\overline{\omega}}\rangle$.

Note that $[\Sigma_5(b_{31})] = \langle[b_t] : 31 \leqslant t \leqslant 50\rangle$. Suppose $h \in P_5(\overline{\omega})$ and $[h]_{\overline{\omega}} \in [\Sigma_5(b_{31})]_{\overline{\omega}}^{\Sigma_5}$. Then we have $h \equiv_{\overline{\omega}} \sum_{t=31}^{50}\gamma_t b_t$. By a direct computation we have

$$\begin{aligned}
\rho_1(h) + h \equiv_{\overline{\omega}}\ & \gamma_{\{31,35\}}b_{31} + \gamma_{\{32,36\}}b_{32} + \gamma_{\{33,37,46\}}b_{33} + \gamma_{\{34,46\}}b_{34} \\
& + \gamma_{\{31,35\}}b_{35} + \gamma_{\{32,36\}}b_{36} + \gamma_{\{33,34,37\}}b_{37} + \gamma_{41}b_{43} \\
& + \gamma_{\{47,50\}}b_{44} + \gamma_{\{42,48,49\}}b_{45} + \gamma_{\{34,46\}}b_{46} + \equiv_{\overline{\omega}} 0, \\
\rho_2(h) + h \equiv_{\overline{\omega}}\ & \gamma_{\{33,34\}}b_{33} + \gamma_{\{33,34\}}b_{34} + \gamma_{\{35,38,41\}}b_{35} + \gamma_{\{36,39,41\}}b_{36} \\
& + \gamma_{\{37,41\}}b_{37} + \gamma_{\{35,37,38\}}b_{38} + \gamma_{\{36,37,39\}}b_{39} + \gamma_{\{37,41\}}b_{41} \\
& + \gamma_{\{43,46\}}b_{43} + \gamma_{\{44,47\}}b_{44} + \gamma_{\{45,48\}}b_{45} + \gamma_{\{43,46\}}b_{46} \\
& + \gamma_{\{44,47\}}b_{47} + \gamma_{\{45,48\}}b_{48} \equiv_{\overline{\omega}} 0, \\
\rho_3(h) + h \equiv_{\overline{\omega}}\ & \gamma_{\{32,33\}}b_{32} + \gamma_{\{32,33\}}b_{33} + \gamma_{\{36,37,46\}}b_{36} + \gamma_{\{36,37,46\}}b_{37} \\
& + \gamma_{\{38,40,47\}}b_{38} + \gamma_{\{39,50\}}b_{39} + \gamma_{\{38,39,40,41\}}b_{40} + \gamma_{\{41,47,50\}}b_{41} \\
& + \gamma_{\{43,44,47,50\}}b_{43} + \gamma_{\{41,43,44\}}b_{44} + \gamma_{\{39,41,47\}}b_{47} + \gamma_{\{48,49\}}b_{48} \\
& + \gamma_{\{48,49\}}b_{49} + \gamma_{\{39,50\}}b_{50} \equiv_{\overline{\omega}} 0, \\
\rho_4(h) + h \equiv_{\overline{\omega}}\ & \gamma_{\{31,32\}}b_{31} + \gamma_{\{31,32\}}b_{32} + \gamma_{\{35,36\}}b_{35} + \gamma_{\{35,36\}}b_{36} \\
& + \gamma_{\{38,39\}}b_{38} + \gamma_{\{38,39\}}b_{39} + \gamma_{\{40,42\}}b_{40} + \gamma_{\{40,42\}}b_{42} \\
& + \gamma_{\{42,44,45\}}b_{44} + \gamma_{\{40,44,45\}}b_{45} + \gamma_{\{42,47,48\}}b_{47} + \gamma_{\{40,47,48\}}b_{48} \\
& + \gamma_{\{49,50\}}b_{49} + \gamma_{\{49,50\}}b_{50} \equiv_{\overline{\omega}} 0.
\end{aligned}$$

Computing from the above relations gives $\gamma_t = 0$ for $t \in \{37, 40, 41, 42\}$ and $\gamma_t = \gamma_{31}$ for $t \neq 37, 40, 41, 42$. Hence, $h \equiv_{\overline{\omega}} h_{10}$. The lemma is proved. □

*Proof of Proposition 3.3.1.* Suppose $g \in P_5(\overline{\omega})$ and $[g]_{\overline{\omega}} \in QP_5(\overline{\omega})^{GL_5}$. Then, $[g]_{\overline{\omega}} \in QP_5(\overline{\omega})^{\Sigma_5}$. By Lemma 3.3.2, we have

$$g \equiv \gamma_8 h_8 + \gamma_9 h_9 + \gamma_{10} h_{10},$$

where $\gamma_s \in \mathbb{F}_2$. By computing $\rho_5(g) + g$ in terms of the admissible monomials, we get

$$\begin{aligned}
\rho_5(g) + g \equiv_{\overline{\omega}}\ & (\gamma_8 + \gamma_9)(x_2^3x_3^5x_4^5x_5^7 + x_2^3x_3^5x_4^7x_5^5 + x_2^3x_3^7x_4^5x_5^5) \\
& \gamma_8\big(x_2^7x_3^3x_4^5x_5^5 + x_1^3x_2^7x_4^5x_5^5 + x_1^3x_2^7x_3^5x_5^5 + x_1^3x_2^7x_3^5x_4^5 \\
& + b_{31} + b_{32} + b_{33} + b_{34} + b_{35} + b_{36} + b_{46}\big) \\
& + \gamma_9\big(b_4 + b_6 + b_7 + b_8 + b_9 + b_{10} + b_{11} + b_{12} + b_{13}
\end{aligned}$$

$$+ b_{23} + b_{24} + b_{25} + b_{43} + b_{44} + b_{45}\big) \equiv_{\overline{\omega}} 0.$$

This equality implies $\gamma_8 = \gamma_9 = 0$. Hence, $[g]_{\overline{\omega}} = [h_{10}]_{\overline{\omega}}$. The proposition is proved. □

## 3.4 Computation of $QP_5((4)|^2|(2))^{GL_5}$

In this subsection we prove the following.

**Proposition 3.4.1.** *We have* $QP_5((4)|^2|(2))^{GL_5} = 0$.

*Proof.* This proposition is proved by the same computations as in the proof of Proposition 3.3.1. We only present here a brief proof.

We have $B_5((4)|^2|(2)) = B_5^0((4)|^2|(2)) \cup \{c_t : 1 \leqslant t \leqslant 91\}$, where the monomials $c_t$ are determined as in Subsection 5.1. Based on this set we have a direct summand decomposition of the $\Sigma_5$-modules

$$QP_5(\omega^*) = [\Sigma_5(w_{50})]_{\omega^*} \bigoplus [\Sigma_5(c_1)]_{\omega^*} \bigoplus [\Sigma_5(c_{31})]_{\omega^*} \bigoplus [\Sigma_5(c_{86})]_{\omega^*},$$

where $\omega^* := (4)|^2|(2)$. By a direct computation we have

$$[\Sigma_5(w_{50}]_{\omega^*}^{\Sigma_5} = \langle [h_{11}]_{\omega^*}\rangle, \quad [\Sigma_5(c_1)]_{\omega^*}^{\Sigma_5} = \langle [h_{12}]_{\omega^*}\rangle,$$
$$[\Sigma_5(c_{31})]_{\omega^*}^{\Sigma_5} = 0, \quad [\Sigma_5(c_{86})]_{\omega^*}^{\Sigma_5} = \langle [h_{13}]_{\omega^*}\rangle,$$

where $h_{11}, h_{12}, h_{13}$ are defined in Subsection 5.3.

Suppose $f \in P_5(\omega^*)$ and $[f]_{\omega^*} \in QP_5(\omega^*)^{GL_5}$. Then, $[f]_{\omega^*} \in QP_5(\omega^*)^{\Sigma_5}$. Hence, we have

$$f \equiv_{\omega^*} \gamma_{11}h_{11} + \gamma_{12}h_{12} + \gamma_{13}h_{13},$$

where $\gamma_{11}, \gamma_{12}, \gamma_{13} \in \mathbb{F}_2$. By computing $\rho_5(f) + f$ in terms of the admissible monomials, we get

$$\rho_5(f) + f \equiv_{\omega^*} \gamma_{11}w_{52} + \gamma_{12}c_4 + \gamma_{13}c_{40} + \text{ other terms } \equiv_{\omega^*} 0.$$

This equality implies $\gamma_{11} = \gamma_{12} = \gamma_{13} = 0$. Hence, $[f]_{\omega^*} = 0$ and the proposition is proved. □

**Remark 3.4.2.** In [15], Phúc stated on Page 16 that $\dim \mathbb{V}^{GL_5} = 1$ with $\mathbb{V} = QP_5(4,2,1,1) \bigoplus QP_5(4,4,2)$. However, by combining Propositions 3.2.2 and 3.4.1, we have $\mathbb{V}^{GL_5} = 0$. Hence, Phúc's result is false.

## 3.5 Proof of Theorem 1.2

We note that by a direct computation we can easily check that $\rho_j(p) + p \equiv 0$ for $1 \leqslant j \leqslant 5$. Hence, $[p] \in (QP_5)_{20}^{GL_5}$.

Suppose $f \in P_5$ and $[f] \in (QP_5)_{20}^{GL_5}$. By Proposition 3.4.1, $[f]_{\omega^*} = 0$, hence $[f]_{\overline{\omega}} \in QP_5(\overline{\omega})^{GL_5}$. By using Proposition 3.3.1, we get

$$f \equiv \gamma h_{10} + \sum_{x \in B((4)|(2)|(1)^2)} \gamma_x . x,$$

where $\gamma$, $\gamma_x \in \mathbb{F}_2$ for all $x \in B((4)|(2)|(1)^2)$. By computing from the relation $\rho_j(f) + f \equiv 0$, we see that $\rho_j(f) + f \equiv 0$ for $1 \leqslant j < 5$ if and only if

$$f \equiv \gamma(h_{10} + h_0) + \sum_{1 \leqslant t \leqslant 7} \gamma_t h_t,$$

where $\gamma_t \in \mathbb{F}_2$ and

$$\begin{aligned} h_0 = x_1x_2x_3^3x_4^3x_5^{12} &+ x_1x_2x_3^3x_4^5x_5^{10} + x_1x_2x_3^3x_4^6x_5^9 + x_1x_2x_3^3x_4^{12}x_5^3 \\ &+ x_1x_2x_3^6x_4^3x_5^9 + x_1x_2x_3^6x_4^9x_5^3 + x_1x_2^3x_3x_4^3x_5^{12} + x_1x_2^3x_3x_4^5x_5^{10} \\ &+ x_1x_2^3x_3x_4^6x_5^9 + x_1x_2^3x_3x_4^{12}x_5^3 + x_1x_2^3x_3^3x_4^4x_5^9 + x_1x_2^3x_3^3x_4^5x_5^8 \\ &+ x_1x_2^3x_3^5x_4^3x_5^8 + x_1x_2^3x_3^5x_4^8x_5^3 + x_1x_2^6x_3x_4^3x_5^9 + x_1x_2^6x_3x_4^9x_5^3 \\ &+ x_1^3x_2x_3^4x_4^3x_5^9 + x_1^3x_2x_3^4x_4^9x_5^3 + x_1^3x_2x_3^5x_4^3x_5^8 + x_1^3x_2x_3^5x_4^8x_5^3 \\ &+ x_1^3x_2^4x_3x_4^3x_5^9 + x_1^3x_2^4x_3x_4^9x_5^3 + x_1^3x_2^5x_3x_4^3x_5^8 + x_1^3x_2^5x_3x_4^8x_5^3. \end{aligned}$$

Note that $h_0 + h_{10} = p$. Since $[p] \in (QP_5)_{20}^{GL_5}$, we get

$$[\sum_{1 \leqslant t \leqslant 7} \gamma_t h_t] \in QP_5((4)|(2)|(1)^2)^{GL_5}.$$

By using Proposition 3.2.2, we obtain $\gamma_t = 0$ for $1 \leqslant t \leqslant 7$. The theorem is proved.

# 4 Proof of Theorem 1.4

Recall from [27] that

$$(QP_5)_{30} \cong QP_5((2)|^4) \bigoplus QP_5(2,4,3,1) \bigoplus QP_5(4,3,3,1).$$

The set of all admissible monomials of the degree 30 in $P_5$ is explicitly presented in [26].

The proof of this theorem is carried out by the same argument as presented in the proof of Theorem 1.2. So, we only present here some main results of the proof.

### 4.1 Computation of $QP_5((2)|^4)^{GL_5}$ and $QP_5(2,4,3,1)^{GL_5}$

In this subsection we explicitly compute $QP_5((2)|^4)^{GL_5}$ and $QP_5(2,4,3,1)^{GL_5}$.

**Proposition 4.1.1.** *We have* $QP_5((2)|^4)^{GL_5} = 0$.

To compute $QP_5((2)|^4)^{GL_5}$ we need the following.

**Lemma 4.1.2.** *We have*

$$QP_5((2)|^4)^{\Sigma_5} = \langle [p_s] : 1 \leqslant s \leqslant 9 \rangle,$$

*where the polynomials* $p_s$ *are determined as in Subsection 5.4.*

*Proof of Proposition 4.1.1.* Suppose $f \in QP_5((2)|^4)$ and $[f] \in P_5((2)|^4)^{GL_5}$. Then, $[f] \in P_5((2)|^4)^{\Sigma_5}$. By Lemma 4.1.2, we have

$$f \equiv \sum_{1 \leqslant s \leqslant 9} \gamma_s p_s,$$

where $\gamma_s \in \mathbb{F}_2$ for $1 \leqslant s \leqslant 9$. By computing $\rho_5(f)+f$ in terms of the admissible monomials, we get

$$\begin{aligned}\rho_5(f) + f \equiv\ & (\gamma_1 + \gamma_2)x_2^{15}x_3^{15} + (\gamma_2 + \gamma_6)x_2x_3^{15}x_4^{14} + \gamma_5 x_2^3x_3x_4^{12}x_5^{14} \\ & + \gamma_6[x_1^3x_2x_3^4x_4^{10}x_5^{12}] + \gamma_7 x_1x_2^3x_3^2x_4^{12}x_5^{12} + \gamma_8 x_1x_2^3x_3^{12}x_4^2x_5^{12} \\ & + (\gamma_2 + \gamma_4)x_2^{15}x_4x_5^{14} + (\gamma_2 + \gamma_3)x_1x_2^{15}x_3^{14} \\ & + (\gamma_5 + \gamma_9)[x_2x_3^3x_4^{12}x_5^{14}] + \text{other terms} \equiv 0.\end{aligned}$$

This equality implies $\gamma_s = 0$ for $1 \leqslant s \leqslant 9$. Hence, $[f] = 0$. The proposition is proved. ☐

Since $QP_5(2,4,3,1)$ is an $GL_5$-module of dimension 1, we get the following.

**Proposition 4.1.3.** *We have*

$$QP_5(2,4,3,1)^{GL_5} = \langle [x_1^3x_2^5x_3^6x_4^6x_5^{10}]_\omega \rangle, \ \textit{with}\ \omega = (2,4,3,1).$$

### 4.2 Computation of $QP_5((4)|(3)|^2|(1))^{GL_5}$

By a direct computation based on the basis of the space $QP_5((4)|(3)|^2|(1))$ as presented in [26] one gets the following result.

**Lemma 4.2.1.** *We have*

$$QP_5((4)|(3)|^2|(1)) = \langle [q_s]_{\omega^*} : 1 \leqslant s \leqslant 7 \rangle,$$

*where the polynomials* $q_s$, $1 \leqslant s \leqslant 7$, *are determined as in Subsection 5.5 and* $\omega^* = (4,3,3,1)$.

*Proof.* We have a direct summand decomposition of the $\Sigma_5$-modules

$$QP_5((4)|(3)|^2|(1)) \cong \Sigma_5(x_1x_2^7x_3^7x_4^{15}) \bigoplus \Sigma_5(x_1^3x_2^5x_3^7x_4^{15}) \bigoplus \Sigma_5(x_1^3x_2^7x_3^7x_4^{13}) \bigoplus \Sigma_5(x_1x_2x_3^6x_4^7x_5^{15}, x_1x_2^2x_3^5x_4^7x_5^{15}) \bigoplus \Sigma_5(x_1x_2^3x_3^5x_4^6x_5^{15}) \bigoplus \mathcal{M},$$

where $\mathcal{M}$ is the subspace of $QP_5((4)|(3)|^2|(1))$ spanned by 330 admissible monomials of $B_5^+(4,3,3,1) \setminus \mathcal{C}$ as listed in [26, Theorem 3.3.7]. By a direct computation one gets

$$\begin{aligned}
&\Sigma_5(x_1x_2^7x_3^7x_4^{15})^{\Sigma_5} = \langle [q_1]_{\omega^*} \rangle\\
&\Sigma_5(x_1^3x_2^5x_3^7x_4^{15})^{\Sigma_5} = \langle [q_2]_{\omega^*} \rangle\\
&\Sigma_5(x_1^3x_2^7x_3^7x_4^{13})^{\Sigma_5} = 0\\
&\Sigma_5(x_1x_2x_3^6x_4^7x_5^{15}, x_1x_2^2x_3^5x_4^7x_5^{15})^{\Sigma_5} = \langle [q_3]_{\omega^*} \rangle\\
&\Sigma_5(x_1x_2^3x_3^5x_4^6x_5^{15})^{\Sigma_5} = \langle [q_4]_{\omega^*} \rangle\\
&\mathcal{M}^{\Sigma_5} = \langle [q_5]_{\omega^*}, [q_6]_{\omega^*}, [q_7]_{\omega^*} \rangle
\end{aligned}$$

