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REGULARITY OF CERTAIN SUBSEMIRINGS OF FULL MATRIX SEMIRINGS

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Abstract

For an additively commutative semiring with zero S, we let $DV_n(S)$ denote the set of all $A \in M_n(S)$ of the form

$$\begin{bmatrix} x_1 & 0 & \cdots & 0 & x_1 \\ 0 & x_2 & \cdots & x_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & x_2 & \cdots & x_2 & 0 \\ x_1 & 0 & \cdots & 0 & x_1 \end{bmatrix}$$

where $M_n(S)$ is the full $n \times n$ matrix semiring over S. We show conditions for being regular semirings, left regular semirings, right regular semirings and intra-regular semirings of $DV_n(S)$.

1 Introduction and Preliminaries

A *semiring* S is an algebraic structure $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups and $\cdot$ is distributive over $+$. An element 0 of S is a *zero* of the semiring $(S, +, \cdot)$ if $x + 0 = x = 0 + x$ and $x \cdot 0 = 0 = 0 \cdot x$ for all $x \in S$. A semiring $(S, +, \cdot)$ is called *additively* [*multiplicatively*] *commutative* if $x + y = y + x$ [$x \cdot y = y \cdot x$] for all $x, y \in S$. We say that $(S, +, \cdot)$ is *commutative* if it is both addtitively commutative and multiplicatively commutative.

A ring R is called a (*Von Neumann*) *regular ring* if for every $a \in R$, $a = axa$ for some $x \in R$. Regular rings was originally introduced by Von Neumann in

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order to clarify certain aspects of operater algebras. For this reason, regular semirings are defined analogously. That is, a semiring S is said to be *regular* if for every $a \in S$, $a = axa$ for some $x \in S$. We can see that the semirings $\mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ are regular but $\mathbb{Z}_0^+$ is not regular. Also, the following semirings in Example 1.1 are regular semirings.

Example 1.1 ([1]). (1) If $S = \{0, 1\}$ with $0+0 = 0$, $0+1 = 1+0 = 1+1 = 1$, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$, then $(S, +, \cdot)$ is a commutative semiring with zero 0 which is not a ring.
(2) Let S be a nonempty subset of $\mathbb{R}$ such that $\min S$ exists. Define

$$x \oplus y = \max\{x, y\} \text{ and } x \odot y = \min\{x, y\} \text{ for all } x, y \in S.$$

Then $(S, \oplus, \odot)$ is a commutative semiring having $\min S$ as its zero. Also, if S contains more than one element, then $(S, \oplus, \odot)$ is not a ring.

Throughout, let S be an additively commutative semiring $(S, +, \cdot)$ with zero. $M_n(S)$ denotes the full $n \times n$ matrix semiring over S, that is, $M_n(S)$ is the set of all $n \times n$ matrices over S which is an additively commutative semiring under the usual addition and multiplication of matrices.

Moreover, various types of regularity have been studied. Left regular semi rings, right regular semirings and intra-regular semirings are defined as follows:

Definition 1.2. A semiring S is called a *left* [*right*] *regular semiring* if for every $a \in S$, $a = xa^2$ [$a = a^2x$] for some $x \in S$.

Definition 1.3. A semiring S is called an *intra-regular semiring* if for every $a \in S$, $a = xa^2y$ for some $x, y \in S$.

In terms of Green's relations, we have that
(1) S is a left [right] regular semiring if and only if $a\mathcal{L}a^2$ [$a\mathcal{R}a^2$] for all $a \in S$ and
(2) S is an intra-regular semiring if and only if $a\mathcal{J}a^2$ for all $a \in S$.

