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HILBERT FUNCTION AND MULTIPLICITY OF IDEALIZTIONS

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Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. We study the relations of the Hilbert function, the multiplicity of an $\mathfrak{m}$-primary ideal of R and these of $\mathfrak{m} \times M$-primary ideal of the idealization. Some applications to the Cohen-Macaulayness of idealization are given.

1 Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and I an $\mathfrak{m}$-primary ideal of R. Let M be a finitely generated R-module of dimension t. We denote by $\ell_R(*)$ the length of an R-module $*$. It is well known that $\ell_R(M/I^{n+1}M)$ agrees with a polynomial function of degree t for all $n \gg 0$. That is, there exist integers $\mathrm{e}_I^0(M), \ldots, \mathrm{e}_I^t(M)$ such that

$$\ell_R(M/I^{n+1}M) = \mathrm{e}_I^0(M)\binom{n+t}{t} - \mathrm{e}_I^1(M)\binom{n+t-1}{n-1} + \cdots + (-1)^t \mathrm{e}_I^t(M)$$

for $n \gg 0$. We call $\mathrm{e}_I^0(M), \ldots, \mathrm{e}_I^t(M)$ the *Hilbert coefficients* of M with respect to I. The leading coefficient $\mathrm{e}_I^0(M)$ is called the *multiplicity* of M with respect to I. The notions of Hilbert function, multiplicity, and Hilbert coefficients

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are well known and have attracted the interest of several researchers, see, for example, [5, Chapter 4].

The purpose of this paper is to study the relations of the Hilbert function, the multiplicity of an $\mathfrak{m}$-primary ideal of R and these of $\mathfrak{m} \times M$-primary ideal of the idealization $R \ltimes M$ of M over R. Recall that the additive group $R \times M$ with multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1),$$

where $r_1, r_2 \in R$ and $m_1, m_2 \in M$ is a commutative ring. This ring is called the *idealization* of M over R or the *trivial extension* of R by M, denoted by $R \ltimes M$. The notion of idealization was introduced by M. Nagata [8] and has diverse applications in Commutative Algebra (see, e.g., [1]-[4], [6],[8], [9]). Note that if $(R, \mathfrak{m})$ is a Noetherian local ring, then $R \ltimes M$ is also a Noetherian local ring with unique maximal ideal $\mathfrak{m} \times M$, and $\dim R \ltimes M = \dim R$ (see [4])). Let

$$\rho : R \ltimes M \to R; (a, m) \mapsto a \quad \text{and} \quad \sigma : R \to R \ltimes M; a \mapsto (a, 0)$$

be the canonical projection and the canonical inclusion, respectively. Then ρ and σ are local ring homomorphisms.

The main result of this paper is as follows.

Theorem 1.1. *1. Let I be an $\mathfrak{m}$-primary ideal of R. Set $J = I \times IM$. Then*

$$\ell_{R \ltimes M}(R \ltimes M/J^{n+1}) = \ell_R(R/I^{n+1}) + \ell_R(M/I^{n+1}M)$$

for all $n \geq 0$. In particular, we have the following.

(i) *If $d = t$, then $\mathrm{e}^i_J(R \ltimes M) = \mathrm{e}^i_I(R) + \mathrm{e}^i_I(M)$ for $i = 0, \ldots, d$.*

(ii) *If $d > t$, then*

$$\mathrm{e}^i_J(R \ltimes M) = \begin{cases} \mathrm{e}^i_I(R) & \text{for } i = 0, \ldots, d-t-1 \text{ and} \\ \mathrm{e}^i_I(R) + (-1)^{d-t}\, \mathrm{e}^{i-(d-t)}_I(M) & \text{for } i = d-t, \ldots, d. \end{cases}$$

2. Let J be an $\mathfrak{m} \times M$-primary ideal of $R \ltimes M$. Set $I = \sigma^{-1}(J)$. Then the following holds true.

