East-West J. of Mathematics, Vol. 25, No. 1 (2024) pp. 61-78
https://doi.org/10.36853/ewjm0401

SELECTIVE INERTIAL BLOCK-ITERATIVE SCHEMES FOR A CLASS OF VARIATIONAL INEQUALITIES AND APPLICATIONS

Nguyen Thi Quynh Anh

The People's Police University of Technology and Logistics.
Thuan Thanh, Bac Ninh, Viet Nam.
e-mail: quynhanhnguyen0178@gmail.com

Abstract *Our purpose in this paper is to present inertial block-iterative schemes with selective technique for finding a solution of a variational inequality problem over the set of common fixed points of a finite family of demiclosed quasi-nonexpansive mappings in Hilbert spaces. First, we introduce a basic scheme and show that any sequence, generated by this scheme, converges weakly to a point in the common fixed point set. Then, based on a specific combination of the scheme with the steepest-descent method, we propose new schemes, strong convergence of which is proved without the approximately shrinking and boundedly regular assumptions on the mappings and their fixed point sets, respectively, that are usually required recently in literature. An application to study a networked system and computational experiments are given for illustration and comparison.*

1. Introduction

Let H be a Hilbert space equipped with the inner product $\langle .,. \rangle$, the corresponding norm $\|.\|$ and with the identity mapping I. Let T_i, for $i \in L := \{1, ..., m\}$ with a finite integer $m \geq 1$, be a demiclosed quasi-nonexpansive mapping on H with the property $\cap_{i\in L}\mathrm{Fix}(T_i) \neq \emptyset$ where $\mathrm{Fix}(T_i) = \{p \in H : p = T_i p\}$, the fixed point set of T_i. The considered problem in this paper is to find a point

$$p_* \in C := \cap_{i\in L}\mathrm{Fix}(T_i) \text{ such that } \langle Fp_*, p_* - p\rangle \leq 0 \ \forall p \in C, \tag{1.1}$$

Key words: Quasi-nonexpansive mapping · fixed point · variational inequality · block-iterative scheme..
2010 AMS Classification: 46N10, 47H10, 47J25,47N10, 65F10,65J99.

where F is η-strongly monotone and l-Lipschitz continuous on H.

When T_i is a nonexpansive mapping, Yamada [31] proposed the hybrid steepest-descent method,

$$x^{k+1} = (I - t_k\mu F)Tx^k, \ k \geq 0, \tag{1.2}$$

where $\mu \in (0, 2\eta/l^2)$ is a fixed number and T is either $T_m T_{m-1}...T_1$ or $\sum_{i\in L}\omega_i T_i$ with $\omega_i \in (0,1)$ and $\sum_{i\in L}\omega_i = 1$, and proved the strong convergence of method (1.2) under two conditions on t_k, one of which is that

(t) $t_k \in (0,1)$ for all $k \geq 1$, $\lim_{k\to\infty} t_k = 0$ and $\sum_{k\geq 0} t_k = \infty$.

In [7], Ng. Buong and L.T.T. Duong gave a modification of (1.2),

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k(I - t_k\mu F)T^k x^k, \ k \geq 0, \tag{1.3}$$

$\alpha_k \in [\varepsilon, 1]$ and $T^k = T_m^k T_{m-1}^k...T_1^k$ where $T_i^k = I + \beta_i^k(T_i - I)$, β_k has the property

(β) $\beta_i^k \in [\underline{\beta}, \overline{\beta}] \subset (0,1)$

and t_k satisfies only condition (t).

When each T_i is quasi-nonexpansive, method (1.2) and its modifications have been recently investigated in [9-15]. Cegielski and Zalas [12] proposed the generalized hybrid steepest-descent method with selective technique, called selective hybrid steepest descent method,