In 2010, Sararnrakskul, Lertvijitsilp, Wassanawichit and Pianskool [3] prove that the ring $D_n(R)$ of all $A \in M_n(R)$ of the form

$$\begin{bmatrix} x_1 & 0 & \cdots & 0 & y_1 \\ 0 & x_2 & \cdots & y_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & y_2 & \cdots & x_2 & 0 \\ y_1 & 0 & \cdots & 0 & x_1 \end{bmatrix}$$

is a maximal commutative subring of the ring $M_n(R)$. In 2014, Chatjaroenporn, Pobpitak, Patlertsin and Sararnrakskul [2] show that the semirings $V_n(S)$ of all $A \in M_n(S)$ of the form

$$\begin{bmatrix} x_1 & 0 & \cdots & 0 & x_1 \\ 0 & x_2 & \cdots & x_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

is a regular commutative subsemiring of the semiring $M_n(S)$.

In this paper, we show the conditions for regularity of the set $DV_n(S)$ consisting of all matrices in $M_n(S)$ of the form

$$\begin{bmatrix} x_1 & 0 & \cdots & 0 & x_1 \\ 0 & x_2 & \cdots & x_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & x_2 & \cdots & x_2 & 0 \\ x_1 & 0 & \cdots & 0 & x_1 \end{bmatrix}.$$

This means that if n is even, then $DV_n(S)$ is the set of all $A \in M_n(S)$ of the form

$$\begin{bmatrix} x_1 & 0 & & & \cdots & & & 0 & x_1 \\ 0 & x_2 & & & & & & x_2 & 0 \\ & & \ddots & & & & \iddots & & \\ \vdots & & & x_m & & x_m & & & \vdots \\ & & \iddots & & & & \ddots & & \\ 0 & x_2 & & & & & & x_2 & 0 \\ x_1 & 0 & & & \cdots & & & 0 & x_1 \end{bmatrix} \quad \text{where} \quad n = 2m$$

and if n is odd, then $DV_n(S)$ is the set of all $A \in M_n(S)$ of the form

$$\begin{bmatrix} x_1 & 0 & & \cdots & & 0 & x_1 \\ 0 & x_2 & & & & x_2 & 0 \\ & & \ddots & & \iddots & & \\ \vdots & & & x_m & & & \vdots \\ & & \iddots & & \ddots & & \\ 0 & x_2 & & & & x_2 & 0 \\ x_1 & 0 & & \cdots & & 0 & x_1 \end{bmatrix} \quad \text{where} \quad n = 2m-1.$$

2 The Subsemiring $DV_n(S)$ of $M_n(S)$

From now on, let S be an additively commutative semiring with zero 0.

Lemma 2.1. *The set $DV_n(S)$ is an additively commutative semiring with zero.*

Proof. We have that $DV_n(S) \subseteq M_n(S)$ and the zero matrix in $DV_n(S)$ is the zero of $DV_n(S)$. Observe that for $A \in M_n(S)$

$$A \in DV_n(S) \Leftrightarrow (i)\ A_{ii} = A_{i,n-i+1} = A_{n-i+1,i} = A_{n-i+1,n-i+1} \quad \text{for all } i \in \{1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil\}$$
$$(ii)\ A_{ij} = 0 \text{ otherwise.}$$

Let $B, C \in DV_n(S)$. Clearly, $B + C = C + B$. Then for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$

$$(B + C)_{ii} = B_{ii} + C_{ii} = B_{i,n-1+1} + C_{i,n-i+1} = (B + C)_{i,n-i+1}.$$

Similarly, $(B + C)_{ii} = (B + C)_{n-i+1,i} = (B + C)_{n-i+1,n-i+1}$. Otherwise $(B+C)_{ij} = B_{ij} + C_{ij} = 0$ where $i, j \in \{1, 2, \ldots, n\}$ with $j \neq i$ and $j \neq n-i+1$. Let $i, j \in \{1, 2, \ldots, n\}$ and $\Omega = \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$. Assume $i \in \Omega$. We have that

$$B_{ii} = B_{i,n-i+1} = B_{n-i+1,i} = B_{n-i+1,n-i+1} \text{ and}$$
$$C_{ii} = C_{i,n-i+1} = C_{n-i+1,i} = C_{n-i+1,n-i+1}.$$

