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ALL MAXIMAL COMPLETELY REGULAR SUBMONOIDS OF $Hyp_G(n)$

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Abstract

A generalized hypersubstitution of type $\tau = (n)$ is a mapping σ which maps the n-ary operation symbol f to the term $\sigma(f)$ which does not necessarily preserve the arity. The set of all generalized hypersubstitutions of type $\tau = (n)$ together with a binary operation defined on this set and the identity hypersubstitution σ_{id} which maps f to the term $f(x_1, ..., x_n)$ forms a monoid. Our motivation in this paper, is to determine all maximal completely regular submonoids of this monoid.

1 Monoid of all Generalized Hypersubstitutions

In Universal Algebra, identities are used to classify algebras into collections called varieties. Hyperidentities are used to classify varieties into collections called hypervarieties. The tool which is used to study hyperidentities and hypervarieties is the concept of a hypersubstitution. The notion of a hypersubstitution was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert ([2]). In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of

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a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively ([4]). The set of all generalized hypersubstitutions together with a binary operation and the identity hypersubstitution forms a monoid.

Let $X := \{x_1, x_2, ...\}$ be the set of countably infinite variables and $X_n := \{x_1, x_2, ..., x_n\}$ which $n \in \mathbb{N}$ is an n-element set. Let $\{f_i | i \in I\}$ be a set of n_i-ary operation symbols indexed by the set I. Every n_i is called the *arity* of f_i and the sequence $\tau := (n_i)_{i\in I}$ of arities of f_i is called the *type*. An n-ary term of type τ is defined inductively, as follows

(i) Every $x_j \in X_n$ is an n-ary term of type τ;

(ii) If $t_1, t_2, ..., t_{n_i}$ are n-ary terms of type τ, then $f_i(t_1, t_2, ..., t_{n_i})$ is an n-ary term of type τ.

The smallest set, which contains $x_1, x_2, ..., x_n$ and is closed under finite application of (ii), is denoted by $W_\tau(X_n)$ and it is called the set of all n-ary terms of type τ. It is clear that every n-ary term is also an m-ary term for all $m \geq n$. Let $W_\tau(X) = \cup_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ.

A generalized hypersubstitution of type $\tau = (n_i)_{i\in I}$ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$, which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on the set of all generalized hypersubstitutions of type τ, we need the concept of a generalized superposition of terms and the extension of a generalized hypersubstitution, which are defined as follows.

Definition 1.1. ([4]) A generalized superposition of terms is a mapping $S^n : W_\tau(X)^{n+1} \to W_\tau(X)$ such that

(i) $S^n(x_j, t_1, ..., t_n) = t_j$, if $1 \leq j \leq n$;

(ii) $S^n(x_j, t_1, ..., t_n) = x_j$, if $n < j$;

(iii) $S^n(t, t_1, ..., t_n) = f_i(S^n(s_1, t_1, ..., t_n), ..., S^n(s_{n_i}, t_1, ..., t_n))$, if $t = f_i(s_1, ..., s_{n_i})$.

We extend every generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ such that

(i) $\hat{\sigma}[x_j] = x_j \in X$;

(ii) $\hat{\sigma}[f_i(t_1, t_2, ..., t_{n_i})] = S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$ for any n_i-ary operation symbol f_i and suppose that $\hat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma \circ_G \alpha := \hat{\sigma} \circ \alpha$ where $\circ$ denotes the usual composition of mappings and $\sigma, \alpha \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, ..., x_{n_i})$.

In 2000, S. Leeratanavalee and K. Denecke proved that for arbitrary terms $t, t_1, t_2, ..., t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \alpha \in Hyp_G(\tau)$, we have

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], ..., \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, ..., t_n)]$;

(*ii*) $(\hat{\sigma} \circ \alpha)\hat{} = \hat{\sigma} \circ \hat{\alpha}$.

Using the previous result, S. Leeratanavalee and K. Deneeke proved that $\underline{Hyp_G(\tau)} := (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid (for more detail in $Hyp_G(\tau)$ see [4]).

2 All maximal completely regular submonoids of $Hyp_G(n)$

The semigroup structure is studied in many fields of Mathematics. Moreover, semigroup theory is used to study formal language and automata theory in Theoretical Computer Science. There are many researchers study on some special elements of semigroup such as regular, left regular, right regular and completely regular elements. The main result of this paper is to determine the set of all maximal completely regular submonoids of the monoid of all generalized hypersubstitutions of type $\tau = (n)$.