We present the detailed proof for the relation $\Sigma_5(x_1x_2^3x_3^5x_4^6x_5^{15})^{\Sigma_5} = \langle [q_4]_{\omega^*} \rangle$. We have $\Sigma_5(x_1x_2^3x_3^5x_4^6x_5^{15}) = \langle [d_t]_{\omega^*} : 1 \leqslant t \leqslant 60 \rangle$, where the monomials $d_t$, $11 \leqslant t \leqslant 60$, are determined as follows:

| | | | |
|---|---|---|---|
| 1. $x_1x_2^3x_3^5x_4^6x_5^{15}$ | 2. $x_1x_2^3x_3^5x_4^{15}x_5^6$ | 3. $x_1x_2^3x_3^6x_4^5x_5^{15}$ | 4. $x_1x_2^3x_3^6x_4^{15}x_5^5$ |
| 5. $x_1x_2^3x_3^{15}x_4^5x_5^6$ | 6. $x_1x_2^3x_3^{15}x_4^6x_5^5$ | 7. $x_1x_2^6x_3^3x_4^5x_5^{15}$ | 8. $x_1x_2^6x_3^3x_4^{15}x_5^5$ |
| 9. $x_1x_2^6x_3^{15}x_4^3x_5^5$ | 10. $x_1x_2^{15}x_3^3x_4^5x_5^6$ | 11. $x_1x_2^{15}x_3^3x_4^6x_5^5$ | 12. $x_1x_2^{15}x_3^6x_4^3x_5^5$ |
| 13. $x_1^3x_2x_3^5x_4^6x_5^{15}$ | 14. $x_1^3x_2x_3^5x_4^{15}x_5^6$ | 15. $x_1^3x_2x_3^6x_4^5x_5^{15}$ | 16. $x_1^3x_2x_3^6x_4^{15}x_5^5$ |
| 17. $x_1^3x_2x_3^{15}x_4^5x_5^6$ | 18. $x_1^3x_2x_3^{15}x_4^6x_5^5$ | 19. $x_1^3x_2^3x_3^4x_4^5x_5^{15}$ | 20. $x_1^3x_2^3x_3^4x_4^{15}x_5^5$ |
| 21. $x_1^3x_2^3x_3^5x_4^4x_5^{15}$ | 22. $x_1^3x_2^3x_3^5x_4^{15}x_5^4$ | 23. $x_1^3x_2^3x_3^{15}x_4^4x_5^5$ | 24. $x_1^3x_2^3x_3^{15}x_4^5x_5^4$ |
| 25. $x_1^3x_2^4x_3^3x_4^5x_5^{15}$ | 26. $x_1^3x_2^4x_3^3x_4^{15}x_5^5$ | 27. $x_1^3x_2^4x_3^{15}x_4^3x_5^5$ | 28. $x_1^3x_2^5x_3x_4^6x_5^{15}$ |
| 29. $x_1^3x_2^5x_3x_4^{15}x_5^6$ | 30. $x_1^3x_2^5x_3^2x_4^5x_5^{15}$ | 31. $x_1^3x_2^5x_3^2x_4^{15}x_5^5$ | 32. $x_1^3x_2^5x_3^3x_4^4x_5^{15}$ |
| 33. $x_1^3x_2^5x_3^3x_4^{15}x_5^4$ | 34. $x_1^3x_2^5x_3^6x_4x_5^{15}$ | 35. $x_1^3x_2^5x_3^6x_4^{15}x_5$ | 36. $x_1^3x_2^5x_3^{15}x_4x_5^6$ |
| 37. $x_1^3x_2^5x_3^{15}x_4^2x_5^5$ | 38. $x_1^3x_2^5x_3^{15}x_4^3x_5^4$ | 39. $x_1^3x_2^5x_3^{15}x_4^6x_5$ | 40. $x_1^3x_2^{15}x_3x_4^5x_5^6$ |
| 41. $x_1^3x_2^{15}x_3x_4^6x_5^5$ | 42. $x_1^3x_2^{15}x_3^3x_4^4x_5^5$ | 43. $x_1^3x_2^{15}x_3^3x_4^5x_5^4$ | 44. $x_1^3x_2^{15}x_3^4x_4^3x_5^5$ |
| 45. $x_1^3x_2^{15}x_3^5x_4x_5^6$ | 46. $x_1^3x_2^{15}x_3^5x_4^2x_5^5$ | 47. $x_1^3x_2^{15}x_3^5x_4^3x_5^4$ | 48. $x_1^3x_2^{15}x_3^5x_4^6x_5$ |
| 49. $x_1^{15}x_2x_3^3x_4^5x_5^6$ | 50. $x_1^{15}x_2x_3^3x_4^6x_5^5$ | 51. $x_1^{15}x_2x_3^6x_4^3x_5^5$ | 52. $x_1^{15}x_2^3x_3x_4^5x_5^6$ |
| 53. $x_1^{15}x_2^3x_3x_4^6x_5^5$ | 54. $x_1^{15}x_2^3x_3^3x_4^4x_5^5$ | 55. $x_1^{15}x_2^3x_3^3x_4^5x_5^4$ | 56. $x_1^{15}x_2^3x_3^4x_4^3x_5^5$ |
| 57. $x_1^{15}x_2^3x_3^5x_4x_5^6$ | 58. $x_1^{15}x_2^3x_3^5x_4^2x_5^5$ | 59. $x_1^{15}x_2^3x_3^5x_4^3x_5^4$ | 60. $x_1^{15}x_2^3x_3^5x_4^6x_5$. |

Suppose $f \in P_5(\omega^*)$ and $[f]_{\omega^*} \in \Sigma_5(x_1x_2^3x_3^5x_4^6x_5^{15})^{\Sigma_5}$. Then we have $f \equiv_{\omega^*} \sum_{t=1}^{60} d_t$ with $\gamma_t \in \mathbb{F}_2$ and $\rho_j(f) + f \equiv_{\omega^*} 0$, $1 \leqslant j \leqslant 4$.

By a direct computation we get

$$\begin{aligned}
\rho_1(f) + f \equiv_{\omega^*} {}& \gamma_{\{1,13\}}d_1 + \gamma_{\{2,14\}}d_2 + \gamma_{\{3,15,25\}}d_3 + \gamma_{\{4,16,26\}}d_4 + \gamma_{\{5,17\}}d_5\\
&+ \gamma_{\{6,18,27\}}d_6 + \gamma_{\{7,25\}}d_7 + \gamma_{\{8,26\}}d_8 + \gamma_{\{9,27\}}d_9 + \gamma_{\{10,49\}}d_{10}
\end{aligned}$$

$$
\begin{aligned}
&+ \gamma_{\{11,50\}}d_{11} + \gamma_{\{12,51\}}d_{12} + \gamma_{\{1,13\}}d_{13} + \gamma_{\{2,14\}}d_{14} + \gamma_{\{3,7,15\}}d_{15}\\
&+ \gamma_{\{4,8,16\}}d_{16} + \gamma_{\{5,17\}}d_{17} + \gamma_{\{6,9,18\}}d_{18} + \gamma_{30}d_{19} + \gamma_{31}d_{20}\\
&+ \gamma_{32}d_{21} + \gamma_{33}d_{22} + \gamma_{37}d_{23} + \gamma_{38}d_{24} + \gamma_{\{7,25\}}d_{25} + \gamma_{\{8,26\}}d_{26}\\
&+ \gamma_{\{9,27\}}d_{27} + \gamma_{\{40,52\}}d_{40} + \gamma_{\{41,53\}}d_{41} + \gamma_{\{42,54\}}d_{42} + \gamma_{\{43,55\}}d_{43}\\
&+ \gamma_{\{44,56\}}d_{44} + \gamma_{\{45,57\}}d_{45} + \gamma_{\{46,58\}}d_{46} + \gamma_{\{47,59\}}d_{47} + \gamma_{\{48,60\}}d_{48}\\
&+ \gamma_{\{10,49\}}d_{49} + \gamma_{\{11,50\}}d_{50} + \gamma_{\{12,51\}}d_{51} + \gamma_{\{40,52\}}d_{52} + \gamma_{\{41,53\}}d_{53}\\
&+ \gamma_{\{42,54\}}d_{54} + \gamma_{\{43,55\}}d_{55} + \gamma_{\{44,56\}}d_{56} + \gamma_{\{45,57\}}d_{57} + \gamma_{\{46,58\}}d_{58}\\
&+ \gamma_{\{47,59\}}d_{59} + \gamma_{\{48,60\}}d_{60} \equiv_{\omega^*} 0,
\end{aligned}
$$

$$
\begin{aligned}
\rho_2(f) + f \equiv_{\omega^*}\ & \gamma_{\{3,7\}}d_3 + \gamma_{\{4,8\}}d_4 + \gamma_{\{5,10\}}d_5 + \gamma_{\{6,11\}}d_6 + \gamma_{\{3,7\}}d_7 + \gamma_{\{4,8\}}d_8\\
&+ \gamma_{\{9,12\}}d_9 + \gamma_{\{5,10\}}d_{10} + \gamma_{\{6,11\}}d_{11} + \gamma_{\{9,12\}}d_{12} + \gamma_{\{13,28,30\}}d_{13}\\
&+ \gamma_{\{14,29,31\}}d_{14} + \gamma_{\{15,30\}}d_{15} + \gamma_{\{16,31\}}d_{16} + \gamma_{\{17,40\}}d_{17}\\
&+ \gamma_{\{18,41\}}d_{18} + \gamma_{\{19,25\}}d_{19} + \gamma_{\{20,26\}}d_{20} + \gamma_{\{21,32,34\}}d_{21}\\
&+ \gamma_{\{22,33,35\}}d_{22} + \gamma_{\{23,42\}}d_{23} + \gamma_{\{24,43\}}d_{24} + \gamma_{\{19,25\}}d_{25}\\
&+ \gamma_{\{20,26\}}d_{26} + \gamma_{\{27,44\}}d_{27} + \gamma_{\{13,15,28\}}d_{28} + \gamma_{\{14,16,29\}}d_{29}\\
&+ \gamma_{\{15,30\}}d_{30} + \gamma_{\{16,31\}}d_{31} + \gamma_{\{21,32,34\}}d_{32} + \gamma_{\{22,33,35\}}d_{33}\\
&+ \gamma_{\{36,45\}}d_{36} + \gamma_{\{37,46\}}d_{37} + \gamma_{\{38,47\}}d_{38} + \gamma_{\{39,48\}}d_{39} + \gamma_{\{17,40\}}d_{40}\\
&+ \gamma_{\{18,41\}}d_{41} + \gamma_{\{23,42\}}d_{42} + \gamma_{\{24,43\}}d_{43} + \gamma_{\{27,44\}}d_{44} + \gamma_{\{36,45\}}d_{45}\\
&+ \gamma_{\{37,46\}}d_{46} + \gamma_{\{38,47\}}d_{47} + \gamma_{\{39,48\}}d_{48} + \gamma_{\{49,52\}}d_{49}\\
&+ \gamma_{\{50,53,56\}}d_{50} + \gamma_{\{51,56\}}d_{51} + \gamma_{\{49,52\}}d_{52} + \gamma_{\{50,51,53\}}d_{53}\\
&+ \gamma_{58}d_{54} + \gamma_{59}d_{55} + \gamma_{\{51,56\}}d_{56} \equiv_{\omega^*} 0,
\end{aligned}
$$

$$
\begin{aligned}
\rho_3(f) + f \equiv_{\omega^*}\ & \gamma_{\{1,3\}}d_1 + \gamma_{\{2,5\}}d_2 + \gamma_{\{1,3\}}d_3 + \gamma_{\{4,6\}}d_4 + \gamma_{\{2,5\}}d_5 + \gamma_{\{4,6\}}d_6\\
&+ \gamma_{\{8,9\}}d_8 + \gamma_{\{8,9\}}d_9 + \gamma_{\{11,12\}}d_{11} + \gamma_{\{11,12\}}d_{12} + \gamma_{\{13,15,25\}}d_{13}\\
&+ \gamma_{\{14,17\}}d_{14} + \gamma_{\{13,15,25\}}d_{15} + \gamma_{\{16,18\}}d_{16} + \gamma_{\{14,17\}}d_{17} + \gamma_{\{16,18\}}d_{18}\\
&+ \gamma_{\{19,21,32\}}d_{19} + \gamma_{\{20,23\}}d_{20} + \gamma_{\{19,21,30\}}d_{21} + \gamma_{\{22,24\}}d_{22}\\
&+ \gamma_{\{20,23\}}d_{23} + \gamma_{\{22,24\}}d_{24} + \gamma_{\{26,27\}}d_{26} + \gamma_{\{26,27\}}d_{27} + \gamma_{\{28,34\}}d_{28}\\
&+ \gamma_{\{29,36\}}d_{29} + \gamma_{\{30,32\}}d_{30} + \gamma_{\{31,37\}}d_{31} + \gamma_{\{30,32\}}d_{32} + \gamma_{\{33,38\}}d_{33}\\
&+ \gamma_{\{28,34\}}d_{34} + \gamma_{\{35,39\}}d_{35} + \gamma_{\{29,36\}}d_{36} + \gamma_{\{31,37\}}d_{37} + \gamma_{\{33,38\}}d_{38}\\
&+ \gamma_{\{35,39\}}d_{39} + \gamma_{\{40,45,46\}}d_{40} + \gamma_{\{41,46\}}d_{41} + \gamma_{\{42,44\}}d_{42}\\
&+ \gamma_{\{43,47,48\}}d_{43} + \gamma_{\{42,44\}}d_{44} + \gamma_{\{40,41,45\}}d_{45} + \gamma_{\{41,46\}}d_{46}\\
&+ \gamma_{\{43,47,48\}}d_{47} + \gamma_{\{50,51\}}d_{50} + \gamma_{\{50,51\}}d_{51} + \gamma_{\{52,57,58\}}d_{52}\\
&+ \gamma_{\{53,58\}}d_{53} + \gamma_{\{54,56\}}d_{54} + \gamma_{\{55,59,60\}}d_{55} + \gamma_{\{54,56\}}d_{56}\\
&+ \gamma_{\{52,53,57\}}d_{57} + \gamma_{\{53,58\}}d_{58} + \gamma_{\{55,59,60\}}d_{59} \equiv_{\omega^*} 0,
\end{aligned}
$$

$$
\begin{aligned}
\rho_4(f) + f \equiv_{\omega^*}\ & \gamma_{\{1,2\}}d_1 + \gamma_{\{1,2\}}d_2 + \gamma_{\{3,4\}}d_3 + \gamma_{\{3,4\}}d_4 + \gamma_{\{5,6\}}d_5 + \gamma_{\{5,6\}}d_6\\
&+ \gamma_{\{7,8\}}d_7 + \gamma_{\{7,8\}}d_8 + \gamma_{\{10,11\}}d_{10} + \gamma_{\{10,11\}}d_{11} + \gamma_{\{13,14\}}d_{13}
\end{aligned}
$$

$$\begin{aligned}
&+ \gamma_{\{13,14\}}d_{14} + \gamma_{\{15,16\}}d_{15} + \gamma_{\{15,16\}}d_{16} + \gamma_{\{17,18,27\}}d_{17}\\
&+ \gamma_{\{17,18,27\}}d_{18} + \gamma_{\{19,20\}}d_{19} + \gamma_{\{19,20\}}d_{20} + \gamma_{\{21,22\}}d_{21}\\
&+ \gamma_{\{21,22\}}d_{22} + \gamma_{\{23,24,38\}}d_{23} + \gamma_{\{23,24,37\}}d_{24} + \gamma_{\{25,26\}}d_{25}\\
&+ \gamma_{\{25,26\}}d_{26} + \gamma_{\{28,29\}}d_{28} + \gamma_{\{28,29\}}d_{29} + \gamma_{\{30,31\}}d_{30} + \gamma_{\{30,31\}}d_{31}\\
&+ \gamma_{\{32,33\}}d_{32} + \gamma_{\{32,33\}}d_{33} + \gamma_{\{34,35\}}d_{34} + \gamma_{\{34,35\}}d_{35} + \gamma_{\{36,39\}}d_{36}\\
&+ \gamma_{\{37,38\}}d_{37} + \gamma_{\{37,38\}}d_{38} + \gamma_{\{36,39\}}d_{39} + \gamma_{\{40,41,44\}}d_{40}\\
&+ \gamma_{\{40,41,44\}}d_{41} + \gamma_{\{42,43,47\}}d_{42} + \gamma_{\{42,43,46\}}d_{43} + \gamma_{\{45,48\}}d_{45}\\
&+ \gamma_{\{46,47\}}d_{46} + \gamma_{\{46,47\}}d_{47} + \gamma_{\{45,48\}}d_{48} + \gamma_{\{49,50\}}d_{49} + \gamma_{\{49,50\}}d_{50}\\
&+ \gamma_{\{52,53,56\}}d_{52} + \gamma_{\{52,53,56\}}d_{53} + \gamma_{\{54,55,59\}}d_{54} + \gamma_{\{54,55,58\}}d_{55}\\
&+ \gamma_{\{57,60\}}d_{57} + \gamma_{\{58,59\}}d_{58} + \gamma_{\{58,59\}}d_{59} + \gamma_{\{57,60\}}d_{60} \equiv_{\omega^*} 0.
\end{aligned}$$

From the above equalities we get $\gamma_t = 0$ for $t \in \mathbb{J} = \{15, 16, 18, 30, 31, 32, 33, 37, 38, 41, 46, 47, 53, 58, 59\}$ and $\gamma_t = \gamma_1$ for $t \notin \mathbb{J}$. Hence $f \equiv_{\omega^*} q_4$. The lemma ia proved. □

**Proposition 4.2.2.** *We have* $QP_5((4)|(3)|^2|(1))^{GL_5} = 0$.

*Proof.* Let $h \in P_5(\omega^*)$ such that $[h]_{\omega^*} \in QP_5((4)|(3)|^2|(1))^{GL_5}$. Then we have $[h]_{\omega^*} \in QP_5((4)|(3)|^2|(1))^{\Sigma_5}$. By Lemma 4.2.1 we have $h \equiv_{\omega^*} \sum_{u=1}^{7} q_u$ with $\gamma_u \in \mathbb{F}_2$. By a direct computation we get

$$\begin{aligned}
\rho_5(h) + h \equiv_{\omega^*}\ &\gamma_1 x_2x_3^7x_4^7x_5^{15} + \gamma_2x_2^3x_3^5x_4^7x_5^{15} + \gamma_3x_1x_2^{15}x_3x_4^6x_5^7\\
&+ \gamma_4x_1x_2^{15}x_3^3x_4^5x_5^6 + \gamma_5x_1^3x_2^7x_3x_4^7x_5^{12} + \gamma_6x_1^3x_2^7x_3x_4^5x_5^{14}\\
&+ \gamma_7x_1^3x_2^7x_3^3x_4^5x_5^{12} + \text{ other terms } \equiv_{\omega^*} 0.
\end{aligned}$$

This equality implies $\gamma_u = 0$ for $1 \leqslant u \leqslant 7$. The proposition is proved. □

## 4.3 Proof of Theorem 1.4

It is not difficult to check that $\rho_t(q) + q \equiv 0$ for $1 \leqslant t \leqslant 5$. Hence, $[q] \in (QP_5)_{30}^{GL_5}$.