If n is even or (n is odd and $i \neq \lceil \frac{n}{2} \rceil$), then

$$\begin{aligned}
(BC)_{ii} &= \sum_{k=1}^{n} B_{ik} C_{ki} \\
&= B_{ii} C_{ii} + B_{i,n-i+1} C_{n-i+1,i} \\
&= B_{ii} C_{i,n-i+1} + B_{i,n-i+1} C_{n-i+1,n-i+1} \\
&= \sum_{k=1}^{n} B_{ik} C_{k,n-i+1} \\
&= (BC)_{i,n-i+1} \\
&= B_{ii} C_{i,n-i+1} + B_{i,n-i+1} C_{n-i+1,n-i+1} \\
&= B_{n-i+1,i} C_{ii} + B_{n-i+1,n-i+1} C_{n-i+1,i} \\
(&= (BC)_{n-i+1,i}) \\
&= B_{n-i+1,i} C_{i,n-i+1} + B_{n-i+1,n-i+1} C_{n-i+1,n-i+1} \\
&= (BC)_{n-i+1,n-i+1}.
\end{aligned}$$

If n is odd and $i = \lceil \frac{n}{2} \rceil$, then $n - i + 1 = i$. Thus $(BC)_{ii} = (BC)_{i,n-i+1} = (BC)_{n-i+1,i} = (BC)_{n-i+1,n-i+1}$.

Next, assume that $i \notin \Omega$. Since $B \in DV_n(S)$, $B_{il} = 0$ for all $l \in \{1, 2, \ldots, n\}$. Hence $(BC)_{ij} = \sum_{k=1}^{n} B_{ik} C_{kj} = 0$. This proves that $DV_n(S)$ is an additively commutative semiring with zero. □

Lemma 2.2. *Let S be commutative. Then $DV_n(S)$ is a commutative subsemi ring of the semiring $M_n(S)$.*

By the proof of Lemma 2.2 and Theorem 2.3 in [3], we have more generalized result for $D_n(S)$ where S is a commutative semiring with zero 0 and unity 1 as the following theorem.

Theorem 2.3. *$D_n(S)$ is a maximal commutative subsemiring of the semiring $M_n(S)$.*

Remark 2.4. By Theorem 2.3, we have $DV_n(S)$ is not a maximal commutative subsemiring of $M_n(S)$ because $DV_n(S) \subsetneq D_n(S)$.

Next, we consider the regularity of $DV_n(S)$. We begin with showing the condition for being regular semirings of $DV_n(S)$ as follows.

Theorem 2.5. *For a positive integer n, the semiring $DV_n(S)$ is regular if and only if S is a regular semiring satisfying the condition that for any $a \in S$, $a = 2x$ for some $x \in S$.*

Proof. Assume $DV_n(S)$ is regular. Let $a \in S$. Let $A \in M_n(S)$ be such that $A_{11} = A_{1n} = A_{n1} = A_{nn} = a$ and $A_{ij} = 0$ otherwise. Then $A \in DV_n(S)$. Thus $A = ABA$ for some $B \in DV_n(S)$. Hence

$$\begin{aligned}
a = A_{11} &= (ABA)_{11} \\
&= \sum_{k=1}^{n} A_{1k}(BA)_{k1} \\
&= A_{11}(BA)_{11} + A_{1n}(BA)_{n1} \\
&= A_{11}(BA)_{11} + A_{11}(BA)_{11} \\
&= A_{11}\sum_{k-1}^{n} B_{1k}A_{k1} + A_{11}\sum_{k-1}^{n} B_{1k}A_{k1} \\
&= A_{11}(B_{11}A_{11} + B_{11}A_{11}) + A_{11}(B_{11}A_{11} + B_{11}A_{11}) \\
&= A_{11}(B_{11} + B_{11} + B_{11} + B_{11})A_{11} \\
&= a(4B_{11})a.
\end{aligned}$$

This means that a is regular. Moreover, $a = a(4B_{11})a = a(2B_{11} + 2B_{11})a = 2(a(2B_{11})a)$.