Henceforth, we introduce some notations which will be used throughout this paper. For a type $\tau = (n)$ with an n-ary operation symbol f and $t \in W_{(n)}(X)$, we denote

σ_t := the generalized hypersubstitution of type $\tau = (n)$ which maps f to the term t;

$leftmost(t)$:= the first variable (from the left) occurs in t;

$rightmost(t)$:= the last variable occurs in t;

$var(t)$:= the set of all variables occur in t.

Next, we recall some definitions which will be used throughout this paper.

Definition 2.1. ([5]) Let $t \in W_{(n)}(X)$ and $i \in \mathbb{N}$ which $1 \leq i \leq n$, an $i - most(t)$ is defined inductively as follows:

(i) if t is a variable, then $i - most(t) = t$;

(ii) if $t = f(t_1, ..., t_n)$, then $i - most(t) = i - most(t_i)$.

Example 2.2. Let $\tau = (3)$ be a type and $t = f(x_2, f(x_8, x_5, x_3), f(x_1, x_6, x_4))$. Then $1 - most(t) = x_2$, $2 - most(t) = 2 - most(f(x_8, x_5, x_3)) = x_5$ and $3 - most(t) = 3 - most(f(x_1, x_6, x_4)) = x_4$.

Note that for $\tau = (n), 1 - most(t) = leftmost(t)$ and $n - most(t) = rightmost(t)$.

Definition 2.3. ([3]) Let S be a semigroup. An element a of a semigroup S is called *completely regular* if there exists $b \in S$ such that $a = aba$ and $ab = ba$.

Let $\sigma_t \in Hyp_G(n)$, we denote

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | t \in W_{(n)}(X) \setminus X$ and $var(t) \cap X_n = \varnothing\}$;

$CR(R_3) := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{\pi(i_1)}, ..., t_{i_m} = x_{\pi(i_m)}$ and π is a bijective map on $\{i_1, ..., i_m\}$ for some $i_1, ..., i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{\pi(i_1)}, ..., x_{\pi(i_m)}\}\}$.

In 2013, A. Boonmee and S. Leeratanavalee proved the following theorem.

Theorem 2.4. ([1]) $CR(Hyp_G(n)) := CR(R_3) \cup R_1 \cup R_2$ is the set of all completely regular elements in $Hyp_G(n)$.

Remark It is easily to see that $R_1, R_2, CR(R_3)$ are pairwise disjoint and R_1, R_2 are subsemigroups of $Hyp_G(n)$ but $CR(R_3)$ is not a submonoid of $Hyp_G(n)$.

Example 2.5. Let $\tau = (3)$ be a type. That means we have only one ternary operation symbol, say f. Let $\sigma_s, \sigma_t \in CR(R_3)$ where $t = f(x_3, x_6, x_1)$ and $s = f(f(x_2, x_5, x_3), x_3, x_2)$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(f(x_2, x_5, x_3), x_3, x_2)] \\ &= S^2(\sigma_t(f), \hat{\sigma}_t[f(x_2, x_5, x_3)], \hat{\sigma}_t[x_3], , \hat{\sigma}_t[x_2]) \\ &= S^2(\sigma_t(f), S^3(\sigma_t(f), \hat{\sigma}_t[x_2], \hat{\sigma}_t[x_5], \hat{\sigma}_t[x_3]), x_3, x_2) \\ &= S^3(\sigma_t(f), S^3(f(x_3, x_6, x_1), x_2, x_5, x_3), x_3, x_2) \\ &= S^3(f(x_3, x_6, x_1), f(x_3, x_6, x_2), x_3, x_2) \\ &= f(x_2, x_6, f(x_3, x_6, x_2)).\end{aligned}$$

Thus $\sigma_t \circ_G \sigma_s \notin CR(R_3)$, so $CR(R_3)$ is not closed under $\circ_G$.

Next, let $\sigma_t \in Hyp_G(n)$, we denote

$CR_1(R_3) := \{\sigma_t | t = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}\}$.

$E := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for some $i_1, ..., i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \neq i_l\}$.

For any $\varnothing \neq I \subset \{1, ..., n\}$, let

$CR_I(R_3) := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_i = x_{\pi(i)}$ for all $i \in I$ and π is a bijective map on $I, var(t) \cap X_n = \{x_{\pi(i)} | i \in I\}\}$.