Suppose $g \in P_5$ and $[g] \in (QP_5)_{30}^{GL_5}$. By Proposition 4.2.2, $[g]_{(4,3,3,1)} = 0$, hence $[g]_{(2,4,3,1)} \in QP_5(2,4,3,1)^{GL_5}$. By using Proposition 4.1.3, we get

$$g \equiv \gamma x_1^3x_2^5x_3^6x_4^6x_5^{10} + \sum_{b\in B((2)^4)} \gamma_b.b,$$

where $\gamma$, $\gamma_b \in \mathbb{F}_2$ for all $b \in B((2)^4)$. By a direct computation, we see that $\rho_t(g) + g \equiv 0$ for $1 \leqslant t < 5$ if and only if

$$f \equiv \gamma(x_1^3x_2^5x_3^6x_4^6x_5^{10} + p_0) + \sum_{1\leqslant s\leqslant 9} \gamma_s p_s,$$

where $\gamma_s \in \mathbb{F}_2$ for $1 \leqslant s \leqslant 9$, and

$$\begin{aligned} p_0 = {} & x_1x_2^2x_3x_4^{14}x_5^{12} + x_1x_2^2x_3^3x_4^{12}x_5^{12} + x_1x_2^2x_3^{12}x_4x_5^{14} + x_1x_2^2x_3^{13}x_4^2x_5^{12} \\ & + x_1x_2^3x_3^2x_4^{12}x_5^{12} + x_1x_2^3x_3^4x_4^{10}x_5^{12} + x_1x_2^3x_3^6x_4^8x_5^{12} + x_1x_2^3x_3^{12}x_4^2x_5^{12} \\ & + x_1x_2^{14}x_3x_4^2x_5^{12} + x_1^3x_2x_3^4x_4^{10}x_5^{12} + x_1^3x_2x_3^6x_4^8x_5^{12} + x_1^3x_2x_3^{12}x_4^2x_5^{12} \\ & + x_1^3x_2^5x_3^2x_4^8x_5^{12}. \end{aligned}$$

Now, computing from the relation $\rho_5(g)+g \equiv 0$, we obtain $\gamma_s = \gamma$ for $1 \leqslant s \leqslant 9$. Then, we have

$$g \equiv \gamma \left( x_1^3x_2^5x_3^6x_4^6x_5^{10} + \sum_{0 \leqslant s \leqslant 9} p_s \right) = \gamma q.$$

The theorem is proved.

# 5 Appendix

## 5.1 The admissible monomials of degree 20 in $P_5$

In this subsection we list the needed admissible monomials of degree 20 in $P_5$. The elements in $B_5^0(20)$ can easily determined by using the results in [24] and the relation 2.3. We have $B_4(20) = \{w_t : 1 \leqslant t \leqslant 55\}$, where

| | | | |
|---|---|---|---|
| $v_1 = x_1x_2x_3^3x_4^{15}$ | $v_2 = x_1x_2x_3^7x_4^{11}$ | $v_3 = x_1x_2x_3^{15}x_4^3$ | $v_4 = x_1x_2^3x_3x_4^{15}$ |
| $v_5 = x_1x_2^3x_3^3x_4^{13}$ | $v_6 = x_1x_2^3x_3^5x_4^{11}$ | $v_7 = x_1x_2^3x_3^7x_4^9$ | $v_8 = x_1x_2^3x_3^{13}x_4^3$ |
| $v_9 = x_1x_2^3x_3^{15}x_4$ | $v_{10} = x_1x_2^7x_3x_4^{11}$ | $v_{11} = x_1x_2^7x_3^3x_4^9$ | $v_{12} = x_1x_2^7x_3^9x_4^3$ |
| $v_{13} = x_1x_2^7x_3^{11}x_4$ | $v_{14} = x_1x_2^{15}x_3x_4^3$ | $v_{15} = x_1x_2^{15}x_3^3x_4$ | $v_{16} = x_1^3x_2x_3x_4^{15}$ |
| $v_{17} = x_1^3x_2x_3^3x_4^{13}$ | $v_{18} = x_1^3x_2x_3^5x_4^{11}$ | $v_{19} = x_1^3x_2x_3^7x_4^9$ | $v_{20} = x_1^3x_2x_3^{13}x_4^3$ |
| $v_{21} = x_1^3x_2x_3^{15}x_4$ | $v_{22} = x_1^3x_2^3x_3x_4^{13}$ | $v_{23} = x_1^3x_2^3x_3^5x_4^9$ | $v_{24} = x_1^3x_2^3x_3^{13}x_4$ |
| $v_{25} = x_1^3x_2^5x_3x_4^{11}$ | $v_{26} = x_1^3x_2^5x_3^3x_4^9$ | $v_{27} = x_1^3x_2^5x_3^9x_4^3$ | $v_{28} = x_1^3x_2^5x_3^{11}x_4$ |
| $v_{29} = x_1^3x_2^7x_3x_4^9$ | $v_{30} = x_1^3x_2^7x_3^9x_4$ | $v_{31} = x_1^3x_2^{13}x_3x_4^3$ | $v_{32} = x_1^3x_2^{13}x_3^3x_4$ |
| $v_{33} = x_1^3x_2^{15}x_3x_4$ | $v_{34} = x_1^7x_2x_3x_4^{11}$ | $v_{35} = x_1^7x_2x_3^3x_4^9$ | $v_{36} = x_1^7x_2x_3^9x_4^3$ |
| $v_{37} = x_1^7x_2x_3^{11}x_4$ | $v_{38} = x_1^7x_2^3x_3x_4^9$ | $v_{39} = x_1^7x_2^3x_3^9x_4$ | $v_{40} = x_1^7x_2^9x_3x_4^3$ |
| $v_{41} = x_1^7x_2^9x_3^3x_4$ | $v_{42} = x_1^7x_2^{11}x_3x_4$ | $v_{43} = x_1^{15}x_2x_3x_4^3$ | $v_{44} = x_1^{15}x_2x_3^3x_4$ |
| $v_{45} = x_1^{15}x_2^3x_3x_4$ | $v_{46} = x_1^3x_2^5x_3^5x_4^7$ | $v_{47} = x_1^3x_2^5x_3^7x_4^5$ | $v_{48} = x_1^3x_2^7x_3^5x_4^5$ |
| $v_{49} = x_1^7x_2^3x_3^5x_4^5$ | $v_{50} = x_1^3x_2^3x_3^7x_4^7$ | $v_{51} = x_1^3x_2^7x_3^3x_4^7$ | $v_{52} = x_1^3x_2^7x_3^7x_4^3$ |
| $v_{53} = x_1^7x_2^3x_3^3x_4^7$ | $v_{54} = x_1^7x_2^3x_3^7x_4^3$ | $v_{55} = x_1^7x_2^7x_3^3x_4^3$ | |

Note that $\omega(w_t) = (4)|(2)|(1)|^2$ for $1 \leqslant t \leqslant 45$, $\omega(w_t) = (4)|(2)|(3)$ for $46 \leqslant t \leqslant 49$ and $\omega(w_t) = (4)|^2|(2)$ for $50 \leqslant t \leqslant 55$. Hence, we get $|B_5^0((4)|(2)|(1)|^2)| = 225$, $|B_5^0((4)|(2)|(3))| = 20$, $|B_5^0((4)|^2|(2))| = 30$ and $|B_5^0(20)| = 275$. So, we list only the elements in

$$B_5^+(20) = B_5^+((4)|(2)|(1)|^2) \bigcup B_5^+((4)|(2)|(3)) \bigcup B_5^+((4)|^2|(2)).$$

$B_5^+((4)|(2)|(1)|^2) = \{a_t = a_t : 1 \leqslant t \leqslant 225\}$, where

| | | | |
|---|---|---|---|
| 1. $x_1x_2x_3x_4^2x_5^{15}$ | 2. $x_1x_2x_3x_4^{15}x_5^2$ | 3. $x_1x_2x_3^2x_4x_5^{15}$ | 4. $x_1x_2x_3^2x_4^{15}x_5$ |
| 5. $x_1x_2x_3^{15}x_4x_5^2$ | 6. $x_1x_2x_3^{15}x_4^2x_5$ | 7. $x_1x_2^2x_3x_4x_5^{15}$ | 8. $x_1x_2^2x_3x_4^{15}x_5$ |
| 9. $x_1x_2^2x_3^{15}x_4x_5$ | 10. $x_1x_2^{15}x_3x_4x_5^2$ | 11. $x_1x_2^{15}x_3x_4^2x_5$ | 12. $x_1x_2^{15}x_3^2x_4x_5$ |
| 13. $x_1^{15}x_2x_3x_4x_5^2$ | 14. $x_1^{15}x_2x_3x_4^2x_5$ | 15. $x_1^{15}x_2x_3^2x_4x_5$ | 16. $x_1x_2x_3x_4^3x_5^{14}$ |
| 17. $x_1x_2x_3x_4^6x_5^{11}$ | 18. $x_1x_2x_3x_4^7x_5^{10}$ | 19. $x_1x_2x_3x_4^{14}x_5^3$ | 20. $x_1x_2x_3^2x_4^3x_5^{13}$ |
| 21. $x_1x_2x_3^2x_4^5x_5^{11}$ | 22. $x_1x_2x_3^2x_4^7x_5^9$ | 23. $x_1x_2x_3^2x_4^{13}x_5^3$ | 24. $x_1x_2x_3^3x_4x_5^{14}$ |
| 25. $x_1x_2x_3^3x_4^2x_5^{13}$ | 26. $x_1x_2x_3^3x_4^3x_5^{12}$ | 27. $x_1x_2x_3^3x_4^4x_5^{11}$ | 28. $x_1x_2x_3^3x_4^5x_5^{10}$ |
| 29. $x_1x_2x_3^3x_4^6x_5^9$ | 30. $x_1x_2x_3^3x_4^7x_5^8$ | 31. $x_1x_2x_3^3x_4^{12}x_5^3$ | 32. $x_1x_2x_3^3x_4^{13}x_5^2$ |
| 33. $x_1x_2x_3^3x_4^{14}x_5$ | 34. $x_1x_2x_3^6x_4x_5^{11}$ | 35. $x_1x_2x_3^6x_4^3x_5^9$ | 36. $x_1x_2x_3^6x_4^9x_5^3$ |
| 37. $x_1x_2x_3^6x_4^{11}x_5$ | 38. $x_1x_2x_3^7x_4x_5^{10}$ | 39. $x_1x_2x_3^7x_4^2x_5^9$ | 40. $x_1x_2x_3^7x_4^3x_5^8$ |
| 41. $x_1x_2x_3^7x_4^8x_5^3$ | 42. $x_1x_2x_3^7x_4^9x_5^2$ | 43. $x_1x_2x_3^7x_4^{10}x_5$ | 44. $x_1x_2x_3^{14}x_4x_5^3$ |
| 45. $x_1x_2x_3^{14}x_4^3x_5$ | 46. $x_1x_2^2x_3x_4^3x_5^{13}$ | 47. $x_1x_2^2x_3x_4^5x_5^{11}$ | 48. $x_1x_2^2x_3x_4^7x_5^9$ |
| 49. $x_1x_2^2x_3x_4^{13}x_5^3$ | 50. $x_1x_2^2x_3^3x_4x_5^{13}$ | 51. $x_1x_2^2x_3^3x_4^5x_5^9$ | 52. $x_1x_2^2x_3^3x_4^{13}x_5$ |
| 53. $x_1x_2^2x_3^5x_4x_5^{11}$ | 54. $x_1x_2^2x_3^5x_4^3x_5^9$ | 55. $x_1x_2^2x_3^5x_4^9x_5^3$ | 56. $x_1x_2^2x_3^5x_4^{11}x_5$ |
| 57. $x_1x_2^2x_3^7x_4x_5^9$ | 58. $x_1x_2^2x_3^7x_4^9x_5$ | 59. $x_1x_2^2x_3^{13}x_4x_5^3$ | 60. $x_1x_2^2x_3^{13}x_4^3x_5$ |
| 61. $x_1x_2^3x_3x_4x_5^{14}$ | 62. $x_1x_2^3x_3x_4^2x_5^{13}$ | 63. $x_1x_2^3x_3x_4^3x_5^{12}$ | 64. $x_1x_2^3x_3x_4^4x_5^{11}$ |
| 65. $x_1x_2^3x_3x_4^5x_5^{10}$ | 66. $x_1x_2^3x_3x_4^6x_5^9$ | 67. $x_1x_2^3x_3x_4^7x_5^8$ | 68. $x_1x_2^3x_3x_4^{12}x_5^3$ |
| 69. $x_1x_2^3x_3x_4^{13}x_5^2$ | 70. $x_1x_2^3x_3x_4^{14}x_5$ | 71. $x_1x_2^3x_3^2x_4x_5^{13}$ | 72. $x_1x_2^3x_3^2x_4^5x_5^9$ |
| 73. $x_1x_2^3x_3^2x_4^{13}x_5$ | 74. $x_1x_2^3x_3^3x_4x_5^{12}$ | 75. $x_1x_2^3x_3^3x_4^4x_5^9$ | 76. $x_1x_2^3x_3^3x_4^5x_5^8$ |
| 77. $x_1x_2^3x_3^3x_4^{12}x_5$ | 78. $x_1x_2^3x_3^4x_4x_5^{11}$ | 79. $x_1x_2^3x_3^4x_4^3x_5^9$ | 80. $x_1x_2^3x_3^4x_4^9x_5^3$ |
| 81. $x_1x_2^3x_3^4x_4^{11}x_5$ | 82. $x_1x_2^3x_3^5x_4x_5^{10}$ | 83. $x_1x_2^3x_3^5x_4^2x_5^9$ | 84. $x_1x_2^3x_3^5x_4^3x_5^8$ |
| 85. $x_1x_2^3x_3^5x_4^8x_5^3$ | 86. $x_1x_2^3x_3^5x_4^9x_5^2$ | 87. $x_1x_2^3x_3^5x_4^{10}x_5$ | 88. $x_1x_2^3x_3^6x_4x_5^9$ |
| 89. $x_1x_2^3x_3^6x_4^9x_5$ | 90. $x_1x_2^3x_3^7x_4x_5^8$ | 91. $x_1x_2^3x_3^7x_4^8x_5$ | 92. $x_1x_2^3x_3^{12}x_4x_5^3$ |
| 93. $x_1x_2^3x_3^{12}x_4^3x_5$ | 94. $x_1x_2^3x_3^{13}x_4x_5^2$ | 95. $x_1x_2^3x_3^{13}x_4^2x_5$ | 96. $x_1x_2^3x_3^{14}x_4x_5$ |
| 97. $x_1x_2^6x_3x_4x_5^{11}$ | 98. $x_1x_2^6x_3x_4^3x_5^9$ | 99. $x_1x_2^6x_3x_4^9x_5^3$ | 100. $x_1x_2^6x_3x_4^{11}x_5$ |
| 101. $x_1x_2^6x_3^3x_4x_5^9$ | 102. $x_1x_2^6x_3^3x_4^9x_5$ | 103. $x_1x_2^6x_3^9x_4x_5^3$ | 104. $x_1x_2^6x_3^9x_4^3x_5$ |
| 105. $x_1x_2^6x_3^{11}x_4x_5$ | 106. $x_1x_2^7x_3x_4x_5^{10}$ | 107. $x_1x_2^7x_3x_4^2x_5^9$ | 108. $x_1x_2^7x_3x_4^3x_5^8$ |
| 109. $x_1x_2^7x_3x_4^8x_5^3$ | 110. $x_1x_2^7x_3x_4^9x_5^2$ | 111. $x_1x_2^7x_3x_4^{10}x_5$ | 112. $x_1x_2^7x_3^2x_4x_5^9$ |
| 113. $x_1x_2^7x_3^2x_4^9x_5$ | 114. $x_1x_2^7x_3^3x_4x_5^8$ | 115. $x_1x_2^7x_3^3x_4^8x_5$ | 116. $x_1x_2^7x_3^8x_4x_5^3$ |
| 117. $x_1x_2^7x_3^8x_4^3x_5$ | 118. $x_1x_2^7x_3^9x_4x_5^2$ | 119. $x_1x_2^7x_3^9x_4^2x_5$ | 120. $x_1x_2^7x_3^{10}x_4x_5$ |
| 121. $x_1x_2^{14}x_3x_4x_5^3$ | 122. $x_1x_2^{14}x_3x_4^3x_5$ | 123. $x_1x_2^{14}x_3^3x_4x_5$ | 124. $x_1^3x_2x_3x_4x_5^{14}$ |
| 125. $x_1^3x_2x_3x_4^2x_5^{13}$ | 126. $x_1^3x_2x_3x_4^3x_5^{12}$ | 127. $x_1^3x_2x_3x_4^4x_5^{11}$ | 128. $x_1^3x_2x_3x_4^5x_5^{10}$ |
| 129. $x_1^3x_2x_3x_4^6x_5^9$ | 130. $x_1^3x_2x_3x_4^7x_5^8$ | 131. $x_1^3x_2x_3x_4^{12}x_5^3$ | 132. $x_1^3x_2x_3x_4^{13}x_5^2$ |
| 133. $x_1^3x_2x_3x_4^{14}x_5$ | 134. $x_1^3x_2x_3^2x_4x_5^{13}$ | 135. $x_1^3x_2x_3^2x_4^5x_5^9$ | 136. $x_1^3x_2x_3^2x_4^{13}x_5$ |
| 137. $x_1^3x_2x_3^3x_4x_5^{12}$ | 138. $x_1^3x_2x_3^3x_4^4x_5^9$ | 139. $x_1^3x_2x_3^3x_4^5x_5^8$ | 140. $x_1^3x_2x_3^3x_4^{12}x_5$ |
| 141. $x_1^3x_2x_3^4x_4x_5^{11}$ | 142. $x_1^3x_2x_3^4x_4^3x_5^9$ | 143. $x_1^3x_2x_3^4x_4^9x_5^3$ | 144. $x_1^3x_2x_3^4x_4^{11}x_5$ |
| 145. $x_1^3x_2x_3^5x_4x_5^{10}$ | 146. $x_1^3x_2x_3^5x_4^2x_5^9$ | 147. $x_1^3x_2x_3^5x_4^3x_5^8$ | 148. $x_1^3x_2x_3^5x_4^8x_5^3$ |
| 149. $x_1^3x_2x_3^5x_4^9x_5^2$ | 150. $x_1^3x_2x_3^5x_4^{10}x_5$ | 151. $x_1^3x_2x_3^6x_4x_5^9$ | 152. $x_1^3x_2x_3^6x_4^9x_5$ |
| 153. $x_1^3x_2x_3^7x_4x_5^8$ | 154. $x_1^3x_2x_3^7x_4^8x_5$ | 155. $x_1^3x_2x_3^{12}x_4x_5^3$ | 156. $x_1^3x_2x_3^{12}x_4^3x_5$ |
| 157. $x_1^3x_2x_3^{13}x_4x_5^2$ | 158. $x_1^3x_2x_3^{13}x_4^2x_5$ | 159. $x_1^3x_2x_3^{14}x_4x_5$ | 160. $x_1^3x_2^3x_3x_4x_5^{12}$ |
| 161. $x_1^3x_2^3x_3x_4^4x_5^9$ | 162. $x_1^3x_2^3x_3x_4^5x_5^8$ | 163. $x_1^3x_2^3x_3x_4^{12}x_5$ | 164. $x_1^3x_2^3x_3^4x_4x_5^9$ |
| 165. $x_1^3x_2^3x_3^4x_4^9x_5$ | 166. $x_1^3x_2^3x_3^5x_4x_5^8$ | 167. $x_1^3x_2^3x_3^5x_4^8x_5$ | 168. $x_1^3x_2^3x_3^{12}x_4x_5$ |