Conversely, assume S is regular and for every $a \in S$, $a = 2x$ for some $x \in S$. Then for every $a \in S$, $a = 2(2x) = 4x$ for some $x \in S$. To show that $DV_n(S)$ is regular, let $A \in DV_n(S)$. For each $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$, let $a_i = A_{ii}$. Then for every $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$, $a_i = a_i x_i a_i$ and $x_i = 2(2u_i) = 4u_i$ for some $x_i, u_i \in S$.

Case 1: n is even.

Let $B \in M_n(S)$ be such that $u_i = B_{ii} = B_{i,n-i+1} = B_{n-i+1,i} = B_{n-i+1,n-i+1}$ for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$ and $B_{ij} = 0$ otherwise. Then

$B \in DV_n(S)$. Let $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$. Then

$$\begin{aligned}
(ABA)_{ii} = \sum_{k=1}^{n} A_{ik}(BA)_{ki} &= A_{ii}(BA)_{ii} + A_{i,n-i+1}(BA)_{n-i+1,i} \\
&= A_{ii}(BA)_{ii} + A_{ii}(BA)_{ii} \\
&= A_{ii}\sum_{k-1}^{n} B_{ik}A_{ki} + A_{ii}\sum_{k-1}^{n} B_{ik}A_{ki} \\
&= A_{ii}(B_{ii}A_{ii} + B_{ii}A_{ii}) + A_{ii}(B_{ii}A_{ii} + B_{ii}A_{ii}) \\
&= A_{ii}(4B_{ii})A_{ii} \\
&= A_{ii}x_iA_{ii} \\
&= A_{ii}.
\end{aligned}$$

Case 2: n is odd.

Let $B \in M_n(S)$ be such that $u_i = B_{ii} = B_{i,n-i+1} = B_{n-i+1,i} = B_{n-i+1,n-i+1}$ for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$, $x_i = B_{ii}$ if $i = \lceil \frac{n}{2} \rceil$ and $B_{ij} = 0$ otherwise. Then $B \in DV_n(S)$. If $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$, then $(ABA)_{ii} = A_{ii}(4B_{ii})A_{ii} = A_{ii}$. If $i = \lceil \frac{n}{2} \rceil$, then $(ABA)_{ii} = A_{ii}(BA)_{ii} = A_{ii}B_{ii}A_{ii} = A_{ii}x_iA_{ii} = A_{ii}$.

In both cases, $(ABA)_{ii} = A_{ii}$ for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$. Thus $(ABA)_{i,n-i+1} = (ABA)_{ii} = A_{ii} = A_{i,n-i+1}$ for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$. Similarly, $(ABA)_{n-i+1,i} = A_{n-i+1,i}$ and $(ABA)_{n-i+1,n-i+1} = A_{n-i+1,n-i+1}$ for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$. Otherwise, for $i, j \in \{1, 2, \ldots, n\}$ such that $j \neq i$ and $j \neq n - i + 1$, we have $A_{ij} = A_{n-i+1,j} = 0$. If n is even or (n is odd and $i \neq \lceil \frac{n}{2} \rceil$), then $(ABA)_{ij} = \sum_{k=1}^{n}(AB)_{ik}A_{kj} = (AB)_{ii}A_{ij} + (AB)_{i,n-i+1}A_{n-i+1,j} = 0 = A_{ij}$. If n is odd and $i = \lceil \frac{n}{2} \rceil$, then $(ABA)_{ij} = (AB)_{ii}A_{ij} = 0 = A_{ij}$. It follows that $(ABA)_{ij} = A_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$. Therefore A is regular. □

Furthermore, we find that the condition in the previous theorem also makes the use of being left regular semirings, right regular semirings and intra-regular semirings.

Theorem 2.6. *Let S be a semiring with zero satisfying for every $a \in S$, $a = 2x$ for some $x \in S$. Then the following statements hold.*
(i) $DV_n(S)$ is regular iff S is regular.
(ii) $DV_n(S)$ is left regular iff S is left regular.
(iii) $DV_n(S)$ is right regular iff S is right regular.
(iv) $DV_n(S)$ is intra-regular iff S is intra-regular.