$CR'_I(R_3) := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_i = x_{\pi(i)}; \pi(i) \in I$ for all $i \in I$ and $t_k = x_{\pi(k)}$ for all $k \in \{1, ..., n\} \setminus I$ and π is a bijective map on $\{1, ..., n\}\}$.

We let

$(MCR)_{Hyp_G(n)} := R_1 \cup R_2 \cup CR_1(R_3)$,

$(MCR_1)_{Hyp_G(n)} := R_1 \cup R_2 \cup E$ and

$(MCR_I)_{Hyp_G(n)} := R_1 \cup R_2 \cup CR_I(R_3) \cup CR'_I(R_3) \cup \{\sigma_{id}\}$.

Theorem 2.6. $(MCR)_{Hyp_G(n)}$ is a completely regular submonoid of $Hyp_G(n)$.

Proof. By Theorem 2.4, we have every element in $(MCR)_{Hyp_G(n)}$ is completely regular. Next we show that $(MCR)_{Hyp_G(n)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR)_{Hyp_G(n)} = R_1 \cup R_2 \cup CR_1(R_3)$. Since R_1, R_2 are closed under $\circ_G$, we will check for closedness only the following cases.

Case 1: $\sigma_t \in R_1, \sigma_s \in R_2 \cup CR_1(R_3)$. Then $t = x_i \in X$.

If $\sigma_s \in R_2$, then $s = f(s_1, ..., s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(s_1, ..., s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= S^n(x_i, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= \begin{cases} \hat{\sigma_t}[s_i], & \text{if } \ i \in \{1, ..., n\}; \\ x_i, & \text{if } \ i > n. \end{cases}\end{aligned}$$

For $1 \leq i \leq n$, since $t = x_i$, we have $\hat{\sigma_t}[s_i] = i - most(s_i) \in X$. Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(n)}$.

If $\sigma_s \in CR_1(R_3)$, then $s = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_{\pi(1)}, ..., x_{\pi(n)})] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[x_{\pi(1)}], ..., \hat{\sigma_t}[x_{\pi(n)}]) \\ &= S^n(x_i, x_{\pi(1)}, ..., x_{\pi(n)}) \\ &= \begin{cases} x_{\pi(i)}, & \text{if } \ i \in \{1, ..., n\} \\ x_i, & \text{if } \ i > n. \end{cases}\end{aligned}$$

Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(n)}$.

Case 2: $\sigma_t \in R_2, \sigma_s \in R_1 \cup CR_1(R_3)$. Then $t \in W_{(n)}(X) \setminus X$ and $var(t) \cap X_n = \varnothing$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(n)}$.

If $\sigma_s \in CR_1(R_3)$, then $s = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_{\pi(1)}, ..., x_{\pi(n)})] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[x_{\pi(1)}], ..., \hat{\sigma_t}[x_{\pi(n)}]) \\ &= S^n(f(t_1, ..., t_n), x_{\pi(1)}, ..., x_{\pi(n)}) \\ &= f(t_1, ..., t_n) \qquad \text{since } \ var(t) \cap X_n = \varnothing.\end{aligned}$$

Hence $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR)_{Hyp_G(n)}$.

Case 3: $\sigma_t \in CR_1(R_3), \sigma_s \in R_1 \cup R_2 \cup CR_1(R_3)$. Then $t = f(x_{\pi_1(1)}, ..., x_{\pi_1(n)})$ where π_1 is a bijective map on $\{1, ..., n\}$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(n)}$.

If $\sigma_s \in R_2$, then $s = f(s_1, ..., s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(s_1, ..., s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= S^n(f(x_{\pi(1)}, ..., x_{\pi(n)}), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= f(\hat{\sigma_t}[s_{\pi(1)}], ..., \hat{\sigma_t}[s_{\pi(n)}]).\end{aligned}$$

Since $var(\hat{\sigma_t}[s_i]) \cap X_n = \varnothing \quad \forall i \in \{1, ..., n\}$, we have $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR)_{Hyp_G(n)}$.