(i) *If $d = t$, then $\mathrm{e}^0_J(R \ltimes M) = \mathrm{e}^0_I(R) + \mathrm{e}^0_I(M)$.*

(ii) *If $d > t$, then $\mathrm{e}^0_J(R \ltimes M) = \mathrm{e}^0_I(R)$.*

Part 1 of Theorem 1.1 focuses on the Hilbert function and the Hilbert coefficients of $R \ltimes M$ with respect to a homogeneous $\mathfrak{m} \times M$-primary ideal $J = I \times IM$. Here, an ideal of $R \ltimes M$ is called *homogeneous* if it has the form $I \times N$, where I is an ideal of R, N is a submodule of M such that $IM \subseteq N$. Note that all prime ideals of $R \ltimes M$ have the form $\mathrm{p} \times M$, where $\mathrm{p} \in \mathrm{Spec}(R)$. So they are homogeneous. However, a primary ideal need not be homogeneous (see [4]). Part 2 of Theorem 1.1 computes the multiplicity of J on $R \ltimes M$ for arbitrary $\mathfrak{m} \times M$-primary ideal J of $R \ltimes M$.

In the next section we prove the main results.

2 Proof of main results

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then $R \ltimes M$ is also a Noetherian local ring with unique maximal ideal $\mathfrak{m} \times M$, and $\dim R \ltimes M = \dim R$ (see [4])). Note that the lengths of L as an R-module and as an $R \ltimes M$-module are the same, that is, $\ell_R(L) = \ell_{R\ltimes M}(L)$.

Proof of Theorem 1.1. 1. Let I be an $\mathfrak{m}$-primary ideal of R. Set $J = I \times IM$. Then J is an $\mathfrak{m}\times M$-primary ideal of $R\ltimes M$ by [4, Theorem 3.6]. By [4, Theorem 3.3 (2)] and by induction we have $J^{n+1} = (I\times IM)^{n+1} = I^{n+1}\times I^{n+1}M$. Hence

$$\begin{aligned} R \ltimes M/J^{n+1} &\cong (R \ltimes M)/(I^{n+1} \ltimes I^{n+1}M) \\ &\cong \big(R/I^{n+1}\big) \ltimes \big(M/I^{n+1}M\big). \end{aligned}$$

These implies that

$$\ell_{R\ltimes M}(R \ltimes M/J^{n+1}) = \ell_R(R/I^{n+1}) + \ell_R(M/I^{n+1}M).$$

So,

$$\begin{aligned} &\mathrm{e}^0_J(R \ltimes M)\binom{n+d}{d} - \mathrm{e}^1_J(R \ltimes M)\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}^d_J(R \ltimes M) \\ &= \mathrm{e}^0_I(R)\binom{n+d}{d} - \mathrm{e}^1_I(R)\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}^d_I(R) + \\ &\quad \mathrm{e}^0_I(M)\binom{n+t}{t} - \mathrm{e}^1_I(M)\binom{n+t-1}{t-1} + \cdots + (-1)^t\mathrm{e}^t_I(M) \end{aligned}$$

for $n \gg 0$. Now we compare the coefficients between the two sides.

(i) If $d = t$, then $\mathrm{e}^i_J(R \ltimes M) = \mathrm{e}^i_I(R) + \mathrm{e}^i_I(M)$ for $i = 0, \ldots, d$.

(ii) Assume that $d > t$. Then $\mathrm{e}^i_J(R \ltimes M) = \mathrm{e}^i_I(R)$ for $i = 0, \ldots, d - t - 1$. For $i = d - t, \ldots, d$, we get that

$$(-1)^{d-t+j}\, \mathrm{e}^{d-t+j}_J(R \ltimes M) = (-1)^{d-t+j}\, \mathrm{e}^{d-t+j}_I(R) + (-1)^j\, \mathrm{e}^j_I(M).$$

This means $\mathrm{e}^i_J(R \ltimes M) = \mathrm{e}^i_I(R) + (-1)^{d-t}\, \mathrm{e}^{i-(d-t)}_I(M)$ for $i = d - t, \ldots, d$.