Proof. Let S be a semiring with zero satisfying for every $a \in S$, $a = 2x$ for some $x \in S$.

(i) is obtained from Theorem 2.5.
(ii) Assume $DV_n(S)$ is left regular. Let $a \in S$. Let $A \in M_n(S)$ be such that $A_{11} = A_{1n} = A_{n1} = A_{nn} = a$ and $A_{ij} = 0$ otherwise. Then $A \in DV_n(S)$. Thus $A = BA^2$ for some $B \in DV_n(S)$. Hence

$$\begin{aligned}
a = A_{11} = (BA^2)_{11} = (BAA)_{11} &\\
&= \sum_{k=1}^{n} B_{1k}(AA)_{k1} \\
&= B_{11}(AA)_{11} + B_{1n}(AA)_{n1} \\
&= B_{11}\sum_{k=1}^{n} A_{1k}A_{k1} + B_{1n}\sum_{k=1}^{n} A_{nk}A_{k1} \\
&= B_{11}(A_{11}A_{11} + A_{1n}A_{n1}) + B_{1n}(A_{n1}A_{11} + A_{nn}A_{n1}) \\
&= B_{11}(aa + aa) + B_{11}(aa + aa) \\
&= B_{11}a^2 + B_{11}a^2 + B_{11}a^2 + B_{11}a^2 \\
&= (4B_{11})a^2.
\end{aligned}$$

Therefore a is left regular for all $a \in S$.

Assume S is left regular. Let $A \in DV_n(S)$. For each $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$, let $a_i = A_{ii}$. Then for every $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$, $a_i = x_i a_i^2$ and $x_i = 4u_i$ for some $x_i, u_i \in S$.

Case 1: n is even.

Let $B \in M_n(S)$ be such that $u_i = B_{ii} = B_{i,n-i+1} = B_{n-i+1,i} = B_{n-i+1,n-i+1}$ for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$ and $B_{ij} = 0$ otherwise. Then $B \in DV_n(S)$. Let $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$. Thus

$$\begin{aligned}
(BAA)_{ii} = \sum_{k=1}^{n} B_{ik}(AA)_{ki} &= B_{ii}(AA)_{ii} + B_{i,n-i+1}(AA)_{n-i+1,i} \\
&= B_{ii}(AA)_{ii} + B_{ii}(AA)_{ii} \\
&= B_{ii}\sum_{k=1}^{n} A_{1k}A_{ki} + B_{ii}\sum_{k=1}^{n} A_{1k}A_{ki} \\
&= B_{ii}(A_{ii}A_{ii} + A_{ii}A_{ii}) + B_{ii}(A_{ii}A_{ii} + A_{ii}A_{ii}) \\
&= (4B_{ii})A_{ii}^2 \\
&= A_{ii}.
\end{aligned}$$

Case 2: n is odd.

Let $B \in DV_n(S)$ be such that $u_i = B_{ii} = B_{i,n-i+1} = B_{n-i+1,i} = B_{n-i+1,n-i+1}$ for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$, $x_i = B_{ii}$ if $i = \lceil \frac{n}{2} \rceil$ and $B_{ij} = 0$ otherwise. If $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$, then $(BAA)_{ii} = (4B_{ii})A_{ii}^2 = A_{ii}$. If $i = \lceil \frac{n}{2} \rceil$, then $(BAA)_{ii} = B_{ii}(AA)_{ii} = x_i A_{ii} A_{ii} = A_{ii}$.