If $\sigma_s \in CR_1(R_3)$, then $s = f(x_{\pi_2(1)}, ..., x_{\pi_2(n)})$ where π_2 is a bijective map on $\{1, ..., n\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_{\pi_2(1)}, ..., x_{\pi_2(n)})] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[x_{\pi_2(1)}], ..., \hat{\sigma_t}[x_{\pi_2(n)}]) \\ &= S^n(f(x_{\pi_1(1)}, ..., x_{\pi_1(n)}), x_{\pi_2(1)}, ..., x_{\pi_2(n)}) \\ &= f(x_{\pi_2(\pi_1(1))}, ..., x_{\pi_2(\pi_1(n))}) \\ &= f(x_{(\pi_2 \circ \pi_1)(1)}, ..., x_{(\pi_2 \circ \pi_1)(n)}).\end{aligned}$$

Since $\pi_1 \circ \pi_2$ is a bijective map on $\{1, ..., n\}$, we have $\sigma_t \circ_G \sigma_s \in CR_1(R_3)$. Therefore $\underline{(MCR)_{Hyp_G(n)}}$ is a completely regular submonoid of $\underline{Hyp_G(n)}$. □

Theorem 2.7. $\underline{(MCR_1)_{Hyp_G(n)}}$ is a completely regular submonoid of $\underline{Hyp_G(n)}$.

Proof. By Theorem 2.4, we have every element in $(MCR_1)_{Hyp_G(n)}$ is completely regular. Next we show that $(MCR_1)_{Hyp_G(n)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR_1)_{Hyp_G(n)} = R_1 \cup R_2 \cup E$. Since R_1, R_2 are closed under $\circ_G$, we will check for closeness only the following cases.

Case 1: $\sigma_t \in R_1, \sigma_s \in R_2 \cup E$. We can prove similarly as in Case 1 of Thorem 2.6, and conclude that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_1)_{Hyp_G(n)}$.

Case 2: $\sigma_t \in R_2, \sigma_s \in R_1 \cup E$. We can prove similarly as in Case 2 of Thorem 2.6, and conclude that $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_1)_{Hyp_G(n)}$.

Case 3: $\sigma_t \in E, \sigma_s \in R_1 \cup R_2 \cup E$. Then $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for some $i_1, ..., i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \neq i_l$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_1)_{Hyp_G(n)}$.

If $\sigma_s \in R_2$, then $s = f(s_1, ..., s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(s_1, ..., s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= S^n(f(t_1, ..., t_n), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= f(w_1, ..., w_n) \quad \text{where} \quad w_i = S^n(t_i, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &\quad \text{for all} \quad i \in \{1, .., n\}.\end{aligned}$$

Since $var(\hat{\sigma_t}[s_i]) \cap X_n = \varnothing \quad \forall i \in \{1, ..., n\}$, we have $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_1)_{Hyp_G(n)}$.

If $\sigma_s \in E$, then $s = f(s_1, ..., s_n)$ where $s_{p_1} = x_{p_1}, ..., s_{p_{m'}} = x_{p_{m'}}$ for some $p_1, ..., p_{m'} \in \{1, ..., n\}$ and $var(s) \cap X_n = \{x_{p_1}, ..., x_{p_{m'}}\}$ and if $x_{p_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$ and $k' \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}\}$, then $j' - most(s_{k'}) \neq x_{p_{l'}}$ for all $j' \neq p_{l'}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(s_1, ..., s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= S^n(f(t_1, ..., t_n), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= f(w_1, ..., w_n) \quad \text{where} \;\; w_i = S^n(t_i, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &\;\;\text{for all} \;\; i \in \{1, .., n\}.\end{aligned}$$

Case 1: $var(t_k) \cap X_n = \varnothing$ for all $k \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$ and $var(s_{k'}) \cap X_n = \varnothing$ for all $k' \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}\}$.

Case 1.1: $i \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$. Then $w_i = S^n(t_i, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) = t_i$.

Case 1.2: $i \in \{i_1, ..., i_m\}$. Then $w_i = S^n(t_i, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) = \hat{\sigma_t}[s_i]$.

If $i \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}\}$, then $var(w_i) \cap X_n = \varnothing$. If $i \in \{p_1, ..., p_{m'}\}$, then $w_i = x_i$. By Case 1.1, 1.2, we have $\sigma_t \circ \sigma_s \in (R_2 \cup E) \subset (MCR_1)_{Hyp_G(n)}$.

Case 2: $var(t_k) \cap X_n = \varnothing$ for all $k \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$ and there exists $x_{p_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$, for all $k' \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}\}$. It can be proved similarly as in Case 1. Hence $\sigma_t \circ \sigma_s \in (R_2 \cup E) \subset (MCR_1)_{Hyp_G(n)}$.