| | | | |
|---|---|---|---|
| 169. $x_1^3x_2^4x_3x_4x_5^{11}$ | 170. $x_1^3x_2^4x_3x_4^3x_5^9$ | 171. $x_1^3x_2^4x_3x_4^{11}x_5$ | 172. $x_1^3x_2^4x_3^3x_4x_5^9$ |
| 173. $x_1^3x_2^4x_3^3x_4^9x_5$ | 174. $x_1^3x_2^4x_3^{11}x_4x_5$ | 175. $x_1^3x_2^5x_3x_4x_5^{10}$ | 176. $x_1^3x_2^5x_3x_4^2x_5^9$ |
| 177. $x_1^3x_2^5x_3x_4^3x_5^8$ | 178. $x_1^3x_2^5x_3x_4^8x_5^3$ | 179. $x_1^3x_2^5x_3x_4^9x_5^2$ | 180. $x_1^3x_2^5x_3x_4^{10}x_5$ |
| 181. $x_1^3x_2^5x_3^2x_4x_5^9$ | 182. $x_1^3x_2^5x_3^2x_4^9x_5$ | 183. $x_1^3x_2^5x_3^3x_4x_5^8$ | 184. $x_1^3x_2^5x_3^3x_4^8x_5$ |
| 185. $x_1^3x_2^5x_3^8x_4x_5^3$ | 186. $x_1^3x_2^5x_3^8x_4^3x_5$ | 187. $x_1^3x_2^5x_3^9x_4x_5^2$ | 188. $x_1^3x_2^5x_3^9x_4^2x_5$ |
| 189. $x_1^3x_2^5x_3^{10}x_4x_5$ | 190. $x_1^3x_2^7x_3x_4x_5^8$ | 191. $x_1^3x_2^7x_3x_4^8x_5$ | 192. $x_1^3x_2^7x_3^8x_4x_5$ |
| 193. $x_1^3x_2^{13}x_3x_4x_5^2$ | 194. $x_1^3x_2^{13}x_3x_4^2x_5$ | 195. $x_1^3x_2^{13}x_3^2x_4x_5$ | 196. $x_1^7x_2x_3x_4x_5^{10}$ |
| 197. $x_1^7x_2x_3x_4^2x_5^9$ | 198. $x_1^7x_2x_3x_4^3x_5^8$ | 199. $x_1^7x_2x_3x_4^8x_5^3$ | 200. $x_1^7x_2x_3x_4^9x_5^2$ |
| 201. $x_1^7x_2x_3x_4^{10}x_5$ | 202. $x_1^7x_2x_3^2x_4x_5^9$ | 203. $x_1^7x_2x_3^2x_4^9x_5$ | 204. $x_1^7x_2x_3^3x_4x_5^8$ |
| 205. $x_1^7x_2x_3^3x_4^8x_5$ | 206. $x_1^7x_2x_3^8x_4x_5^3$ | 207. $x_1^7x_2x_3^8x_4^3x_5$ | 208. $x_1^7x_2x_3^9x_4x_5^2$ |
| 209. $x_1^7x_2x_3^9x_4^2x_5$ | 210. $x_1^7x_2x_3^{10}x_4x_5$ | 211. $x_1^7x_2^3x_3x_4x_5^8$ | 212. $x_1^7x_2^3x_3x_4^8x_5$ |
| 213. $x_1^7x_2^3x_3^8x_4x_5$ | 214. $x_1^7x_2^9x_3x_4x_5^2$ | 215. $x_1^7x_2^9x_3x_4^2x_5$ | 216. $x_1^3x_2^4x_3x_4^9x_5^3$ |
| 217. $x_1^3x_2^4x_3^9x_4x_5^3$ | 218. $x_1^3x_2^4x_3^9x_4^3x_5$ | 219. $x_1^3x_2^{12}x_3x_4x_5^3$ | 220. $x_1^3x_2^{12}x_3x_4^3x_5$ |
| 221. $x_1^3x_2^{12}x_3^3x_4x_5$ | 222. $x_1^7x_2^8x_3x_4x_5^3$ | 223. $x_1^7x_2^8x_3x_4^3x_5$ | 224. $x_1^7x_2^8x_3^3x_4x_5$ |
| 225. $x_1^7x_2^9x_3^2x_4x_5$ | | | |

We have $B_5^+(4,2,3) = \{b_t : 1 \leqslant t \leqslant 50\}$, where

| | | | |
|---|---|---|---|
| 1. $x_1x_2^2x_3^5x_4^5x_5^7$ | 2. $x_1x_2^2x_3^5x_4^7x_5^5$ | 3. $x_1x_2^2x_3^7x_4^5x_5^5$ | 4. $x_1x_2^7x_3^2x_4^5x_5^5$ |
| 5. $x_1^7x_2x_3^2x_4^5x_5^5$ | 6. $x_1x_2^3x_3^4x_4^5x_5^7$ | 7. $x_1x_2^3x_3^4x_4^7x_5^5$ | 8. $x_1x_2^3x_3^5x_4^4x_5^7$ |
| 9. $x_1x_2^3x_3^5x_4^7x_5^4$ | 10. $x_1x_2^3x_3^7x_4^4x_5^5$ | 11. $x_1x_2^3x_3^7x_4^5x_5^4$ | 12. $x_1x_2^7x_3^3x_4^4x_5^5$ |
| 13. $x_1x_2^7x_3^3x_4^5x_5^4$ | 14. $x_1^3x_2x_3^4x_4^5x_5^7$ | 15. $x_1^3x_2x_3^4x_4^7x_5^5$ | 16. $x_1^3x_2x_3^5x_4^4x_5^7$ |
| 17. $x_1^3x_2x_3^5x_4^7x_5^4$ | 18. $x_1^3x_2x_3^7x_4^4x_5^5$ | 19. $x_1^3x_2x_3^7x_4^5x_5^4$ | 20. $x_1^3x_2^5x_3x_4^4x_5^7$ |
| 21. $x_1^3x_2^5x_3x_4^7x_5^4$ | 22. $x_1^3x_2^5x_3^7x_4x_5^4$ | 23. $x_1^3x_2^7x_3x_4^4x_5^5$ | 24. $x_1^3x_2^7x_3x_4^5x_5^4$ |
| 25. $x_1^3x_2^7x_3^5x_4x_5^4$ | 26. $x_1^7x_2x_3^3x_4^4x_5^5$ | 27. $x_1^7x_2x_3^3x_4^5x_5^4$ | 28. $x_1^7x_2^3x_3x_4^4x_5^5$ |
| 29. $x_1^7x_2^3x_3x_4^5x_5^4$ | 30. $x_1^7x_2^3x_3^5x_4x_5^4$ | 31. $x_1x_2^3x_3^5x_4^5x_5^6$ | 32. $x_1x_2^3x_3^5x_4^6x_5^5$ |
| 33. $x_1x_2^3x_3^6x_4^5x_5^5$ | 34. $x_1x_2^6x_3^3x_4^5x_5^5$ | 35. $x_1^3x_2x_3^5x_4^5x_5^6$ | 36. $x_1^3x_2x_3^5x_4^6x_5^5$ |
| 37. $x_1^3x_2x_3^6x_4^5x_5^5$ | 38. $x_1^3x_2^5x_3x_4^5x_5^6$ | 39. $x_1^3x_2^5x_3x_4^6x_5^5$ | 40. $x_1^3x_2^5x_3^5x_4x_5^6$ |
| 41. $x_1^3x_2^5x_3^2x_4^5x_5^5$ | 42. $x_1^3x_2^5x_3^5x_4^5x_5^2$ | 43. $x_1^3x_2^3x_3^4x_4^5x_5^5$ | 44. $x_1^3x_2^3x_3^5x_4^4x_5^5$ |
| 45. $x_1^3x_2^3x_3^5x_4^5x_5^4$ | 46. $x_1^3x_2^4x_3^3x_4^5x_5^5$ | 47. $x_1^3x_2^5x_3^3x_4^4x_5^5$ | 48. $x_1^3x_2^5x_3^3x_4^5x_5^4$ |
| 49. $x_1^3x_2^5x_3^5x_4^3x_5^4$ | 50. $x_1^3x_2^5x_3^5x_4^4x_5^3$ | | |

Note that these monomials have also been explicitly determined in [15].

$B_5^+(4,4,2) = \{c_t : 1 \leqslant t \leqslant 91\}$, where

| | | | |
|---|---|---|---|
| 1. $x_1x_2^2x_3^3x_4^7x_5^7$ | 2. $x_1x_2^2x_3^7x_4^3x_5^7$ | 3. $x_1x_2^2x_3^7x_4^7x_5^3$ | 4. $x_1x_2^3x_3^2x_4^7x_5^7$ |
| 5. $x_1x_2^3x_3^7x_4^2x_5^7$ | 6. $x_1x_2^3x_3^7x_4^7x_5^2$ | 7. $x_1x_2^7x_3^2x_4^3x_5^7$ | 8. $x_1x_2^7x_3^2x_4^7x_5^3$ |
| 9. $x_1x_2^7x_3^3x_4^2x_5^7$ | 10. $x_1x_2^7x_3^3x_4^7x_5^2$ | 11. $x_1x_2^7x_3^7x_4^2x_5^3$ | 12. $x_1x_2^7x_3^7x_4^3x_5^2$ |
| 13. $x_1^3x_2x_3^2x_4^7x_5^7$ | 14. $x_1^3x_2x_3^7x_4^2x_5^7$ | 15. $x_1^3x_2x_3^7x_4^7x_5^2$ | 16. $x_1^3x_2^7x_3x_4^2x_5^7$ |
| 17. $x_1^3x_2^7x_3x_4^7x_5^2$ | 18. $x_1^3x_2^7x_3^7x_4x_5^2$ | 19. $x_1^7x_2x_3^2x_4^3x_5^7$ | 20. $x_1^7x_2x_3^2x_4^7x_5^3$ |
| 21. $x_1^7x_2x_3^3x_4^2x_5^7$ | 22. $x_1^7x_2x_3^3x_4^7x_5^2$ | 23. $x_1^7x_2x_3^7x_4^2x_5^3$ | 24. $x_1^7x_2x_3^7x_4^3x_5^2$ |
| 25. $x_1^7x_2^3x_3x_4^2x_5^7$ | 26. $x_1^7x_2^3x_3x_4^7x_5^2$ | 27. $x_1^7x_2^3x_3^7x_4x_5^2$ | 28. $x_1^7x_2^7x_3x_4^2x_5^3$ |
| 29. $x_1^7x_2^7x_3x_4^3x_5^2$ | 30. $x_1^7x_2^7x_3^3x_4x_5^2$ | 31. $x_1x_2^3x_3^3x_4^6x_5^7$ | 32. $x_1x_2^3x_3^3x_4^7x_5^6$ |
| 33. $x_1x_2^3x_3^6x_4^3x_5^7$ | 34. $x_1x_2^3x_3^6x_4^7x_5^3$ | 35. $x_1x_2^3x_3^7x_4^3x_5^6$ | 36. $x_1x_2^3x_3^7x_4^6x_5^3$ |
| 37. $x_1x_2^6x_3^3x_4^3x_5^7$ | 38. $x_1x_2^6x_3^3x_4^7x_5^3$ | 39. $x_1x_2^6x_3^7x_4^3x_5^3$ | 40. $x_1x_2^7x_3^3x_4^3x_5^6$ |

| | | | |
|---|---|---|---|
| 41. $x_1x_2^7x_3^3x_4^6x_5^3$ | 42. $x_1x_2^7x_3^6x_4^3x_5^3$ | 43. $x_1^3x_2x_3^3x_4^6x_5^7$ | 44. $x_1^3x_2x_3^3x_4^7x_5^6$ |
| 45. $x_1^3x_2x_3^6x_4^3x_5^7$ | 46. $x_1^3x_2x_3^6x_4^7x_5^3$ | 47. $x_1^3x_2x_3^7x_4^3x_5^6$ | 48. $x_1^3x_2x_3^7x_4^6x_5^3$ |
| 49. $x_1^3x_2^3x_3x_4^6x_5^7$ | 50. $x_1^3x_2^3x_3x_4^7x_5^6$ | 51. $x_1^3x_2^3x_3^7x_4x_5^6$ | 52. $x_1^3x_2^7x_3x_4^3x_5^6$ |
| 53. $x_1^3x_2^7x_3x_4^6x_5^3$ | 54. $x_1^3x_2^7x_3^3x_4x_5^6$ | 55. $x_1^7x_2x_3^3x_4^3x_5^6$ | 56. $x_1^7x_2x_3^3x_4^6x_5^3$ |
| 57. $x_1^7x_2x_3^6x_4^3x_5^3$ | 58. $x_1^7x_2^3x_3x_4^3x_5^6$ | 59. $x_1^7x_2^3x_3x_4^6x_5^3$ | 60. $x_1^7x_2^3x_3^3x_4x_5^6$ |
| 61. $x_1^3x_2^3x_3^5x_4^2x_5^7$ | 62. $x_1^3x_2^3x_3^5x_4^7x_5^2$ | 63. $x_1^3x_2^3x_3^7x_4^5x_5^2$ | 64. $x_1^3x_2^5x_3^2x_4^3x_5^7$ |
| 65. $x_1^3x_2^5x_3^2x_4^7x_5^3$ | 66. $x_1^3x_2^5x_3^3x_4^2x_5^7$ | 67. $x_1^3x_2^5x_3^3x_4^7x_5^2$ | 68. $x_1^3x_2^5x_3^7x_4^2x_5^3$ |
| 69. $x_1^3x_2^5x_3^7x_4^3x_5^2$ | 70. $x_1^3x_2^7x_3^3x_4^5x_5^2$ | 71. $x_1^3x_2^7x_3^5x_4^2x_5^3$ | 72. $x_1^3x_2^7x_3^5x_4^3x_5^2$ |
| 73. $x_1^7x_2^3x_3^3x_4^5x_5^2$ | 74. $x_1^7x_2^3x_3^5x_4^2x_5^3$ | 75. $x_1^7x_2^3x_3^5x_4^3x_5^2$ | 76. $x_1^3x_2^3x_3^3x_4^4x_5^7$ |
| 77. $x_1^3x_2^3x_3^3x_4^7x_5^4$ | 78. $x_1^3x_2^3x_3^4x_4^3x_5^7$ | 79. $x_1^3x_2^3x_3^4x_4^7x_5^3$ | 80. $x_1^3x_2^3x_3^7x_4^3x_5^4$ |
| 81. $x_1^3x_2^3x_3^7x_4^4x_5^3$ | 82. $x_1^3x_2^7x_3^3x_4^3x_5^4$ | 83. $x_1^3x_2^7x_3^3x_4^4x_5^3$ | 84. $x_1^7x_2^3x_3^3x_4^3x_5^4$ |
| 85. $x_1^7x_2^3x_3^3x_4^4x_5^3$ | 86. $x_1^3x_2^3x_3^3x_4^5x_5^6$ | 87. $x_1^3x_2^3x_3^5x_4^3x_5^6$ | 88. $x_1^3x_2^3x_3^5x_4^6x_5^3$ |
| 89. $x_1^3x_2^5x_3^3x_4^3x_5^6$ | 90. $x_1^3x_2^5x_3^3x_4^6x_5^3$ | 91. $x_1^3x_2^5x_3^6x_4^3x_5^3$ | |

## 5.2 A basis of the space $(\widetilde{SF}_5)_{20}$

We have $(\widetilde{SF}_5)_{20} = \langle [p_u] : 1 \leqslant u \leqslant 11\rangle$, where

$$\begin{aligned}
g_1 &= x_1x_2x_3x_4^3x_5^{14} + x_1x_2x_3x_4^{14}x_5^3 + x_1x_2x_3^6x_4^3x_5^9 + x_1x_2x_3^6x_4^9x_5^3\\
&\quad + x_1x_2^6x_3x_4^3x_5^9 + x_1x_2^6x_3x_4^9x_5^3 + x_1^3x_2x_3x_4^3x_5^{12} + x_1^3x_2x_3x_4^5x_5^{10}\\
&\quad + x_1^3x_2x_3x_4^6x_5^9 + x_1^3x_2x_3x_4^{12}x_5^3 + x_1^3x_2x_3^4x_4^3x_5^9 + x_1^3x_2x_3^4x_4^9x_5^3\\
&\quad + x_1^3x_2^4x_3x_4^3x_5^9 + x_1^3x_2^4x_3x_4^9x_5^3,\\
g_2 &= x_1x_2x_3^3x_4x_5^{14} + x_1x_2x_3^3x_4^{12}x_5^3 + x_1x_2x_3^{14}x_4x_5^3 + x_1x_2^6x_3^3x_4x_5^9\\
&\quad + x_1x_2^6x_3^9x_4x_5^3 + x_1^3x_2x_3^3x_4x_5^{12} + x_1^3x_2x_3^3x_4^4x_5^9 + x_1^3x_2x_3^5x_4x_5^{10}\\
&\quad + x_1^3x_2x_3^5x_4^8x_5^3 + x_1^3x_2x_3^6x_4x_5^9 + x_1^3x_2x_3^{12}x_4x_5^3 + x_1^3x_2^4x_3^3x_4x_5^9\\
&\quad + x_1^3x_2^4x_3^9x_4x_5^3,\\
g_3 &= x_1x_2x_3^3x_4^3x_5^{12} + x_1x_2x_3^3x_4^{14}x_5 + x_1x_2x_3^{14}x_4^3x_5 + x_1x_2^6x_3^3x_4^9x_5\\
&\quad + x_1x_2^6x_3^9x_4^3x_5 + x_1^3x_2x_3^3x_4^5x_5^8 + x_1^3x_2x_3^3x_4^{12}x_5 + x_1^3x_2x_3^5x_4^3x_5^8\\
&\quad + x_1^3x_2x_3^5x_4^{10}x_5 + x_1^3x_2x_3^6x_4^9x_5 + x_1^3x_2x_3^{12}x_4^3x_5 + x_1^3x_2^4x_3^3x_4^9x_5\\
&\quad + x_1^3x_2^4x_3^9x_4^3x_5,\\
g_4 &= x_1x_2^3x_3x_4x_5^{14} + x_1x_2^3x_3x_4^{12}x_5^3 + x_1x_2^3x_3^{12}x_4x_5^3 + x_1x_2^{14}x_3x_4x_5^3\\
&\quad + x_1^3x_2^3x_3x_4x_5^{12} + x_1^3x_2^3x_3x_4^4x_5^9 + x_1^3x_2^3x_3^4x_4x_5^9 + x_1^3x_2^5x_3x_4^2x_5^9\\
&\quad + x_1^3x_2^5x_3x_4^8x_5^3 + x_1^3x_2^5x_3^2x_4x_5^9 + x_1^3x_2^5x_3^8x_4x_5^3 + x_1^3x_2^{12}x_3x_4x_5^3,\\
g_5 &= x_1x_2^3x_3x_4^3x_5^{12} + x_1x_2^3x_3x_4^{14}x_5 + x_1x_2^3x_3^{12}x_4^3x_5 + x_1x_2^{14}x_3x_4^3x_5\\
&\quad + x_1^3x_2^3x_3x_4^5x_5^8 + x_1^3x_2^3x_3x_4^{12}x_5 + x_1^3x_2^3x_3^4x_4^9x_5 + x_1^3x_2^5x_3x_4^3x_5^8\\
&\quad + x_1^3x_2^5x_3x_4^9x_5^2 + x_1^3x_2^5x_3^2x_4^9x_5 + x_1^3x_2^5x_3^8x_4^3x_5 + x_1^3x_2^{12}x_3x_4^3x_5,\\
g_6 &= x_1x_2^3x_3^3x_4x_5^{12} + x_1x_2^3x_3^3x_4^{12}x_5 + x_1x_2^3x_3^{14}x_4x_5 + x_1x_2^{14}x_3^3x_4x_5
\end{aligned}$$