Thus $(BAA)_{i,n-i+1} = (BAA)_{ii} = A_{ii} = A_{i,n-i+1}$ for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$. Also, $(BAA)_{n-i+1,i} = A_{n-i+1,i}$ and $(BAA)_{n-i+1,n-i+1} = A_{n-i+1,n-i+1}$. Otherwise, for $i, j \in \{1, 2, \ldots, n\}$ such that $j \neq i$ and $j \neq n-i+1$, we have $(BAA)_{ij} = 0 = A_{ij}$.
(iii) analogous to (ii).
(iv) Assume $DV_n(S)$ is intra-regular. Let $a \in S$. Let $A \in M_n(S)$ be such that $A_{11} = A_{1n} = A_{n1} = A_{nn} = a$ and $A_{ij} = 0$ otherwise. Then $A \in DV_n(S)$, so $A = BA^2C$ for some $B, C \in DV_n(S)$. It follows that

$$\begin{aligned}
a = A_{11} = (BA^2C)_{11} &= \sum_{k=1}^{n} B_{1k}(A^2C)_{k1} \\
&= B_{11}(A^2C)_{11} + B_{1n}(A^2C)_{n1} \\
&= B_{11}(A^2C)_{11} + B_{11}(A^2C)_{11} \\
&= B_{11}\sum_{k=1}^{n}(A^2)_{1k}C_{k1} + B_{11}\sum_{k=1}^{n}(A^2)_{1k}C_{k1} \\
&= B_{11}[(A^2)_{11}C_{11} + (A^2)_{11}C_{11}] + B_{11}[(A^2)_{11}C_{11} + (A^2)_{11}C_{11}] \\
&= (4B_{11})[(A^2)_{11}C_{11}] \\
&= (4B_{11})[\sum_{k=1}^{n} A_{1k}A_{k1}](C_{11}) \\
&= (4B_{11})[(A^2)_{11} + (A^2)_{11}](C_{11}) \\
&= (4B_{11})(A^2)_{11}(2C_{11}) \\
&= (4B_{11})a^2(2C_{11}).
\end{aligned}$$

Also, $a = (4B_{11})a^2(2C_{11}) = 2[(2B_{11})a^2(2C_{11})]$.

Conversely, assume S is intra-regular and for every $a \in S, a = 2x$ for some $x \in S$. For each $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$, let $a_i = A_{ii}$. Then for every $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$ there exist $x_i, y_i, u_i, v_i \in S$ such that $a_i = x_i a_i^2 y_i, x_i = 4u_i$ and $y_i = 2v_i$.

Case 1: n is even.

Let $B, C \in M_n(S)$ be such that

$$\begin{aligned}
&u_i = B_{ii} = B_{i,n-i+1} = B_{n-i+1,i} = B_{n-i+1,n-i+1} \text{ and} \\
&v_i = C_{ii} = C_{i,n-i+1} = C_{n-i+1,i} = C_{n-i+1,n-i+1}
\end{aligned}$$

for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$ and $B_{ij} = 0 = C_{ij}$ otherwise. Then $B \in$

$DV_n(S)$. Let $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$. Thus

$$\begin{aligned}
(BA^2C)_{ii} &= B_{ii}(A^2C)_{ii} + B_{i,n-i+1}(A^2C)_{n-i+1,i} \\
&= B_{ii}(A^2C)_{ii} + B_{ii}(A^2C)_{ii} \\
&= B_{ii}[(A^2)_{ii}C_{ii} + (A^2)_{ii}C_{ii}] + B_{ii}[(A^2)_{ii}C_{ii} + (A^2)_{ii}C_{ii}] \\
&= B_{ii}[(A_{ii}A_{ii} + A_{ii}A_{ii})C_{ii} + (A_{ii}A_{ii} + A_{ii}A_{ii})C_{ii}] \\
&\quad + B_{ii}[(A_{ii}A_{ii} + A_{ii}A_{ii})C_{ii} + (A_{ii}A_{ii} + A_{ii}A_{ii})C_{ii}] \\
&= (4B_{ii})(A_{ii}A_{ii} + A_{ii}A_{ii})C_{ii} \\
&= (4B_{ii})(A_{ii})^2(2C_{ii}) \\
&= x_i a_i^2 y_i \\
&= a_i \\
&= A_{ii}.
\end{aligned}$$