Case 3: There exists $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$, for all $k \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$ and there exists $x_{p_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$, for all $k' \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}\}$.

Case 3.1: $i \in \{i_1, ..., i_m\}$. Then $w_i = \hat{\sigma_t}[s_i]$.

For $i \in \{p_1, ..., p_{m'}\}$, we have $w_i = x_i$.

For $i = k'$, we have $w_i = \hat{\sigma_t}[s_i]$. If $i_l = p_{l'}$, then $w_{i_l} = x_{i_l}$. If $i_l \neq p_{l'}$, then $var(w_i) \cap \{x_{i_l}\} = \varnothing$.

For $i \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}, k'\}$. Then $var(w_i) \cap X_n = \varnothing$.

Case 3.2: $i \in \{1, ..., n\} \backslash \{i_1, ..., i_m, k\}$. Then $var(w_i) \cap X_n = \varnothing$. By Case 3.1,3.2, we have $\sigma_t \circ \sigma_s \in (R_2 \cup E) \subset (MCR_1)_{Hyp_G(n)}$. □

Theorem 2.8. $\underline{(MCR_I)_{Hyp_G(n)}}$ is a completely regular submonoid of $\underline{Hyp_G(n)}$.

Proof. By Theorem 2.4, we have every element in $(MCR_I)_{Hyp_G(n)}$ is completely regular. Next we show that $(MCR_I)_{Hyp_G(n)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR_I)_{Hyp_G(n)} = R_1 \cup R_2 \cup CR_I(R_3) \cup CR'_I(R_3) \cup \{\sigma_{id}\}$. Since R_1, R_2 are closed under $\circ_G$ and σ_{id} is an identity element, we will check for closeness only the following cases.

Case 1: $\sigma_t \in R_1, \sigma_s \in R_2 \cup CR_I(R_3) \cup CR'_I(R_3)$. We can prove similarly as in Case 1 of Thorem 2.6, and conclude that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_I)_{Hyp_G(n)}$.

Case 2: $\sigma_t \in R_2, \sigma_s \in R_1 \cup CR_I(R_3) \cup CR'_I(R_3)$. We can prove similarly as in Case 2 of Thorem 2.6, and conclude that $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_I)_{Hyp_G(n)}$.

Case 3: $\sigma_t \in CR_I(R_3), \sigma_s \in R_1 \cup R_2 \cup CR_I(R_3) \cup CR'_I(R_3)$. Then $t = f(t_1, ..., t_n)$ where $t_i = x_{\pi_1(i)}$ for all $i \in I$ and π_1 is a bijective map on I, $var(t) \cap X_n = \{x_{\pi(i)} \mid i \in I\}$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_I)_{HypG(n)}$.

If $\sigma_s \in R_2$, then $s = f(s_1, ..., s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, ..., s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\ &= S^n(f(t_1, ..., t_n), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\ &= f(w_1, ..., w_n) \quad \text{where} \quad w_i = S^n(t_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\ &\quad \text{for all} \quad i \in \{1, .., n\}.\end{aligned}$$

Since $var(\hat{\sigma}_t[s_i]) \cap X_n = \varnothing$ for all $i \in \{1, ..., n\}$, we have $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_I)_{HypG(n)}$.

If $\sigma_s \in CR_I(R_3)$, then $s = f(s_1, ..., s_n)$ where $s_i = x_{\pi_2(i)}$ for all $i \in I$ and π_2 is a bijective map on I, $var(s) \cap X_n = \{x_{\pi_2(i)} \mid i \in I\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, ..., s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\ &= S^n(f(t_1, ..., t_n), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\ &= f(w_1, ..., w_n) \quad \text{where} \quad w_i = S^n(t_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\ &\quad \text{for all} \quad i \in \{1, .., n\}.\end{aligned}$$

For any $i_l \in I$, since π_1, π_2 are bijective maps on I there exist $i_p, i_q \in I$ such that $\pi_1(i_l) = i_p$ and $\pi_2(i_p) = i_q$. Then $w_{i_l} = S^n(t_{i_l}, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) = S^n(x_{\pi_1(i_l)}, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) = \hat{\sigma}_t[s_{i_p}] = \hat{\sigma}_t[x_{\pi_2(i_p)}] = x_{i_q}$.