$$
\begin{aligned}
&+ x_1^3x_2^3x_3^5x_4x_5^8 + x_1^3x_2^3x_3^5x_4^8x_5 + x_1^3x_2^3x_3^{12}x_4x_5 + x_1^3x_2^5x_3^3x_4x_5^8\\
&+ x_1^3x_2^5x_3^3x_4^8x_5 + x_1^3x_2^5x_3^9x_4x_5^2 + x_1^3x_2^5x_3^9x_4^2x_5 + x_1^3x_2^{12}x_3^3x_4x_5,\\
g_7 &= x_1^3x_2x_3x_4x_5^{14} + x_1^3x_2x_3x_4^{12}x_5^3 + x_1^3x_2x_3^{12}x_4x_5^3 + x_1^3x_2^{12}x_3x_4x_5^3\\
&+ x_1^7x_2x_3x_4x_5^{10} + x_1^7x_2x_3x_4^8x_5^3 + x_1^7x_2x_3^8x_4x_5^3 + x_1^7x_2^8x_3x_4x_5^3,\\
g_8 &= x_1^3x_2x_3x_4^3x_5^{12} + x_1^3x_2x_3x_4^{14}x_5 + x_1^3x_2x_3^{12}x_4^3x_5 + x_1^3x_2^{12}x_3x_4^3x_5\\
&+ x_1^7x_2x_3x_4^3x_5^8 + x_1^7x_2x_3x_4^{10}x_5 + x_1^7x_2x_3^8x_4^3x_5 + x_1^7x_2^8x_3x_4^3x_5,\\
g_9 &= x_1^3x_2x_3^3x_4x_5^{12} + x_1^3x_2x_3^3x_4^{12}x_5 + x_1^3x_2x_3^{14}x_4x_5 + x_1^3x_2^{12}x_3^3x_4x_5\\
&+ x_1^7x_2x_3^3x_4x_5^8 + x_1^7x_2x_3^3x_4^8x_5 + x_1^7x_2x_3^{10}x_4x_5 + x_1^7x_2^8x_3^3x_4x_5,\\
g_{10} &= x_1^3x_2^3x_3x_4x_5^{12} + x_1^3x_2^3x_3x_4^{12}x_5 + x_1^3x_2^3x_3^{12}x_4x_5 + x_1^3x_2^{13}x_3x_4x_5^2\\
&+ x_1^3x_2^{13}x_3x_4^2x_5 + x_1^3x_2^{13}x_3^2x_4x_5 + x_1^7x_2^3x_3x_4x_5^8 + x_1^7x_2^3x_3x_4^8x_5\\
&+ x_1^7x_2^3x_3^8x_4x_5 + x_1^7x_2^9x_3x_4x_5^2 + x_1^7x_2^9x_3x_4^2x_5 + x_1^7x_2^9x_3^2x_4x_5,\\
g_{11} &= x_1x_2^3x_3^5x_4^5x_5^6 + x_1x_2^3x_3^5x_4^6x_5^5 + x_1x_2^3x_3^6x_4^5x_5^5 + x_1x_2^6x_3^3x_4^5x_5^5\\
&+ x_1^3x_2x_3^5x_4^5x_5^6 + x_1^3x_2x_3^5x_4^6x_5^5 + x_1^3x_2^5x_3x_4^5x_5^6 + x_1^3x_2^5x_3x_4^6x_5^5\\
&+ x_1^3x_2^3x_3^4x_4^5x_5^5 + x_1^3x_2^3x_3^5x_4^4x_5^5 + x_1^3x_2^3x_3^5x_4^5x_5^4 + x_1^3x_2^4x_3^3x_4^5x_5^5\\
&+ x_1^3x_2^5x_3^3x_4^4x_5^5 + x_1^3x_2^5x_3^3x_4^5x_5^4 + x_1^3x_2^5x_3^5x_4^3x_5^4 + x_1^3x_2^5x_3^5x_4^4x_5^3.
\end{aligned}
$$

## 5.3 $\Sigma_5$-invariants of degree 20

$QP_5((4)|(2)|(1)|^2)^{\Sigma_5} = \langle [h_t] : 1 \leqslant t \leqslant 7\rangle$, where

$$
\begin{aligned}
h_1 &= x_2x_3x_4^{15}x_5^3 + x_2x_3x_4^3x_5^{15} + x_2x_3^{15}x_4x_5^3 + x_2x_3^{15}x_4^3x_5 + x_2x_3^3x_4x_5^{15}\\
&+ x_2x_3^3x_4^{15}x_5 + x_2^{15}x_3x_4x_5^3 + x_2^{15}x_3x_4^3x_5 + x_2^{15}x_3^3x_4x_5 + x_2^3x_3x_4x_5^{15}\\
&+ x_2^3x_3x_4^{15}x_5 + x_2^3x_3^{15}x_4x_5 + x_1x_3x_4^{15}x_5^3 + x_1x_3x_4^3x_5^{15} + x_1x_3^{15}x_4x_5^3\\
&+ x_1x_3^{15}x_4^3x_5 + x_1x_3^3x_4x_5^{15} + x_1x_3^3x_4^{15}x_5 + x_1x_2x_4^{15}x_5^3 + x_1x_2x_4^3x_5^{15}\\
&+ x_1x_2x_3^{15}x_5^3 + x_1x_2x_3^{15}x_4^3 + x_1x_2x_3^3x_5^{15} + x_1x_2x_3^3x_4^{15} + x_1x_2^{15}x_4x_5^3\\
&+ x_1x_2^{15}x_4^3x_5 + x_1x_2^{15}x_3x_5^3 + x_1x_2^{15}x_3x_4^3 + x_1x_2^{15}x_3^3x_5 + x_1x_2^{15}x_3^3x_4\\
&+ x_1x_2^3x_4x_5^{15} + x_1x_2^3x_4^{15}x_5 + x_1x_2^3x_3x_5^{15} + x_1x_2^3x_3x_4^{15} + x_1x_2^3x_3^{15}x_5\\
&+ x_1x_2^3x_3^{15}x_4 + x_1^{15}x_3x_4x_5^3 + x_1^{15}x_3x_4^3x_5 + x_1^{15}x_3^3x_4x_5 + x_1^{15}x_2x_4x_5^3\\
&+ x_1^{15}x_2x_4^3x_5 + x_1^{15}x_2x_3x_5^3 + x_1^{15}x_2x_3x_4^3 + x_1^{15}x_2x_3^3x_5 + x_1^{15}x_2x_3^3x_4\\
&+ x_1^{15}x_2^3x_4x_5 + x_1^{15}x_2^3x_3x_5 + x_1^{15}x_2^3x_3x_4 + x_1^3x_3x_4x_5^{15} + x_1^3x_3x_4^{15}x_5\\
&+ x_1^3x_3^{15}x_4x_5 + x_1^3x_2x_4x_5^{15} + x_1^3x_2x_4^{15}x_5 + x_1^3x_2x_3x_5^{15} + x_1^3x_2x_3x_4^{15}\\
&+ x_1^3x_2x_3^{15}x_5 + x_1^3x_2x_3^{15}x_4 + x_1^3x_2^{15}x_4x_5 + x_1^3x_2^{15}x_3x_5 + x_1^3x_2^{15}x_3x_4,\\
h_2 &= x_2x_3x_4^7x_5^{11} + x_2x_3^7x_4x_5^{11} + x_2x_3^7x_4^{11}x_5 + x_2^7x_3x_4x_5^{11} + x_2^7x_3x_4^{11}x_5\\
&+ x_2^7x_3^{11}x_4x_5 + x_1x_3x_4^7x_5^{11} + x_1x_3^7x_4x_5^{11} + x_1x_3^7x_4^{11}x_5 + x_1x_2x_4^7x_5^{11}
\end{aligned}
$$

$$
\begin{aligned}
&+ x_1x_2x_3^7x_5^{11} + x_1x_2x_3^7x_4^{11} + x_1x_2^7x_4x_5^{11} + x_1x_2^7x_4^{11}x_5 + x_1x_2^7x_3x_5^{11}\\
&+ x_1x_2^7x_3^{11}x_5 + x_1x_2^7x_3x_4^{11} + x_1x_2^7x_3^{11}x_4 + x_1^7x_3x_4x_5^{11} + x_1^7x_3x_4^{11}x_5\\
&+ x_1^7x_3^{11}x_4x_5 + x_1^7x_2x_4x_5^{11} + x_1^7x_2x_4^{11}x_5 + x_1^7x_2x_3x_5^{11} + x_1^7x_2^{11}x_4x_5\\
&+ x_1^7x_2x_3^{11}x_5 + x_1^7x_2^{11}x_3x_5 + x_1^7x_2x_3x_4^{11} + x_1^7x_2x_3^{11}x_4 + x_1^7x_2^{11}x_3x_4,\\
h_3 = &\ x_2x_3^3x_4^{13}x_5^3 + x_2x_3^3x_4^3x_5^{13} + x_2^3x_3x_4^{13}x_5^3 + x_2^3x_3x_4^3x_5^{13} + x_2^3x_3^{13}x_4x_5^3\\
&+ x_2^3x_3^{13}x_4^3x_5 + x_2^3x_3^3x_4x_5^{13} + x_2^3x_3^3x_4^{13}x_5 + x_1x_3^3x_4^{13}x_5^3 + x_1x_3^3x_4^3x_5^{13}\\
&+ x_1x_2^3x_4^{13}x_5^3 + x_1x_2^3x_4^3x_5^{13} + x_1x_2^3x_3^{13}x_5^3 + x_1x_2^3x_3^{13}x_4^3 + x_1x_2^3x_3^3x_5^{13}\\
&+ x_1x_2^3x_3^3x_4^{13} + x_1^3x_3x_4^{13}x_5^3 + x_1^3x_3x_4^3x_5^{13} + x_1^3x_3^{13}x_4x_5^3 + x_1^3x_3^{13}x_4^3x_5\\
&+ x_1^3x_3^3x_4x_5^{13} + x_1^3x_3^3x_4^{13}x_5 + x_1^3x_2x_4^{13}x_5^3 + x_1^3x_2x_4^3x_5^{13} + x_1^3x_2x_3^{13}x_5^3\\
&+ x_1^3x_2x_3^{13}x_4^3 + x_1^3x_2x_3^3x_5^{13} + x_1^3x_2^{13}x_4x_5^3 + x_1^3x_2^{13}x_4^3x_5 + x_1^3x_2^{13}x_3x_5^3\\
&+ x_1^3x_2x_3^3x_4^{13} + x_1^3x_2^{13}x_3x_4^3 + x_1^3x_2^{13}x_3^3x_5 + x_1^3x_2^{13}x_3^3x_4 + x_1^3x_2^3x_4x_5^{13}\\
&+ x_1^3x_2^3x_4^{13}x_5 + x_1^3x_2^3x_3x_5^{13} + x_1^3x_2^3x_3x_4^{13} + x_1^3x_2^3x_3^{13}x_5 + x_1^3x_2^3x_3^{13}x_4\\
&+ x_2x_3^7x_4^3x_5^9 + x_2x_3^7x_4^9x_5^3 + x_2^7x_3x_4^3x_5^9 + x_2^7x_3x_4^9x_5^3 + x_2^7x_3^3x_4x_5^9\\
&+ x_2^7x_3^3x_4^9x_5 + x_2^7x_3^9x_4x_5^3 + x_2^7x_3^9x_4^3x_5 + x_1x_3^7x_4^3x_5^9 + x_1x_3^7x_4^9x_5^3\\
&+ x_1x_2^7x_4^3x_5^9 + x_1x_2^7x_4^9x_5^3 + x_1x_2^7x_3^3x_5^9 + x_1x_2^7x_3^3x_4^9 + x_1x_2^7x_3^9x_5^3\\
&+ x_1x_2^7x_3^9x_4^3 + x_1^7x_3x_4^3x_5^9 + x_1^7x_3x_4^9x_5^3 + x_1^7x_3^3x_4x_5^9 + x_1^7x_3^3x_4^9x_5\\
&+ x_1^7x_3^9x_4x_5^3 + x_1^7x_3^9x_4^3x_5 + x_1^7x_2x_4^3x_5^9 + x_1^7x_2x_4^9x_5^3 + x_1^7x_2x_3^3x_5^9\\
&+ x_1^7x_2x_3^3x_4^9 + x_1^7x_2x_3^9x_5^3 + x_1^7x_2x_3^9x_4^3 + x_1^7x_2^3x_4x_5^9 + x_1^7x_2^3x_4^9x_5\\
&+ x_1^7x_2^3x_3x_5^9 + x_1^7x_2^3x_3x_4^9 + x_1^7x_2^3x_3^9x_5 + x_1^7x_2^3x_3^9x_4 + x_1^7x_2^9x_4x_5^3\\
&+ x_1^7x_2^9x_4^3x_5 + x_1^7x_2^9x_3x_5^3 + x_1^7x_2^9x_3x_4^3 + x_1^7x_2^9x_3^3x_5 + x_1^7x_2^9x_3^3x_4,\\
h_4 = &\ x_2^3x_3^3x_4^5x_5^9 + x_2^3x_3^5x_4^3x_5^9 + x_2^3x_3^5x_4^9x_5^3 + x_1^3x_3^3x_4^5x_5^9 + x_1^3x_3^5x_4^3x_5^9\\
&+ x_1^3x_3^5x_4^9x_5^3 + x_1^3x_2^3x_4^5x_5^9 + x_1^3x_2^3x_3^5x_5^9 + x_1^3x_2^3x_3^5x_4^9 + x_1^3x_2^5x_4^3x_5^9\\
&+ x_1^3x_2^5x_4^9x_5^3 + x_1^3x_2^5x_3^3x_5^9 + x_1^3x_2^5x_3^3x_4^9 + x_1^3x_2^5x_3^9x_5^3 + x_1^3x_2^5x_3^9x_4^3,\\
h_5 = &\ x_1x_2x_3x_4^3x_5^{14} + x_1x_2x_3x_4^6x_5^{11} + x_1x_2x_3x_4^7x_5^{10} + x_1x_2x_3x_4^{14}x_5^3\\
&+ x_1x_2x_3^3x_4x_5^{14} + x_1x_2x_3^3x_4^{14}x_5 + x_1x_2x_3^6x_4x_5^{11} + x_1x_2x_3^6x_4^{11}x_5\\
&+ x_1x_2x_3^7x_4x_5^{10} + x_1x_2x_3^7x_4^{10}x_5 + x_1x_2x_3^{14}x_4x_5^3 + x_1x_2x_3^{14}x_4^3x_5\\
&+ x_1x_2^3x_3x_4x_5^{14} + x_1x_2^3x_3x_4^{14}x_5 + x_1x_2^3x_3^{14}x_4x_5 + x_1x_2^6x_3x_4x_5^{11}\\
&+ x_1x_2^6x_3x_4^{11}x_5 + x_1x_2^6x_3^{11}x_4x_5 + x_1x_2^7x_3x_4x_5^{10} + x_1x_2^7x_3x_4^{10}x_5\\
&+ x_1x_2^7x_3^{10}x_4x_5 + x_1x_2^{14}x_3x_4x_5^3 + x_1x_2^{14}x_3x_4^3x_5 + x_1x_2^{14}x_3^3x_4x_5\\
&+ x_1^3x_2x_3x_4^4x_5^{11} + x_1^3x_2x_3x_4^7x_5^8 + x_1^3x_2x_3^4x_4x_5^{11} + x_1^3x_2x_3^4x_4^{11}x_5\\
&+ x_1^3x_2x_3^7x_4x_5^8 + x_1^3x_2x_3^7x_4^8x_5 + x_1^3x_2^4x_3x_4x_5^{11} + x_1^3x_2^4x_3x_4^{11}x_5\\
&+ x_1^3x_2^4x_3^{11}x_4x_5 + x_1^3x_2^7x_3x_4x_5^8 + x_1^3x_2^7x_3x_4^8x_5 + x_1^3x_2^7x_3^8x_4x_5
\end{aligned}
$$