Case 2: n is odd.
Let $B, C \in DV_n(S)$ be such that

$$\begin{aligned}
u_i = B_{ii} = B_{i,n-i+1} = B_{n-i+1,i} = B_{n-i+1,n-i+1}, \\
v_i = C_{ii} = C_{i,n-i+1} = C_{n-i+1,i} = C_{n-i+1,n-i+1}
\end{aligned}$$

for all $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$, $x_i = B_{ii}$ and $y_i = C_{ii}$ if $i = \lceil \frac{n}{2} \rceil$ and $B_{ij} = 0 = C_{ij}$ otherwise. If $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$, then $(BA^2C)_{ii} = (4B_{ii})(A_{ii})^2(2C_{ii}) = A_{ii}$. If $i = \lceil \frac{n}{2} \rceil$, then $(BA^2C)_{ii} = B_{ii}(A_{ii})^2C_{ii} = A_{ii}$.

Hence for every $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$, $(BA^2C)_{i,n-i+1} = (BA^2C)_{ii} = A_{ii} = A_{i,n-i+1}$, $(BA^2C)_{n-i+1,i} = A_{n-i+1,i}$ and $(BA^2C)_{n-i+1,n-i+1} = A_{n-i+1,n-i+1}$. Otherwise, for $i, j \in \{1, 2, \ldots, n\}$ such that $j \neq i$ and $j \neq n - i + 1$, we have $(BA^2C)_{ij} = 0 = A_{ij}$. This completes the proof. □

Remark 2.7. Let F be a field. The following remarkable properties of $DV_n(F)$ are shown.
(1) If F is a finite field of order q, then $|M_n(F)| = q^{n^2}$ while

$$|DV_n(F)| = \begin{cases} q^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ q^{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

(2) As vector spaces over F, $\dim M_n(F) = n^2$, $DV_n(F)$ is a subspace of $M_n(F)$ and

$$\dim DV_n(F) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

For each $k \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, let $B^{(k)} \in M_n(F)$ be defined by

$$B^{(k)}_{ij} = \begin{cases} 1 & \text{if } i, j \in \{k, n-k+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

If n is odd, let $K \in M_n(F)$ be as follows:

$$\begin{bmatrix} 0 & & \cdots & 0 & \cdots & & 0 \\ 0 & \ddots & & \vdots & & \iddots & \\ & & 0 & 0 & 0 & & \\ \vdots & & 0 & 1 & 0 & & \vdots \\ & & 0 & 0 & 0 & & \\ & \iddots & & & & \ddots & \\ 0 & & \cdots & 0 & \cdots & & 0 \end{bmatrix}.$$

It is clear that if n is even, then $\{B^{(1)}, \ldots, B^{(\frac{n}{2})}\}$ is a basis of $DV_n(F)$ over F and if n is odd, then $\{B^{(1)}, \ldots, B^{(\frac{n-1}{2})}, K\}$ is a basis of $DV_n(F)$. Observe that for $A \in DV_n(F)$,

$$\begin{aligned} A &= A_{11}B^{(1)} + \cdots + A_{\frac{n}{2},\frac{n}{2}}B^{(\frac{n}{2})} && \text{if } n \text{ is even,} \\ A &= A_{11}B^{(1)} + \cdots + A_{\frac{n-1}{2},\frac{n-1}{2}}B^{(\frac{n-1}{2})} + A_{\frac{n+1}{2},\frac{n+1}{2}}K && \text{if } n \text{ is odd.} \end{aligned}$$

(3) For each $k \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$, we let

$$DV_n^{(k)}(F) = \{ A \in DV_n(F) \mid A_{kk} = A_{k,n-k+1} = A_{n-k+1,k} = A_{n-k+1,n-k+1} = 0 \}.$$

Then for every $k \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$, $DV_n^{(k)}(F)$ is a subspace of $DV_n(F)$ over F and $DV_n(F)/DV_n^{(k)}(F) \cong F$.
(4) $DV_n^{(k)}(F)$ are also ideals of the ring $DV_n(F)$ for all $k \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$.

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