For any $j \in \{1, ..., n\} \setminus I$, let $t_j = f(u_1, ..., u_n)$. Consider

$$\begin{aligned}w_j &= S^n(t_j, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\ &= S^n(f(u_1, .., u_n), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\ &= f(w'_1, ..., w'_n) \quad \text{where} \quad w'_k = S^n(t_k, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\ &\quad \text{for all} \quad k \in \{1, .., n\}.\end{aligned}$$

If $var(u_k) \cap X_n = \varnothing$, then $w'_k = u_k$. If $u_k = x_{\pi_1(i_l)}$ and $\pi_1(i_l) = i_p, \pi_2(i_p) = i_q$, then $w'_k = S^n(u_k, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) = S^n(x_{\pi_1(i_l)}, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) = \hat{\sigma}_t[s_{i_p}] = x_{\pi_2(i_p)} = x_{i_q}; i_q \in I$. Hence $\sigma_t \circ_G \sigma_s \in CR_I(R_3) \subset (MCR_I)_{HypG(n)}$.

If $\sigma_s \in CR'_I(R_3)$, we can prove as in the previous proof. Hence $\sigma_t \circ_G \sigma_s \in CR_I(R_3) \subset (MCR_I)_{HypG(n)}$.

Case 4: $\sigma_t \in CR'_I(R_3), \sigma_s \in R_1 \cup R_2 \cup CR_I(R_3) \cup CR'_I(R_3)$. It can be proved similarly as in Case 3. Hence $\sigma_t \circ_G \sigma_s \in (MCR_I)_{HypG(n)}$. Therefore $\underline{(MCR_I)_{HypG(n)}}$ is a completely regular submonoid of $\underline{HypG(n)}$. □

Theorem 2.9. $\underline{(MCR)_{Hyp_G(n)}}$ is a maximal completely regular submonoid of $\underline{Hyp_G(n)}$.

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $\underline{Hyp_G(n)}$ such that $(MCR)_{Hyp_G(n)} \subseteq K \subset Hyp_G(n)$. Let $\sigma_t \in K$ where $\sigma_t \in \overline{CR(R_3) \setminus} CR_1(R_3)$. Then $t = f(t_1, ..., t_n)$ where $t_{i_l} = x_{\pi(i_l)}$ for all $i_l \in I$ and π is a bijective map on I, $var(t) \cap X_n = \{x_{\pi(i_l)} \mid i_l \in I\}$. Choose $\sigma_s \in CR_1(R_3)$ such that $s = f(x_{\pi'(1)}, ..., x_{\pi'(n)})$ where π' is a bijective map on $\{1, ..., n\}$ and $\pi' = (\pi'(1)...\pi'(n))$ is a cycle. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_{\pi'(1)}, ..., x_{\pi'(n)})] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[x_{\pi'(1)}], ..., \hat{\sigma_t}[x_{\pi'(n)}]) \\ &= S^n(f(t_1, ..., t_n), x_{\pi'(1)}, ..., x_{\pi'(n)}) \\ &= f(w_1, ..., w_n) \quad \text{where} \quad w_j = S^n(t_j, x_{\pi'(1)}, ..., x_{\pi'(n)}) \\ &\quad \text{for all} \quad j \in \{1, .., n\}.\end{aligned}$$

Since $I \subset \{1, ..., n\}$, there exist $i_p \in I$, $i_q \in \{1, ..., n\} \setminus I$ such that $\pi'(i_p) = i_q$ and $\pi(i_l) = i_p$, for some $i_l \in I$, then

$$\begin{aligned}w_{i_l} &= S^n(t_{i_l}, \hat{\sigma_t}[x_{\pi'(1)}], ..., \hat{\sigma_t}[x_{\pi'(n)}]) \\ &= S^n(x_{\pi(i_l)}, x_{\pi'(1)}, ..., x_{\pi'(n)}) \\ &= x_{\pi'(i_p)} \\ &= x_{i_q}.\end{aligned}$$

By Theorem 2.4, $\sigma_s \circ_G \sigma_t$ is not completely regular, so $\sigma_t \in (MCR)_{Hyp_G(n)}$. Therefore $K \subseteq (MCR)_{Hyp_G(n)}$ and thus $\underline{K} = \underline{(MCR)_{Hyp_G(n)}}$. □

Theorem 2.10. $\underline{(MCR_1)_{Hyp_G(n)}}$ is a maximal completely regular submonoid of $\underline{Hyp_G(n)}$.