$$
\begin{aligned}
&+ x_1^7x_2x_3x_4^3x_5^8 + x_1^7x_2x_3x_4^8x_5^3 + x_1^7x_2x_3^3x_4x_5^8 + x_1^7x_2x_3^3x_4^8x_5\\
&+ x_1^7x_2x_3^8x_4x_5^3 + x_1^7x_2x_3^8x_4^3x_5 + x_1^7x_2^3x_3x_4x_5^8 + x_1^7x_2^3x_3x_4^8x_5\\
&+ x_1^7x_2^3x_3^8x_4x_5 + x_1^7x_2^8x_3x_4x_5^3 + x_1^7x_2^8x_3x_4^3x_5 + x_1^7x_2^8x_3^3x_4x_5,\\
h_6 &= x_1x_2x_3x_4^6x_5^{11} + x_1x_2x_3x_4^7x_5^{10} + x_1x_2x_3^6x_4x_5^{11} + x_1x_2x_3^6x_4^{11}x_5\\
&+ x_1x_2x_3^7x_4x_5^{10} + x_1x_2x_3^7x_4^{10}x_5 + x_1x_2^6x_3x_4x_5^{11} + x_1x_2^6x_3x_4^{11}x_5\\
&+ x_1x_2^6x_3^{11}x_4x_5 + x_1x_2^7x_3x_4x_5^{10} + x_1x_2^7x_3x_4^{10}x_5 + x_1x_2^7x_3^{10}x_4x_5\\
&+ x_1^3x_2x_3x_4x_5^{14} + x_1^3x_2x_3x_4^4x_5^{11} + x_1^3x_2x_3x_4^7x_5^8 + x_1^3x_2x_3x_4^{14}x_5\\
&+ x_1^3x_2x_3^4x_4x_5^{11} + x_1^3x_2x_3^4x_4^{11}x_5 + x_1^3x_2x_3^7x_4x_5^8 + x_1^3x_2x_3^7x_4^8x_5\\
&+ x_1^3x_2x_3^{14}x_4x_5 + x_1^3x_2^4x_3x_4x_5^{11} + x_1^3x_2^4x_3x_4^{11}x_5 + x_1^3x_2^4x_3^{11}x_4x_5\\
&+ x_1^3x_2^7x_3x_4x_5^8 + x_1^3x_2^7x_3x_4^8x_5 + x_1^3x_2^7x_3^8x_4x_5 + x_1^3x_2^{13}x_3x_4x_5^2\\
&+ x_1^3x_2^{13}x_3x_4^2x_5 + x_1^3x_2^{13}x_3^2x_4x_5 + x_1^7x_2x_3x_4x_5^{10} + x_1^7x_2x_3x_4^{10}x_5\\
&+ x_1^7x_2x_3^{10}x_4x_5 + x_1^7x_2^9x_3x_4x_5^2 + x_1^7x_2^9x_3x_4^2x_5 + x_1^7x_2^9x_3^2x_4x_5,\\
h_7 &= x_1x_2x_3^3x_4^3x_5^{12} + x_1x_2x_3^3x_4^{12}x_5^3 + x_1x_2x_3^6x_4^3x_5^9 + x_1x_2x_3^6x_4^9x_5^3\\
&+ x_1x_2^3x_3x_4^3x_5^{12} + x_1x_2^3x_3x_4^{12}x_5^3 + x_1x_2^3x_3^3x_4x_5^{12} + x_1x_2^3x_3^3x_4^{12}x_5\\
&+ x_1x_2^3x_3^{12}x_4x_5^3 + x_1x_2^3x_3^{12}x_4^3x_5 + x_1x_2^6x_3x_4^3x_5^9 + x_1x_2^6x_3x_4^9x_5^3\\
&+ x_1x_2^6x_3^3x_4x_5^9 + x_1x_2^6x_3^3x_4^9x_5 + x_1x_2^6x_3^9x_4x_5^3 + x_1x_2^6x_3^9x_4^3x_5\\
&+ x_1^3x_2x_3x_4^5x_5^{10} + x_1^3x_2x_3x_4^6x_5^9 + x_1^3x_2x_3^3x_4^4x_5^9 + x_1^3x_2x_3^3x_4^5x_5^8\\
&+ x_1^3x_2x_3^4x_4^3x_5^9 + x_1^3x_2x_3^4x_4^9x_5^3 + x_1^3x_2x_3^5x_4x_5^{10} + x_1^3x_2x_3^5x_4^3x_5^8\\
&+ x_1^3x_2x_3^5x_4^8x_5^3 + x_1^3x_2x_3^5x_4^{10}x_5 + x_1^3x_2x_3^6x_4x_5^9 + x_1^3x_2x_3^6x_4^9x_5\\
&+ x_1^3x_2^3x_3x_4^4x_5^9 + x_1^3x_2^3x_3x_4^5x_5^8 + x_1^3x_2^3x_3^4x_4x_5^9 + x_1^3x_2^3x_3^4x_4^9x_5\\
&+ x_1^3x_2^3x_3^5x_4x_5^8 + x_1^3x_2^3x_3^5x_4^8x_5 + x_1^3x_2^4x_3x_4^3x_5^9 + x_1^3x_2^4x_3x_4^9x_5^3\\
&+ x_1^3x_2^4x_3^3x_4x_5^9 + x_1^3x_2^4x_3^3x_4^9x_5 + x_1^3x_2^4x_3^9x_4x_5^3 + x_1^3x_2^4x_3^9x_4^3x_5\\
&+ x_1^3x_2^5x_3x_4^2x_5^9 + x_1^3x_2^5x_3x_4^3x_5^8 + x_1^3x_2^5x_3x_4^8x_5^3 + x_1^3x_2^5x_3x_4^9x_5^2\\
&+ x_1^3x_2^5x_3^2x_4x_5^9 + x_1^3x_2^5x_3^2x_4^9x_5 + x_1^3x_2^5x_3^3x_4x_5^8 + x_1^3x_2^5x_3^3x_4^8x_5\\
&+ x_1^3x_2^5x_3^8x_4x_5^3 + x_1^3x_2^5x_3^8x_4^3x_5 + x_1^3x_2^5x_3^9x_4x_5^2 + x_1^3x_2^5x_3^9x_4^2x_5.
\end{aligned}
$$

$QP_5((4)|(2)|(3))^{\Sigma_5} = \langle [h_t]_{\overline{\omega}} : 8 \leqslant t \leqslant 10\rangle$, where $\overline{\omega} = (4)|(2)|(3)$ and

$$
\begin{aligned}
h_8 &= x_2^3x_3^5x_4^5x_5^7 + x_2^3x_3^5x_4^7x_5^5 + x_2^3x_3^7x_4^5x_5^5 + x_2^7x_3^3x_4^5x_5^5 + x_1^3x_3^5x_4^5x_5^7\\
&+ x_1^3x_3^5x_4^7x_5^5 + x_1^3x_3^7x_4^5x_5^5 + x_1^3x_2^5x_4^5x_5^7 + x_1^3x_2^5x_4^7x_5^5 + x_1^3x_2^5x_3^5x_5^7\\
&+ x_1^3x_2^5x_3^5x_4^7 + x_1^3x_2^5x_3^7x_5^5 + x_1^3x_2^5x_3^7x_4^5 + x_1^3x_2^7x_4^5x_5^5 + x_1^3x_2^7x_3^5x_5^5\\
&+ x_1^3x_2^7x_3^5x_4^5 + x_1^7x_3^3x_4^5x_5^5 + x_1^7x_2^3x_4^5x_5^5 + x_1^7x_2^3x_3^5x_5^5 + x_1^7x_2^3x_3^5x_4^5,\\
h_9 &= x_1x_2^2x_3^5x_4^5x_5^7 + x_1x_2^2x_3^5x_4^7x_5^5 + x_1x_2^2x_3^7x_4^5x_5^5 + x_1x_2^7x_3^2x_4^5x_5^5\\
&+ x_1^7x_2x_3^2x_4^5x_5^5 + x_1x_2^3x_3^4x_4^5x_5^7 + x_1x_2^3x_3^4x_4^7x_5^5 + x_1x_2^3x_3^5x_4^4x_5^7
\end{aligned}
$$

$$
\begin{aligned}
&+ x_1x_2^3x_3^5x_4^7x_5^4 + x_1x_2^3x_3^7x_4^4x_5^5 + x_1x_2^3x_3^7x_4^5x_5^4 + x_1x_2^7x_3^3x_4^4x_5^5\\
&+ x_1x_2^7x_3^3x_4^5x_5^4 + x_1^3x_2x_3^4x_4^5x_5^7 + x_1^3x_2x_3^4x_4^7x_5^5 + x_1^3x_2x_3^5x_4^4x_5^7\\
&+ x_1^3x_2x_3^5x_4^7x_5^4 + x_1^3x_2x_3^7x_4^4x_5^5 + x_1^3x_2x_3^7x_4^5x_5^4 + x_1^3x_2^5x_3x_4^4x_5^7\\
&+ x_1^3x_2^5x_3x_4^7x_5^4 + x_1^3x_2^5x_3^7x_4x_5^4 + x_1^3x_2^7x_3x_4^4x_5^5 + x_1^3x_2^7x_3x_4^5x_5^4\\
&+ x_1^3x_2^7x_3^5x_4x_5^4 + x_1^7x_2x_3^3x_4^4x_5^5 + x_1^7x_2x_3^3x_4^5x_5^4 + x_1^7x_2^3x_3x_4^4x_5^5\\
&+ x_1^7x_2^3x_3x_4^5x_5^4 + x_1^7x_2^3x_3^5x_4x_5^4,\\
h_{10} = {}& x_1x_2^3x_3^5x_4^5x_5^6 + x_1x_2^3x_3^5x_4^6x_5^5 + x_1x_2^3x_3^6x_4^5x_5^5 + x_1x_2^6x_3^3x_4^5x_5^5\\
&+ x_1^3x_2x_3^5x_4^5x_5^6 + x_1^3x_2x_3^5x_4^6x_5^5 + x_1^3x_2^5x_3x_4^5x_5^6 + x_1^3x_2^5x_3x_4^6x_5^5\\
&+ x_1^3x_2^3x_3^4x_4^5x_5^5 + x_1^3x_2^3x_3^5x_4^4x_5^5 + x_1^3x_2^3x_3^5x_4^5x_5^4 + x_1^3x_2^4x_3^3x_4^5x_5^5\\
&+ x_1^3x_2^5x_3^3x_4^4x_5^5 + x_1^3x_2^5x_3^3x_4^5x_5^4 + x_1^3x_2^5x_3^5x_4^3x_5^4 + x_1^3x_2^5x_3^5x_4^4x_5^3.
\end{aligned}
$$

$QP_5((4)|^2|(2))^{\Sigma_5} = \langle [h_t]_{\omega^*} : 11 \leqslant t \leqslant 13\rangle$, where $\omega^* = (4)|^2|(2)$ and

$$
\begin{aligned}
h_{11} = {}& x_2^3x_3^3x_4^7x_5^7 + x_2^3x_3^7x_4^3x_5^7 + x_2^3x_3^7x_4^7x_5^3 + x_2^7x_3^3x_4^3x_5^7 + x_2^7x_3^3x_4^7x_5^3\\
&+ x_2^7x_3^7x_4^3x_5^3 + x_1^3x_3^3x_4^7x_5^7 + x_1^3x_3^7x_4^3x_5^7 + x_1^3x_3^7x_4^7x_5^3 + x_1^3x_2^3x_4^7x_5^7\\
&+ x_1^3x_2^3x_3^7x_5^7 + x_1^3x_2^3x_3^7x_4^7 + x_1^3x_2^7x_4^3x_5^7 + x_1^3x_2^7x_4^7x_5^3 + x_1^3x_2^7x_3^3x_5^7\\
&+ x_1^3x_2^7x_3^3x_4^7 + x_1^3x_2^7x_3^7x_5^3 + x_1^3x_2^7x_3^7x_4^3 + x_1^7x_3^3x_4^3x_5^7 + x_1^7x_3^3x_4^7x_5^3\\
&+ x_1^7x_3^7x_4^3x_5^3 + x_1^7x_2^3x_4^3x_5^7 + x_1^7x_2^3x_4^7x_5^3 + x_1^7x_2^3x_3^3x_5^7 + x_1^7x_2^3x_3^3x_4^7\\
&+ x_1^7x_2^3x_3^7x_5^3 + x_1^7x_2^3x_3^7x_4^3 + x_1^7x_2^7x_4^3x_5^3 + x_1^7x_2^7x_3^3x_5^3 + x_1^7x_2^7x_3^3x_4^3,\\
h_{12} = {}& x_1x_2^2x_3^3x_4^7x_5^7 + x_1x_2^2x_3^7x_4^3x_5^7 + x_1x_2^2x_3^7x_4^7x_5^3 + x_1x_2^3x_3^2x_4^7x_5^7\\
&+ x_1x_2^3x_3^7x_4^2x_5^7 + x_1x_2^3x_3^7x_4^7x_5^2 + x_1x_2^7x_3^2x_4^3x_5^7 + x_1x_2^7x_3^2x_4^7x_5^3\\
&+ x_1x_2^7x_3^3x_4^2x_5^7 + x_1x_2^7x_3^3x_4^7x_5^2 + x_1x_2^7x_3^7x_4^2x_5^3 + x_1x_2^7x_3^7x_4^3x_5^2\\
&+ x_1^3x_2x_3^2x_4^7x_5^7 + x_1^3x_2x_3^7x_4^2x_5^7 + x_1^3x_2x_3^7x_4^7x_5^2 + x_1^3x_2^7x_3x_4^2x_5^7\\
&+ x_1^3x_2^7x_3x_4^7x_5^2 + x_1^3x_2^7x_3^7x_4x_5^2 + x_1^7x_2x_3^2x_4^3x_5^7 + x_1^7x_2x_3^2x_4^7x_5^3\\
&+ x_1^7x_2x_3^3x_4^2x_5^7 + x_1^7x_2x_3^3x_4^7x_5^2 + x_1^7x_2x_3^7x_4^2x_5^3 + x_1^7x_2x_3^7x_4^3x_5^2\\
&+ x_1^7x_2^3x_3x_4^2x_5^7 + x_1^7x_2^3x_3x_4^7x_5^2 + x_1^7x_2^3x_3^7x_4x_5^2 + x_1^7x_2^7x_3x_4^2x_5^3\\
&+ x_1^7x_2^7x_3x_4^3x_5^2 + x_1^7x_2^7x_3^3x_4x_5^2,\\
h_{13} = {}& x_1^3x_2^3x_3^3x_4^5x_5^6 + x_1^3x_2^3x_3^5x_4^3x_5^6 + x_1^3x_2^3x_3^5x_4^6x_5^3 + x_1^3x_2^5x_3^3x_4^3x_5^6\\
&+ x_1^3x_2^5x_3^3x_4^6x_5^3 + x_1^3x_2^5x_3^6x_4^3x_5^3.
\end{aligned}
$$

## 5.4 $\Sigma_5$-invariants of $QP_5((2)|^4)$

$QP_5((4)|(2)|(1)|^2)^{\Sigma_5} = \langle [p_t] : 1 \leqslant t \leqslant 9\rangle$, where

$$p_1 = x_4^{15}x_5^{15} + x_3^{15}x_5^{15} + x_3^{15}x_4^{15} + x_2^{15}x_5^{15} + x_2^{15}x_4^{15}$$

$$
\begin{aligned}
&+ x_2^{15}x_3^{15} + x_1^{15}x_5^{15} + x_1^{15}x_4^{15} + x_1^{15}x_3^{15} + x_1^{15}x_2^{15},\\
p_2 &= x_3x_4^{14}x_5^{15} + x_3x_4^{15}x_5^{14} + x_3^{15}x_4x_5^{14} + x_2x_4^{14}x_5^{15} + x_2x_4^{15}x_5^{14} + x_2x_3^{14}x_5^{15}\\
&+ x_2x_3^{14}x_4^{15} + x_2x_3^{15}x_5^{14} + x_2x_3^{15}x_4^{14} + x_2^{15}x_4x_5^{14} + x_2^{15}x_3x_5^{14} + x_2^{15}x_3x_4^{14}\\
&+ x_1x_4^{14}x_5^{15} + x_1x_4^{15}x_5^{14} + x_1x_3^{14}x_5^{15} + x_1x_3^{14}x_4^{15} + x_1x_3^{15}x_5^{14} + x_1x_3^{15}x_4^{14}\\
&+ x_1x_2^{14}x_5^{15} + x_1x_2^{14}x_4^{15} + x_1x_2^{14}x_3^{15} + x_1x_2^{15}x_5^{14} + x_1x_2^{15}x_4^{14} + x_1x_2^{15}x_3^{14}\\
&+ x_1^{15}x_4x_5^{14} + x_1^{15}x_3x_5^{14} + x_1^{15}x_3x_4^{14} + x_1^{15}x_2x_5^{14} + x_1^{15}x_2x_4^{14} + x_1^{15}x_2x_3^{14},\\
p_3 &= x_3^3x_4^{13}x_5^{14} + x_2^3x_4^{13}x_5^{14} + x_2^3x_3^{13}x_5^{14} + x_2^3x_3^{13}x_4^{14} + x_1^3x_4^{13}x_5^{14}\\
&+ x_1^3x_3^{13}x_5^{14} + x_1^3x_3^{13}x_4^{14} + x_1^3x_2^{13}x_5^{14} + x_1^3x_2^{13}x_4^{14} + x_1^3x_2^{13}x_3^{14},\\
p_4 &= x_2x_3x_4^{14}x_5^{14} + x_2x_3^{14}x_4x_5^{14} + x_2^3x_3^5x_4^{10}x_5^{12} + x_1x_3x_4^{14}x_5^{14} + x_1x_3^{14}x_4x_5^{14}\\
&+ x_1x_2x_4^{14}x_5^{14} + x_1x_2x_3^{14}x_5^{14} + x_1x_2x_3^{14}x_4^{14} + x_1x_2^{14}x_4x_5^{14} + x_1x_2^{14}x_3x_5^{14}\\
&+ x_1x_2^{14}x_3x_4^{14} + x_1^3x_3^5x_4^{10}x_5^{12} + x_1^3x_2^5x_4^{10}x_5^{12} + x_1^3x_2^5x_3^{10}x_5^{12} + x_1^3x_2^5x_3^{10}x_4^{12},\\
p_5 &= x_2x_3^2x_4^{13}x_5^{14} + x_2x_3^3x_4^{12}x_5^{14} + x_2x_3^3x_4^{14}x_5^{12} + x_2^3x_3x_4^{12}x_5^{14} + x_2^3x_3x_4^{14}x_5^{12}\\
&+ x_2^3x_3^{13}x_4^2x_5^{12} + x_1x_3^2x_4^{13}x_5^{14} + x_1x_3^3x_4^{12}x_5^{14} + x_1x_3^3x_4^{14}x_5^{12} + x_1x_2^2x_4^{13}x_5^{14}\\
&+ x_1x_2^2x_3^{13}x_5^{14} + x_1x_2^2x_3^{13}x_4^{14} + x_1x_2^3x_4^{12}x_5^{14} + x_1x_2^3x_4^{14}x_5^{12} + x_1x_2^3x_3^{12}x_5^{14}\\
&+ x_1x_2^3x_3^{12}x_4^{14} + x_1x_2^3x_3^{14}x_5^{12} + x_1x_2^3x_3^{14}x_4^{12} + x_1^3x_3x_4^{12}x_5^{14} + x_1^3x_3x_4^{14}x_5^{12}\\
&+ x_1^3x_3^{13}x_4^2x_5^{12} + x_1^3x_2x_4^{12}x_5^{14} + x_1^3x_2x_4^{14}x_5^{12} + x_1^3x_2x_3^{12}x_5^{14} + x_1^3x_2x_3^{12}x_4^{14}\\
&+ x_1^3x_2x_3^{14}x_5^{12} + x_1^3x_2x_3^{14}x_4^{12} + x_1^3x_2^{13}x_4^2x_5^{12} + x_1^3x_2^{13}x_3^2x_5^{12} + x_1^3x_2^{13}x_3^2x_4^{12},\\
p_6 &= x_2x_3^2x_4^{12}x_5^{15} + x_2x_3^2x_4^{15}x_5^{12} + x_2x_3^{15}x_4^2x_5^{12} + x_2^{15}x_3x_4^2x_5^{12} + x_1x_3^2x_4^{12}x_5^{15}\\
&+ x_1x_3^2x_4^{15}x_5^{12} + x_1x_3^{15}x_4^2x_5^{12} + x_1x_2^2x_4^{12}x_5^{15} + x_1x_2^2x_4^{15}x_5^{12} + x_1x_2^2x_3^{12}x_5^{15}\\
&+ x_1x_2^2x_3^{12}x_4^{15} + x_1x_2^2x_3^{15}x_5^{12} + x_1x_2^2x_3^{15}x_4^{12} + x_1x_2^{15}x_4^2x_5^{12} + x_1x_2^{15}x_3^2x_5^{12}\\
&+ x_1x_2^{15}x_3^2x_4^{12} + x_1^{15}x_3x_4^2x_5^{12} + x_1^{15}x_2x_4^2x_5^{12} + x_1^{15}x_2x_3^2x_5^{12} + x_1^{15}x_2x_3^2x_4^{12},\\
p_7 &= x_1x_2^2x_3^4x_4^8x_5^{15} + x_1x_2^2x_3^4x_4^{15}x_5^8 + x_1x_2^2x_3^{15}x_4^4x_5^8 + x_1x_2^{15}x_3^2x_4^4x_5^8\\
&+ x_1^{15}x_2x_3^2x_4^4x_5^8,\\
p_8 &= x_1x_2x_3^2x_4^{14}x_5^{12} + x_1x_2x_3^6x_4^{10}x_5^{12} + x_1x_2x_3^{14}x_4^2x_5^{12} + x_1x_2^2x_3x_4^{12}x_5^{14}\\
&+ x_1x_2^2x_3^5x_4^{10}x_5^{12} + x_1x_2^2x_3^{12}x_4x_5^{14} + x_1x_2^3x_3^2x_4^{12}x_5^{12} + x_1x_2^3x_3^4x_4^{10}x_5^{12}\\
&+ x_1x_2^3x_3^6x_4^8x_5^{12} + x_1x_2^3x_3^6x_4^{12}x_5^8 + x_1x_2^{14}x_3x_4^2x_5^{12} + x_1^3x_2x_3^4x_4^{10}x_5^{12}\\
&+ x_1^3x_2x_3^6x_4^8x_5^{12} + x_1^3x_2x_3^6x_4^{12}x_5^8 + x_1^3x_2^5x_3^2x_4^8x_5^{12} + x_1^3x_2^5x_3^2x_4^{12}x_5^8\\
&+ x_1^3x_2^5x_3^{10}x_4^4x_5^8,\\
p_9 &= x_1x_2^2x_3x_4^{12}x_5^{14} + x_1x_2^2x_3x_4^{14}x_5^{12} + x_1x_2^2x_3^4x_4^9x_5^{14} + x_1x_2^2x_3^5x_4^8x_5^{14}\\
&+ x_1x_2^2x_3^5x_4^{14}x_5^8 + x_1x_2^2x_3^{12}x_4x_5^{14} + x_1x_2^3x_3^4x_4^8x_5^{14} + x_1x_2^3x_3^4x_4^{14}x_5^8\\
&+ x_1x_2^3x_3^{14}x_4^4x_5^8 + x_1x_2^{14}x_3x_4^2x_5^{12} + x_1^3x_2x_3^4x_4^8x_5^{14} + x_1^3x_2x_3^4x_4^{14}x_5^8\\
&+ x_1^3x_2x_3^{14}x_4^4x_5^8 + x_1^3x_2^{13}x_3^2x_4^4x_5^8,
\end{aligned}
$$