2. Since $I = \sigma^{-1}(J)$, I is an $\mathfrak{m}$-primary ideal of R. Set $J' = J + (0) \times M$. Then $J' = I \times M$. We have $J \subseteq J'$. Since $((0) \times M)^2 = 0$, we get that

$$(J')^{n+1} = (J + (0) \times M)^{n+1} = J(J')^n,$$

for all $n \geq 1$. Hence J is a reduction of J'. This implies that

$$e^0_J(R \ltimes M) = e^0_{J'}(R \ltimes M) = e^0_{I\times M}(R \ltimes M).$$

Since $(I \times M)^{n+1} = I^{n+1} \times I^n M$ [4, Theorem 3.3 (2)] and by induction,

$$R \ltimes M/(I \times M)^{n+1} \cong \big(R/I^{n+1}\big) \ltimes \big(M/I^n M\big).$$

So,

$$\begin{aligned}
&\mathrm{e}_J^0(R \ltimes M)\binom{n+d}{d} - \mathrm{e}_J^1(R \ltimes M)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_J^d(R \ltimes M) \\
&= \mathrm{e}_I^0(R)\binom{n+d}{d} - \mathrm{e}_I^1(R)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(R) + \\
&\mathrm{e}_I^0(M)\binom{n+t-1}{t} - \mathrm{e}_I^1(M)\binom{n+t-2}{t-1} + \cdots + (-1)^t \mathrm{e}_I^t(M)
\end{aligned}$$

for $n \gg 0$. Then

(i) If $d = t$, then $\mathrm{e}_J^0(R \ltimes M) = \mathrm{e}_I^0(R) + \mathrm{e}_I^0(M)$.

(ii) If $d > t$, then $\mathrm{e}_J^0(R \ltimes M) = \mathrm{e}_I^0(R)$.

□

Remark 2.1. Let $Q = (r_1, \ldots, r_d)$ be a parameter ideal of R. Then $\ell_R(M/QM) < \infty$. Let $\overline{Q}$ be an ideal of $R \ltimes M$ generated by elements $(r_1, 0), \ldots, (r_d, 0)$. It is easy to check that $\overline{Q} = Q \times QM$. Since

$$R \ltimes M/\overline{Q} \cong R/Q \ltimes M/QM,$$

we get that $\ell_{R\ltimes M}(R \ltimes M/\overline{Q}) = \ell_R(R/Q) + \ell_R(M/QM) < \infty$. Hence $\overline{Q}$ is a parameter ideal of $R \ltimes M$.

Cohen-Macaulayness of idealizations was considered in [4] by considering the depth and the dimension of idealizations. Using the Theorem 1.1, Remark 2.1, and [7, Theorem 17.11], we have the following corollary.

Corollary 2.2. *The following holds true.*

(i) *Assume $d = t$. Then $R \ltimes M$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and M are Cohen-Macaulay.*

(ii) *Assume $d > t$. Then $R \ltimes M$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and $M = 0$.*

Proof. (i) Assume $d = t$. By Theorem 1.1,

$$\mathrm{e}(\overline{Q}, R \ltimes M) = \mathrm{e}(Q \times QM, R \ltimes M) = \mathrm{e}(Q, R) + \mathrm{e}(Q, M).$$

Then

$$\ell(R \ltimes M/\overline{Q}) - \mathrm{e}(\overline{Q}, R \ltimes M) = \ell(R/Q) - \mathrm{e}(Q, R) + \ell(M/QM) - e(Q, M).$$

This implies that $R \ltimes M$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and M are Cohen-Macaulay.

(ii) Assume that $d > t$. Then

$$\ell(R \ltimes M/\overline{Q}) - \mathrm{e}(\overline{Q}, R \ltimes M) = \ell(R/Q) - \mathrm{e}(Q, R) + \ell(M/QM).$$

It follows that $R \ltimes M$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and $M = 0$. □

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