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $\underline{Hyp_G(n)}$ such that $(MCR_1)_{Hyp_G(n)} \subseteq K \subset Hyp_G(n)$. Let $\sigma_t \in K$, then σ_t is a completely regular element.

Case 1: $\sigma_t \in CR_1(R_3)$. Then $t = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}$.

Case 1.1: $t = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}$ and $(\pi(1)...\pi(n))$ is a cycle. Choose $\sigma_s \in E$ then $s = f(s_1, ..., s_n)$ where $s_{i_1} = x_{i_1}, ..., s_{i_m} = x_{i_m}$ and $s_j \in X \setminus X_n$, for all $j \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma_s}[f(x_{\pi(1)}, ..., x_{\pi(n)})] \\ &= S^n(\sigma_s(f), \hat{\sigma_s}[x_{\pi(1)}], ..., \hat{\sigma_s}[x_{\pi(n)}]) \\ &= S^n(f(s_1, ..., s_n), x_{\pi(1)}, ..., x_{\pi(n)}) \\ &= f(w_1, ..., w_n) \quad \text{where} \quad w_j = S^n(s_j, x_{\pi(1)}, ..., x_{\pi(n)}) \\ &\quad \text{for all} \quad j \in \{1, .., n\}.\end{aligned}$$

If $i_l \in \{i_1, ..., i_m\}$, then $w_{i_l} = x_{\pi(i_l)}$. Since $(\pi(1)...\pi(n))$ is a cycle, we have that $x_{\pi(i_l)} = x_{i_q}$; $i_q \in \{i_1, ...i_m\}\backslash\{i_l\}$. If $j \in \{1, ..., n\}\backslash\{i_1, ..., i_m\}$, then $w_j = s_j$. By Theorem 2.4, we have $\sigma_s \circ_G \sigma_t$ is not completely regular.

Case 1.2: $t = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}$ and there is $P = \{R_1, ..., R_l\}$ is a partition of $\{1, ..., n\}$ such that $R_1 = \{r_{11}, ..., r_{1f}\}, ..., R_l = \{r_{l1}, ..., r_{lh}\}$ and $(r_{11}...r_{1f})...(r_{l1}, ..., r_{lh})$. Let $d \in R_k$ for some $k \in \{1, ..., l\}$ and $|R_k| > 1$. Choose $\sigma_s \in E$, then $s = f(s_1, ..., s_n)$ where $s_d = x_d$ and $s_q \in X \setminus X_n$, for all $q \in \{1, ..., n\} \setminus \{d\}$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma_s}[f(x_{\pi(1)}, ..., x_{\pi(n)})]\\ &= S^n(\sigma_s(f), \hat{\sigma_s}[x_{\pi(1)}], ..., \hat{\sigma_s}[x_{\pi(n)}])\\ &= S^n(f(s_1, ..., s_n), x_{\pi(1)}, ..., x_{\pi(n)})\\ &= f(w_1, ..., w_n) \quad \text{where } \ w_j = S^n(s_j, x_{\pi(1)}, ..., x_{\pi(n)})\\ &\ \ \text{for all } \ j \in \{1, .., n\}.\end{aligned}$$

Then $w_j = x_{\pi(j)}$; $q \in \{1, ..., n\}\backslash\{i\}$. Since $d \in R_k$ for some $k \in \{1, ..., l\}$ and $|R_k| > 1$, we have $x_{\pi(j)} = x_q$ and $w_i = s_i$, $i \in \{1, ..., n\}\backslash\{d\}$. By Theorem 2.4, we have $\sigma_s \circ_G \sigma_t$ is not completely regular.