## 5.5 $\Sigma_5$-invariants of $QP_5((4)|(3)|^2|(1))$

$QP_5((4)|((4)|(3)|^2|(1))^{\Sigma_5} = \langle [q_t]_{(4)|(3)|^2|(1)} : 1 \leqslant t \leqslant 7\rangle$, where

$$\begin{aligned}
q_1 = {} & x_2x_3^7x_4^7x_5^{15} + x_2x_3^7x_4^{15}x_5^7 + x_2x_3^{15}x_4^7x_5^7 + x_2^7x_3x_4^7x_5^{15} + x_2^7x_3x_4^{15}x_5^7 \\
& + x_2^7x_3^7x_4x_5^{15} + x_2^7x_3^7x_4^{15}x_5 + x_2^7x_3^{15}x_4x_5^7 + x_2^7x_3^{15}x_4^7x_5 + x_2^{15}x_3x_4^7x_5^7 \\
& + x_2^{15}x_3^7x_4x_5^7 + x_2^{15}x_3^7x_4^7x_5 + x_1x_3^7x_4^7x_5^{15} + x_1x_3^7x_4^{15}x_5^7 + x_1x_3^{15}x_4^7x_5^7 \\
& + x_1x_2^7x_4^7x_5^{15} + x_1x_2^7x_4^{15}x_5^7 + x_1x_2^7x_3^7x_5^{15} + x_1x_2^7x_3^7x_4^{15} + x_1x_2^7x_3^{15}x_5^7 \\
& + x_1x_2^7x_3^{15}x_4^7 + x_1x_2^{15}x_4^7x_5^7 + x_1x_2^{15}x_3^7x_5^7 + x_1x_2^{15}x_3^7x_4^7 + x_1^7x_3x_4^7x_5^{15} \\
& + x_1^7x_3x_4^{15}x_5^7 + x_1^7x_3^7x_4x_5^{15} + x_1^7x_3^7x_4^{15}x_5 + x_1^7x_3^{15}x_4x_5^7 + x_1^7x_3^{15}x_4^7x_5 \\
& + x_1^7x_2x_4^7x_5^{15} + x_1^7x_2x_4^{15}x_5^7 + x_1^7x_2x_3^7x_5^{15} + x_1^7x_2x_3^7x_4^{15} + x_1^7x_2x_3^{15}x_5^7 \\
& + x_1^7x_2x_3^{15}x_4^7 + x_1^7x_2^7x_4x_5^{15} + x_1^7x_2^7x_4^{15}x_5 + x_1^7x_2^7x_3x_5^{15} + x_1^7x_2^7x_3x_4^{15} \\
& + x_1^7x_2^7x_3^{15}x_5 + x_1^7x_2^7x_3^{15}x_4 + x_1^7x_2^{15}x_4x_5^7 + x_1^7x_2^{15}x_4^7x_5 + x_1^7x_2^{15}x_3x_5^7 \\
& + x_1^7x_2^{15}x_3x_4^7 + x_1^7x_2^{15}x_3^7x_5 + x_1^7x_2^{15}x_3^7x_4 + x_1^{15}x_3x_4^7x_5^7 + x_1^{15}x_3^7x_4x_5^7 \\
& + x_1^{15}x_3^7x_4^7x_5 + x_1^{15}x_2x_4^7x_5^7 + x_1^{15}x_2x_3^7x_5^7 + x_1^{15}x_2x_3^7x_4^7 + x_1^{15}x_2^7x_4x_5^7 \\
& + x_1^{15}x_2^7x_4^7x_5 + x_1^{15}x_2^7x_3x_5^7 + x_1^{15}x_2^7x_3x_4^7 + x_1^{15}x_2^7x_3^7x_5 + x_1^{15}x_2^7x_3^7x_4, \\
q_2 = {} & x_2^3x_3^5x_4^7x_5^{15} + x_2^3x_3^5x_4^{15}x_5^7 + x_2^3x_3^7x_4^5x_5^{15} + x_2^3x_3^7x_4^{15}x_5^5 + x_2^3x_3^{15}x_4^5x_5^7 \\
& + x_2^3x_3^{15}x_4^7x_5^5 + x_2^7x_3^3x_4^5x_5^{15} + x_2^7x_3^3x_4^{15}x_5^5 + x_2^7x_3^{15}x_4^3x_5^5 + x_2^{15}x_3^3x_4^5x_5^7 \\
& + x_2^{15}x_3^3x_4^7x_5^5 + x_2^{15}x_3^7x_4^3x_5^5 + x_1^3x_3^5x_4^7x_5^{15} + x_1^3x_3^5x_4^{15}x_5^7 + x_1^3x_3^7x_4^5x_5^{15} \\
& + x_1^3x_3^7x_4^{15}x_5^5 + x_1^3x_3^{15}x_4^5x_5^7 + x_1^3x_3^{15}x_4^7x_5^5 + x_1^3x_2^5x_4^7x_5^{15} + x_1^3x_2^5x_4^{15}x_5^7 \\
& + x_1^3x_2^5x_3^7x_5^{15} + x_1^3x_2^5x_3^7x_4^{15} + x_1^3x_2^5x_3^{15}x_5^7 + x_1^3x_2^5x_3^{15}x_4^7 + x_1^3x_2^7x_4^5x_5^{15} \\
& + x_1^3x_2^7x_4^{15}x_5^5 + x_1^3x_2^7x_3^5x_5^{15} + x_1^3x_2^7x_3^5x_4^{15} + x_1^3x_2^7x_3^{15}x_5^5 + x_1^3x_2^7x_3^{15}x_4^5 \\
& + x_1^3x_2^{15}x_4^5x_5^7 + x_1^3x_2^{15}x_4^7x_5^5 + x_1^3x_2^{15}x_3^5x_5^7 + x_1^3x_2^{15}x_3^5x_4^7 + x_1^3x_2^{15}x_3^7x_5^5 \\
& + x_1^3x_2^{15}x_3^7x_4^5 + x_1^7x_3^3x_4^5x_5^{15} + x_1^7x_3^3x_4^{15}x_5^5 + x_1^7x_3^{15}x_4^3x_5^5 + x_1^7x_2^3x_4^5x_5^{15} \\
& + x_1^7x_2^3x_4^{15}x_5^5 + x_1^7x_2^3x_3^5x_5^{15} + x_1^7x_2^3x_3^5x_4^{15} + x_1^7x_2^3x_3^{15}x_5^5 + x_1^7x_2^3x_3^{15}x_4^5 \\
& + x_1^7x_2^{15}x_4^3x_5^5 + x_1^7x_2^{15}x_3^3x_5^5 + x_1^7x_2^{15}x_3^3x_4^5 + x_1^{15}x_3^3x_4^5x_5^7 + x_1^{15}x_3^3x_4^7x_5^5 \\
& + x_1^{15}x_3^7x_4^3x_5^5 + x_1^{15}x_2^3x_4^5x_5^7 + x_1^{15}x_2^3x_4^7x_5^5 + x_1^{15}x_2^3x_3^5x_5^7 + x_1^{15}x_2^3x_3^5x_4^7 \\
& + x_1^{15}x_2^3x_3^7x_5^5 + x_1^{15}x_2^3x_3^7x_4^5 + x_1^{15}x_2^7x_4^3x_5^5 + x_1^{15}x_2^7x_3^3x_5^5 + x_1^{15}x_2^7x_3^3x_4^5, \\
q_3 = {} & x_1x_2x_3^6x_4^7x_5^{15} + x_1x_2x_3^6x_4^{15}x_5^7 + x_1x_2x_3^7x_4^6x_5^{15} + x_1x_2x_3^7x_4^{15}x_5^6 \\
& + x_1x_2x_3^{15}x_4^6x_5^7 + x_1x_2x_3^{15}x_4^7x_5^6 + x_1x_2^6x_3x_4^7x_5^{15} + x_1x_2^6x_3x_4^{15}x_5^7 \\
& + x_1x_2^6x_3^7x_4x_5^{15} + x_1x_2^6x_3^7x_4^{15}x_5 + x_1x_2^6x_3^{15}x_4x_5^7 + x_1x_2^6x_3^{15}x_4^7x_5 \\
& + x_1x_2^7x_3x_4^6x_5^{15} + x_1x_2^7x_3x_4^{15}x_5^6 + x_1x_2^7x_3^6x_4x_5^{15} + x_1x_2^7x_3^6x_4^{15}x_5 \\
& + x_1x_2^7x_3^{15}x_4x_5^6 + x_1x_2^7x_3^{15}x_4^6x_5 + x_1x_2^{15}x_3x_4^6x_5^7 + x_1x_2^{15}x_3x_4^7x_5^6 \\
& + x_1x_2^{15}x_3^6x_4x_5^7 + x_1x_2^{15}x_3^6x_4^7x_5 + x_1x_2^{15}x_3^7x_4x_5^6 + x_1x_2^{15}x_3^7x_4^6x_5
\end{aligned}$$

$$
\begin{aligned}
&+ x_1^7x_2x_3x_4^6x_5^{15} + x_1^7x_2x_3x_4^{15}x_5^6 + x_1^7x_2x_3^6x_4x_5^{15} + x_1^7x_2x_3^6x_4^{15}x_5\\
&+ x_1^7x_2x_3^{15}x_4x_5^6 + x_1^7x_2x_3^{15}x_4^6x_5 + x_1^7x_2^{15}x_3x_4x_5^6 + x_1^7x_2^{15}x_3x_4^6x_5\\
&+ x_1^{15}x_2x_3x_4^6x_5^7 + x_1^{15}x_2x_3x_4^7x_5^6 + x_1^{15}x_2x_3^6x_4x_5^7 + x_1^{15}x_2x_3^6x_4^7x_5\\
&+ x_1^{15}x_2x_3^7x_4x_5^6 + x_1^{15}x_2x_3^7x_4^6x_5 + x_1^{15}x_2^7x_3x_4x_5^6 + x_1^{15}x_2^7x_3x_4^6x_5\\
&+ x_1^3x_2x_3^4x_4^7x_5^{15} + x_1^3x_2x_3^4x_4^{15}x_5^7 + x_1^3x_2x_3^7x_4^4x_5^{15} + x_1^3x_2x_3^7x_4^{15}x_5^4\\
&+ x_1^3x_2x_3^{15}x_4^4x_5^7 + x_1^3x_2x_3^{15}x_4^7x_5^4 + x_1^3x_2^4x_3x_4^7x_5^{15} + x_1^3x_2^4x_3x_4^{15}x_5^7\\
&+ x_1^3x_2^4x_3^7x_4x_5^{15} + x_1^3x_2^4x_3^7x_4^{15}x_5 + x_1^3x_2^4x_3^{15}x_4x_5^7 + x_1^3x_2^4x_3^{15}x_4^7x_5\\
&+ x_1^3x_2^7x_3x_4^4x_5^{15} + x_1^3x_2^7x_3x_4^{15}x_5^4 + x_1^3x_2^7x_3^4x_4x_5^{15} + x_1^3x_2^7x_3^4x_4^{15}x_5\\
&+ x_1^3x_2^7x_3^{15}x_4x_5^4 + x_1^3x_2^7x_3^{15}x_4^4x_5 + x_1^3x_2^{15}x_3x_4^4x_5^7 + x_1^3x_2^{15}x_3x_4^7x_5^4\\
&+ x_1^3x_2^{15}x_3^4x_4x_5^7 + x_1^3x_2^{15}x_3^4x_4^7x_5 + x_1^3x_2^{15}x_3^7x_4x_5^4 + x_1^3x_2^{15}x_3^7x_4^4x_5\\
&+ x_1^7x_2^3x_3x_4^4x_5^{15} + x_1^7x_2^3x_3x_4^{15}x_5^4 + x_1^7x_2^3x_3^4x_4x_5^{15} + x_1^7x_2^3x_3^4x_4^{15}x_5\\
&+ x_1^7x_2^3x_3^{15}x_4x_5^4 + x_1^7x_2^3x_3^{15}x_4^4x_5 + x_1^7x_2^{15}x_3^3x_4x_5^4 + x_1^7x_2^{15}x_3^3x_4^4x_5\\
&+ x_1^{15}x_2^3x_3x_4^4x_5^7 + x_1^{15}x_2^3x_3x_4^7x_5^4 + x_1^{15}x_2^3x_3^4x_4x_5^7 + x_1^{15}x_2^3x_3^4x_4^7x_5\\
&+ x_1^{15}x_2^3x_3^7x_4x_5^4 + x_1^{15}x_2^3x_3^7x_4^4x_5 + x_1^{15}x_2^7x_3^3x_4x_5^4 + x_1^{15}x_2^7x_3^3x_4^4x_5,
\end{aligned}
$$

$$
\begin{aligned}
q_4 = {}& x_1x_2^3x_3^5x_4^6x_5^{15} + x_1x_2^3x_3^5x_4^{15}x_5^6 + x_1x_2^3x_3^6x_4^5x_5^{15} + x_1x_2^3x_3^6x_4^{15}x_5^5\\
&+ x_1x_2^3x_3^{15}x_4^5x_5^6 + x_1x_2^3x_3^{15}x_4^6x_5^5 + x_1x_2^6x_3^3x_4^5x_5^{15} + x_1x_2^6x_3^3x_4^{15}x_5^5\\
&+ x_1x_2^6x_3^{15}x_4^3x_5^5 + x_1x_2^{15}x_3^3x_4^5x_5^6 + x_1x_2^{15}x_3^3x_4^6x_5^5 + x_1x_2^{15}x_3^6x_4^3x_5^5\\
&+ x_1^3x_2x_3^5x_4^6x_5^{15} + x_1^3x_2x_3^5x_4^{15}x_5^6 + x_1^3x_2x_3^{15}x_4^5x_5^6 + x_1^3x_2^5x_3x_4^6x_5^{15}\\
&+ x_1^3x_2^5x_3x_4^{15}x_5^6 + x_1^3x_2^5x_3^6x_4x_5^{15} + x_1^3x_2^5x_3^6x_4^{15}x_5 + x_1^3x_2^5x_3^{15}x_4x_5^6\\
&+ x_1^3x_2^5x_3^{15}x_4^6x_5 + x_1^3x_2^{15}x_3x_4^5x_5^6 + x_1^3x_2^{15}x_3^5x_4x_5^6 + x_1^3x_2^{15}x_3^5x_4^6x_5\\
&+ x_1^{15}x_2x_3^3x_4^5x_5^6 + x_1^{15}x_2x_3^3x_4^6x_5^5 + x_1^{15}x_2x_3^6x_4^3x_5^5 + x_1^{15}x_2^3x_3x_4^5x_5^6\\
&+ x_1^{15}x_2^3x_3^5x_4x_5^6 + x_1^{15}x_2^3x_3^5x_4^6x_5 + x_1^3x_2^3x_3^4x_4^5x_5^{15} + x_1^3x_2^3x_3^4x_4^{15}x_5^5\\
&+ x_1^3x_2^3x_3^5x_4^4x_5^{15} + x_1^3x_2^3x_3^5x_4^{15}x_5^4 + x_1^3x_2^3x_3^{15}x_4^4x_5^5 + x_1^3x_2^3x_3^{15}x_4^5x_5^4\\
&+ x_1^3x_2^4x_3^3x_4^5x_5^{15} + x_1^3x_2^4x_3^3x_4^{15}x_5^5 + x_1^3x_2^4x_3^{15}x_4^3x_5^5 + x_1^3x_2^{15}x_3^3x_4^4x_5^5\\
&+ x_1^3x_2^{15}x_3^3x_4^5x_5^4 + x_1^3x_2^{15}x_3^4x_4^3x_5^5 + x_1^{15}x_2^3x_3^3x_4^4x_5^5 + x_1^{15}x_2^3x_3^3x_4^5x_5^4\\
&+ x_1^{15}x_2^3x_3^4x_4^3x_5^5,
\end{aligned}
$$