Case 2: $\sigma_t \in CR_I(R_3)\backslash E$. Then $t = f(t_1, ..., t_n)$ where $t_i = x_{\pi(i)}$ for all $i \in I$ and π is a bijective map on I, $var(t) \cap X_n = \{x_{\pi(i)} \mid i \in I\}$. Choose $\sigma_s \in E$ where $s = f(x_k, ..., x_k)$ for some $k \in \{1, ..., n\}\backslash I, x_k \neq x_{\pi(i)}$ for all $i \in I$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_k, ..., x_k)]\\ &= S^n(\sigma_t(f), \hat{\sigma_t}[x_k], ..., \hat{\sigma_t}[x_k])\\ &= S^n(f(t_1, ..., t_n), x_k, ..., x_k)\\ &= f(w_1, ..., w_n) \quad \text{where } \ w_j = S^n(t_j, x_k, ..., x_k)\\ &\ \ \text{for all } \ j \in \{1, .., n\}.\end{aligned}$$

Then $w_k = t'_k$ where t'_k is a new term derived by substituting x_{i_l} for all $i_l \in I$ which occur in t_k by x_k. By Theorem 2.4, we have $\sigma_s \circ_G \sigma_t$ is not completely regular. Hence $\sigma_t \in (MCR_1)_{Hyp_G(n)}$. Therefore $K \subseteq (MCR_1)_{Hyp_G(n)}$ and thus $\underline{K} = \underline{(MCR_1)_{Hyp_G(n)}}$. □

Theorem 2.11. $\underline{(MCR_I)_{Hyp_G(n)}}$ is a maximal completely regular submonoid of $\underline{Hyp_G(n)}$.

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $\underline{Hyp_G(n)}$ such that $(MCR_I)_{Hyp_G(n)} \subseteq K \subset Hyp_G(n)$. Let $\sigma_t \in K$ where $\sigma_t \in CR(R_3) \setminus (CR_I(R_3) \cup CR'_I(R_3) \cup E)$ then $t = f(t_1, ..., t_n)$ where $t_i = x_{\pi(i)}$ and π is a bijective map on $\{1, ..., n\}$. Choose $\sigma_s \in CR_I(R_3)$ then $s = f(s_1, ..., s_n)$ where $s_i =$

$x_{\pi'(i)}$ for all $i \in I$ and π' is a bijective map on I, $var(s) \cap X_n = \{x_{\pi'(i)} \mid i \in I\}$ and $s_j \in X \setminus X_n$ for all $j \in \{1, ..., n\} \setminus I$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma_s}[f(x_{\pi(1)}, ..., x_{\pi(n)})] \\ &= S^n(\sigma_s(f), \hat{\sigma_s}[x_{\pi(1)}], ..., \hat{\sigma_s}[x_{\pi(n)}]) \\ &= S^n(f(s_1, ..., s_n), x_{\pi(1)}, ..., x_{\pi(n)}) \\ &= f(w_1, ..., w_n) \quad \text{where} \quad w_j = S^n(s_j, x_{\pi(1)}, ..., x_{\pi(n)}) \\ &\quad \text{for all} \quad j \in \{1, .., n\}.\end{aligned}$$

Since $I \subset \{1, ..., n\}$ there exist $i_p \in I$, $i_q \in \{1, ..., n\} \setminus I$ such that $\pi(i_p) = i_q$ and $\pi'(i_r) = i_p;\ i_r \in I$. Then

$$\begin{aligned}w_{i_r} &= S^n(s_{i_r}, x_{\pi(1)}, ..., x_{\pi(n)}) \\ &= S^n(x_{i_p}, x_{\pi(1)}, ..., x_{\pi(n)}) \\ &= x_{\pi(i_p)} \\ &= x_{i_q}.\end{aligned}$$

By Theorem 2.4, $\sigma_s \circ_G \sigma_t$ is not completely regular, so $\sigma_t \in (MCR_I)_{Hyp_G(n)}$. Therefore $K \subseteq (MCR_I)_{Hyp_G(n)}$ and thus $\underline{K} = \underline{(MCR_I)_{Hyp_G(n)}}$. □

Corollary 2.12. $\{\underline{(MCR)_{Hyp_G(n)}}, \underline{(MCR_1)_{Hyp_G(n)}}\} \cup \{\underline{(MCR_I)_{Hyp_G(n)}} \mid \varnothing \neq I \subset \{1, ..., n\}\}$ is the set of all maximal completely regular submonoids of $\underline{Hyp_G(n)}$.

Proof. By using Theorem 2.9 to Theorem 2.11, we have $\{\underline{(MCR)_{Hyp_G(n)}}, \underline{(MCR_1)_{Hyp_G(n)}}\} \cup \{\underline{(MCR_I)_{Hyp_G(n)}} \mid \varnothing \neq I \subset \{1, ..., n\}\}$ is the set of all maximal completely regular submonoids of $\underline{Hyp_G(n)}$. □

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