$$
\begin{aligned}
q_5 = {}& x_1x_2^3x_3^5x_4^7x_5^{14} + x_1x_2^3x_3^5x_4^{14}x_5^7 + x_1x_2^3x_3^6x_4^7x_5^{13} + x_1x_2^3x_3^6x_4^{13}x_5^7\\
&+ x_1x_2^6x_3^3x_4^7x_5^{13} + x_1x_2^6x_3^3x_4^{13}x_5^7 + x_1x_2^6x_3^7x_4^7x_5^9 + x_1x_2^6x_3^7x_4^9x_5^7\\
&+ x_1x_2^7x_3^6x_4^7x_5^9 + x_1x_2^7x_3^6x_4^9x_5^7 + x_1x_2^7x_3^7x_4^7x_5^8 + x_1x_2^7x_3^7x_4^8x_5^7\\
&+ x_1^3x_2x_3^5x_4^7x_5^{14} + x_1^3x_2x_3^5x_4^{14}x_5^7 + x_1^3x_2x_3^7x_4^7x_5^{12} + x_1^3x_2x_3^7x_4^{12}x_5^7\\
&+ x_1^3x_2^3x_3^4x_4^7x_5^{13} + x_1^3x_2^3x_3^4x_4^{13}x_5^7 + x_1^3x_2^3x_3^5x_4^7x_5^{12} + x_1^3x_2^3x_3^5x_4^{12}x_5^7\\
&+ x_1^3x_2^3x_3^7x_4^4x_5^{13} + x_1^3x_2^3x_3^7x_4^{13}x_5^4 + x_1^3x_2^3x_3^{13}x_4^4x_5^7 + x_1^3x_2^3x_3^{13}x_4^7x_5^4
\end{aligned}
$$

$$
\begin{aligned}
&+ x_1^3x_2^4x_3^3x_4^7x_5^{13} + x_1^3x_2^4x_3^3x_4^{13}x_5^7 + x_1^3x_2^4x_3^7x_4^7x_5^9 + x_1^3x_2^4x_3^7x_4^9x_5^7\\
&+ x_1^3x_2^5x_3x_4^7x_5^{14} + x_1^3x_2^5x_3x_4^{14}x_5^7 + x_1^3x_2^5x_3^6x_4^7x_5^9 + x_1^3x_2^5x_3^6x_4^9x_5^7\\
&+ x_1^3x_2^5x_3^7x_4x_5^{14} + x_1^3x_2^5x_3^7x_4^7x_5^8 + x_1^3x_2^5x_3^7x_4^8x_5^7 + x_1^3x_2^5x_3^7x_4^{14}x_5\\
&+ x_1^3x_2^5x_3^{14}x_4x_5^7 + x_1^3x_2^5x_3^{14}x_4^7x_5 + x_1^3x_2^7x_3x_4^7x_5^{12} + x_1^3x_2^7x_3x_4^{12}x_5^7\\
&+ x_1^3x_2^7x_3^4x_4^7x_5^9 + x_1^3x_2^7x_3^4x_4^9x_5^7 + x_1^3x_2^7x_3^7x_4x_5^{12} + x_1^3x_2^7x_3^7x_4^4x_5^9\\
&+ x_1^3x_2^7x_3^7x_4^9x_5^4 + x_1^3x_2^7x_3^7x_4^{12}x_5 + x_1^3x_2^7x_3^9x_4^4x_5^7 + x_1^3x_2^7x_3^9x_4^7x_5^4\\
&+ x_1^3x_2^7x_3^{12}x_4x_5^7 + x_1^3x_2^7x_3^{12}x_4^7x_5 + x_1^7x_2x_3^6x_4^7x_5^9 + x_1^7x_2x_3^6x_4^9x_5^7\\
&+ x_1^7x_2x_3^7x_4^7x_5^8 + x_1^7x_2x_3^7x_4^8x_5^7 + x_1^7x_2^3x_3x_4^7x_5^{12} + x_1^7x_2^3x_3x_4^{12}x_5^7\\
&+ x_1^7x_2^3x_3^4x_4^7x_5^9 + x_1^7x_2^3x_3^4x_4^9x_5^7 + x_1^7x_2^3x_3^7x_4x_5^{12} + x_1^7x_2^3x_3^7x_4^4x_5^9\\
&+ x_1^7x_2^3x_3^7x_4^9x_5^4 + x_1^7x_2^3x_3^7x_4^{12}x_5 + x_1^7x_2^3x_3^9x_4^4x_5^7 + x_1^7x_2^3x_3^9x_4^7x_5^4\\
&+ x_1^7x_2^3x_3^{12}x_4x_5^7 + x_1^7x_2^3x_3^{12}x_4^7x_5 + x_1^7x_2^7x_3x_4^7x_5^8 + x_1^7x_2^7x_3x_4^8x_5^7\\
&+ x_1^7x_2^7x_3^7x_4x_5^8 + x_1^7x_2^7x_3^7x_4^8x_5 + x_1^7x_2^7x_3^8x_4x_5^7 + x_1^7x_2^7x_3^8x_4^7x_5,\\
q_6 = {}& x_1x_2^3x_3^6x_4^7x_5^{13} + x_1x_2^3x_3^6x_4^{13}x_5^7 + x_1x_2^3x_3^7x_4^7x_5^{12} + x_1x_2^3x_3^7x_4^{12}x_5^7\\
&+ x_1x_2^6x_3^3x_4^7x_5^{13} + x_1x_2^6x_3^3x_4^{13}x_5^7 + x_1x_2^6x_3^7x_4^3x_5^{13} + x_1x_2^6x_3^7x_4^{11}x_5^5\\
&+ x_1x_2^6x_3^{11}x_4^5x_5^7 + x_1x_2^6x_3^{11}x_4^7x_5^5 + x_1x_2^7x_3^3x_4^7x_5^{12} + x_1x_2^7x_3^3x_4^{12}x_5^7\\
&+ x_1x_2^7x_3^6x_4^7x_5^9 + x_1x_2^7x_3^6x_4^9x_5^7 + x_1x_2^7x_3^7x_4^3x_5^{12} + x_1x_2^7x_3^7x_4^7x_5^8\\
&+ x_1x_2^7x_3^7x_4^8x_5^7 + x_1x_2^7x_3^7x_4^{10}x_5^5 + x_1x_2^7x_3^{10}x_4^5x_5^7 + x_1x_2^7x_3^{10}x_4^7x_5^5\\
&+ x_1^3x_2x_3^7x_4^6x_5^{13} + x_1^3x_2x_3^7x_4^7x_5^{12} + x_1^3x_2x_3^7x_4^{12}x_5^7 + x_1^3x_2x_3^7x_4^{14}x_5^5\\
&+ x_1^3x_2x_3^{14}x_4^5x_5^7 + x_1^3x_2x_3^{14}x_4^7x_5^5 + x_1^3x_2^3x_3^4x_4^7x_5^{13} + x_1^3x_2^3x_3^4x_4^{13}x_5^7\\
&+ x_1^3x_2^3x_3^7x_4^4x_5^{13} + x_1^3x_2^3x_3^7x_4^{13}x_5^4 + x_1^3x_2^3x_3^{13}x_4^4x_5^7 + x_1^3x_2^3x_3^{13}x_4^7x_5^4\\
&+ x_1^3x_2^4x_3^3x_4^7x_5^{13} + x_1^3x_2^4x_3^3x_4^{13}x_5^7 + x_1^3x_2^4x_3^7x_4^3x_5^{13} + x_1^3x_2^4x_3^7x_4^{11}x_5^5\\
&+ x_1^3x_2^4x_3^{11}x_4^5x_5^7 + x_1^3x_2^4x_3^{11}x_4^7x_5^5 + x_1^3x_2^7x_3x_4^5x_5^{14} + x_1^3x_2^7x_3x_4^6x_5^{13}\\
&+ x_1^3x_2^7x_3x_4^{13}x_5^6 + x_1^3x_2^7x_3x_4^{14}x_5^5 + x_1^3x_2^7x_3^3x_4^4x_5^{13} + x_1^3x_2^7x_3^3x_4^{13}x_5^4\\
&+ x_1^3x_2^7x_3^4x_4^7x_5^9 + x_1^3x_2^7x_3^4x_4^9x_5^7 + x_1^3x_2^7x_3^7x_4^4x_5^9 + x_1^3x_2^7x_3^7x_4^8x_5^5\\
&+ x_1^3x_2^7x_3^8x_4^5x_5^7 + x_1^3x_2^7x_3^8x_4^7x_5^5 + x_1^3x_2^7x_3^{13}x_4^2x_5^5 + x_1^3x_2^7x_3^{13}x_4^3x_5^4\\
&+ x_1^3x_2^{13}x_3^2x_4^5x_5^7 + x_1^3x_2^{13}x_3^2x_4^7x_5^5 + x_1^3x_2^{13}x_3^3x_4^4x_5^7 + x_1^3x_2^{13}x_3^3x_4^7x_5^4\\
&+ x_1^3x_2^{13}x_3^7x_4^2x_5^5 + x_1^3x_2^{13}x_3^7x_4^3x_5^4 + x_1^7x_2x_3^3x_4^7x_5^{12} + x_1^7x_2x_3^3x_4^{12}x_5^7\\
&+ x_1^7x_2x_3^6x_4^7x_5^9 + x_1^7x_2x_3^6x_4^9x_5^7 + x_1^7x_2x_3^7x_4^3x_5^{12} + x_1^7x_2x_3^7x_4^7x_5^8\\
&+ x_1^7x_2x_3^7x_4^8x_5^7 + x_1^7x_2x_3^7x_4^{10}x_5^5 + x_1^7x_2x_3^{10}x_4^5x_5^7 + x_1^7x_2x_3^{10}x_4^7x_5^5\\
&+ x_1^7x_2^3x_3^4x_4^7x_5^9 + x_1^7x_2^3x_3^4x_4^9x_5^7 + x_1^7x_2^3x_3^7x_4^4x_5^9 + x_1^7x_2^3x_3^7x_4^9x_5^4\\
&+ x_1^7x_2^3x_3^9x_4^4x_5^7 + x_1^7x_2^3x_3^9x_4^7x_5^4 + x_1^7x_2^7x_3x_4^3x_5^{12} + x_1^7x_2^7x_3x_4^6x_5^9\\
&+ x_1^7x_2^7x_3x_4^9x_5^6 + x_1^7x_2^7x_3x_4^{10}x_5^5 + x_1^7x_2^7x_3^3x_4^4x_5^9 + x_1^7x_2^7x_3^3x_4^9x_5^4
\end{aligned}
$$

$$
\begin{aligned}
&+ x_1^7x_2^7x_3^9x_4^2x_5^5 + x_1^7x_2^7x_3^9x_4^3x_5^4 + x_1^7x_2^9x_3^2x_4^5x_5^7 + x_1^7x_2^9x_3^2x_4^7x_5^5\\
&+ x_1^7x_2^9x_3^3x_4^4x_5^7 + x_1^7x_2^9x_3^3x_4^7x_5^4 + x_1^7x_2^9x_3^7x_4^2x_5^5 + x_1^7x_2^9x_3^7x_4^3x_5^4,\\
q_7 = {}& x_1^3x_2^3x_3^5x_4^6x_5^{13} + x_1^3x_2^3x_3^5x_4^{14}x_5^5 + x_1^3x_2^3x_3^{13}x_4^6x_5^5 + x_1^3x_2^5x_3^3x_4^5x_5^{14}\\
&+ x_1^3x_2^5x_3^3x_4^6x_5^{13} + x_1^3x_2^5x_3^3x_4^{13}x_5^6 + x_1^3x_2^5x_3^3x_4^{14}x_5^5 + x_1^3x_2^5x_3^6x_4^3x_5^{13}\\
&+ x_1^3x_2^5x_3^6x_4^{11}x_5^5 + x_1^3x_2^5x_3^{11}x_4^5x_5^6 + x_1^3x_2^5x_3^{11}x_4^6x_5^5 + x_1^3x_2^5x_3^{14}x_4^3x_5^5\\
&+ x_1^3x_2^7x_3^3x_4^5x_5^{12} + x_1^3x_2^7x_3^3x_4^{12}x_5^5 + x_1^3x_2^7x_3^5x_4^6x_5^9 + x_1^3x_2^7x_3^5x_4^9x_5^6\\
&+ x_1^3x_2^7x_3^9x_4^5x_5^6 + x_1^3x_2^7x_3^{12}x_4^3x_5^5 + x_1^7x_2^3x_3^3x_4^5x_5^{12} + x_1^7x_2^3x_3^3x_4^{12}x_5^5\\
&+ x_1^7x_2^3x_3^5x_4^6x_5^9 + x_1^7x_2^3x_3^5x_4^9x_5^6 + x_1^7x_2^3x_3^9x_4^5x_5^6 + x_1^7x_2^3x_3^{12}x_4^3x_5^5\\
&+ x_1^7x_2^7x_3^3x_4^5x_5^8 + x_1^7x_2^7x_3^3x_4^8x_5^5 + x_1^7x_2^7x_3^8x_4^3x_5^5.
\end{aligned}
$$

# References


[1] J.M. Boardman, *Modular representations on the homology of power of real projective space*, in: M. C. Tangora (Ed.), Algebraic Topology, Oaxtepec, 1991, in: Contemp. Math., vol. 146, 1993, pp. 49-70, MR1224907.

[2] R.R. Bruner, L.M. Hà and N.H.V. Hưng, *On behavior of the algebraic transfer*, Trans. Amer. Math. Soc. 357 (2005), 473-487, MR2095619.

[3] T. W. Chen, *Determination of* $\mathrm{Ext}_{\mathcal{A}}^{5,*}(\mathbb{Z}/2,\mathbb{Z}/2)$, Topology Appl., 158 (2011), 660-689, MR2774051.

[4] P.H. Chơn and L.M. Hà, *Lambda algebra and the Singer transfer*, C. R. Math. Acad. Sci. Paris 349 (2011) 21-23, MR2755689.

[5] P.H. Chơn and L.M. Hà, *On May spectral sequence and the algebraic transfer*, Manuscripta Math. 138 (2012) 141-160, MR2898751.

[6] P. H. Chơn and L. M. Hà, *On the May spectral sequence and the algebraic transfer II*, Topology Appl. 178 (2014) 372-383, MR3276753.

[7] L.M. Hà, *Sub-Hopf algebras of the Steenrod algebra and the Singer transfer*, "Proceedings of the International School and Conference in Algebraic Topology, Hà Nội 2004", Geom. Topol. Monogr., Geom. Topol. Publ., Coventry, vol. 11 (2007), 81-105, MR2402802.

[8] N.H.V. Hưng, *The cohomology of the Steenrod algebra and representations of the general linear groups*, Trans. Amer. Math. Soc. 357 (2005), 4065-4089, MR2159700.

[9] N.H.V. Hưng and V. T. N. Quỳnh, *The image of Singer's fourth transfer,* C. R. Math. Acad. Sci. Paris 347 (2009) 1415-1418, MR2588792.

[10] M.Kameko, *Products of projective spaces as Steenrod modules*, PhD. Thesis, The Johns Hopkins University, ProQuest LLC, Ann Arbor, MI, 1990. 29 pp. MR2638633.

[11] W.H. Lin, $\mathrm{Ext}_{\mathcal{A}}^{4,*}(\mathbb{Z}/2,\mathbb{Z}/2)$ *and* $\mathrm{Ext}_{\mathcal{A}}^{5,*}(\mathbb{Z}/2,\mathbb{Z}/2)$, Topology Appl., 155 (2008) 459-496, MR2380930.

[12] N. Minami, *The iterated transfer analogue of the new doomsday conjecture*, Trans. Amer. Math. Soc. 351 (1999) 2325-2351, MR1443884.

[13] T.N. Nam, *Transfert algébrique et action du groupe linéaire sur les puissances divisées modulo 2*, Ann. Inst. Fourier (Grenoble) 58 (2008) 1785-1837, MR2445834.

[14] D.V. Phúc, *The affirmative answer to Singer's conjecture on the algebraic transfer of rank four*, Proc. R. Soc. Edinb., Sect. A 153 (2023) 1529–1542, MR4638620.

[15] D. V. Phúc, *On the fifth Singer algebraic transfer in a generic family of internal degree characterized by* $\mu(n) = 4$, J. Pure Appl. Algebra 228 (2024) 107658.

[16] D. V. Phúc, *A note on the modular representation on the $\mathbb{Z}/2$-homology groups of the fourth power of real projective space and its application*, Preprint 2021, available online at: https://arxiv.org/abs/2106.14606v11.

[17] D. V. Phúc and N. Sum, *On the generators of the polynomial algebra as a module over the Steenrod algebra*, C. R. Acad. Sci. Paris, Ser. I 353 (2015) 1035-1040, MR3419856.

[18] V. T. N. Quỳnh, *On behavior of the fifth algebraic transfer*, "Proceedings of the International School and Conference in Algebraic Topology, Hà Nội 2004", Geom. Topol. Monogr., Geom. Topol. Publ., Coventry, vol. 11 (2007), 309-326, MR2402811.

[19] W.M. Singer, *The transfer in homological algebra*, Math. Zeit. 202 (1989) 493-523, MR1022818.

[20] W. M. Singer, *On the action of the Steenrod squares on polynomial algebras*, Proc. Amer. Math. Soc. 111 (1991), 577-583, MR1045150.

[21] N. E. Steenrod and D.B.A. Epstein, *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press, Princeton N.J (1962), MR0145525.

[22] N. Sum, *The negative answer to Kameko's conjecture on the hit problem*, Adv. Math. 225 (2010), 2365-2390, MR2680169.

[23] N. Sum, *On the Peterson hit problem of five variables and its applications to the fifth Singer transfer*, East-West J. of Mathematics, 16 (2014) 47-62, MR3409252.

[24] N. Sum, *On the Peterson hit problem*, Adv. Math. 274 (2015) 432-489, MR3318156.

[25] N. Sum, *The squaring operation and the Singer algebraic transfer*, Vietnam J. Math. 49 (2021) 1079-1096, MR4319539.

[26] N. Sum and L. X. Mong, *The hit problem of five variable in the degree thirty*,East-West J. Math., 23 (2022), no. 2, 79-99.

[27] N. Sum, *The hit problem of rank five in a generic degree*, East-West J. Math., 25 (2024), no. 1, 1-60.

[28] N. Sum, *A counter-example to Singer's conjecture for the algebraic transfer*. Preprint 2024, available online at: http://arxiv.org/abs/2408.06669.

[29] N. K. Tin, *A note on the Peterson hit problem for the Steenrod algebra*, Proc. Japan Acad. Ser. A Math. Sci., 97 (2021), no. 4, 25-28, MR4255188.

[30] N. K. Tin, *A note on the $\mathcal{A}$-generators of the polynomial algebra of six variables and applications*, Turkish J. Math. 46 (2022), no. 5, 1911-1926. MR4453895.