$$x^{k+1} = (I - t_k F)T_{i_k}x^k, \tag{1.4}$$

where i_k is selected by

$$i_k = \arg\max_{i\in L} \|T_i x^k - x^k\|. \tag{1.5}$$

The strong convergence of (1.4)–(1.5) is guaranteed when the mapping T_i is approximately shrinking for each $i \in L$ and the family $\mathcal{F} := \{\mathrm{Fix}(T_i) : i \in L\}$ is boundedly regular. A combination of method (1.4)–(1.5) with the outer approximations, has been presented by Gibali et al. [15] with the same properties of T_i and $\mathcal{F}$ as the above. The last two conditions are deleted by He and Tian [17], when T_i is nonexpansive and then T_{i_k} in the equivalent form to (1.4), $x^{k+1} = T_{i_k}(I - t_k F)x^k$, is replaced by $I + \beta_i^k(T_{i_k} - I)$. Very recently, Ng. Buong [8] gave a scheme, that is a specific combination of the block-iterative method, introduced by Aleyner and Reich [1] for the convex feasibility problem, with the steepest-descent method and proved the strong convergence without the approximately shrinking and boundedly regular assumptions, where μ is still chosen in dependence of η and l.

To speed up the convergence of the Krasnoselskii [20]- Mann [25] iterative method of finding a point in C, for the general case $m = \infty$, Maingé [22,23] suggested a combination of this method with an inertial effect, that is

$$\begin{aligned} u^k &= x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} &= (1-\alpha_k)x^k + \alpha_k T^k u^k, \end{aligned} \tag{1.6}$$

where $T^k = (1/\gamma^k)\sum_{i\geq 1}\gamma_i T_i$ with $\gamma_i > 0, \sum_{i\geq 1}\gamma_i = 1$ and $\gamma^k = \sum_{i=1}^k \gamma_i$, $\alpha_k \in (0,2)$ and θ_k is chosen such that

(c1) $\{\theta_k\} \subset [0,\theta]$, where $\theta \in [0,1)$.

(c2) $\sum_{k\geq 1}\theta_k\|x^k - x^{k-1}\|^2 < \infty$.

When $m = 1$ and T is nonexpasnive on H, Bot et al. [6] also studied the convergence of (1.6) with removing condition (c2). But, they required a strict condition on θ_k and α_k. Shehu [27] combined the inertial effect u_k in (1.6) with the Ishikawa [19] iterative method under weaker conditions on the inertial factor θ_k and iterative parameters α_k than those in [6]. Recently, in order to obtain a strongly convergent sequence, Tan et al. [29], by combining the inertial Krasnoselskii-Mann iterative method with the Halpern [16] and Moudafi [26] viscosity approximation methods under a new condition on θ_k,

(c3) $\lim_{k\to\infty}\frac{\theta_k}{t_k}\|x^k - x^{k-1}\| = 0$.

In this paper, based on method (1.3) and a reconstruction of a method in [1] with the inertial effect u_k, we first propose an inertial block-iterative scheme with selective technique, that converges weakly to a point in C. Next, for solving (1.1), we introduce new block-iterative schemes and prove their strong convergence without the approximately shrinking and boundedly regular assumptions on T_i and $\mathcal{F}$, respectively. Moreover, the parameter μ in our schemes is chosen through an adaptive way.

We organize the rest of this paper as follows. In Section 2, we list some terminologies, using in this paper, and related facts, that will be used in the proof of our results. In Section 3, we suggest two block-iterative schemes with several modifications and give convergence theorems for the schemes and modifications. An application to a networked system and computational experiments are given for illustration and comparison.

2. Preliminaries

We remember that an operator T in H is called (see, [14]):

- nonexpansive or contractive if $\|Tx - Ty\| \leq a\|x - y\|$ with $a = 1$ or $a \in [0,1)$, respectively, for all $x, y \in H$.

- quasi-nonexpansive, if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in H$ and $p \in \mathrm{Fix}(T)$.

- a cutter, if $\mathrm{Fix}(T) \neq \emptyset$ and $\langle p - Tx, x - Tx\rangle \leq 0$ for all $x \in H$ and $p \in \mathrm{Fix}(T)$.

- ρ-strongly quasi-nonexpansive, if $\mathrm{Fix}(T) \neq \emptyset$ and, for all $x \in H$ and $p \in \mathrm{Fix}(T)$, $\|Tx - p\|^2 \leq \|x - p\|^2 - \rho\|Tx - x\|^2$ where the real number $\rho \geq 0$.

- approximately shrinking on a subset $D \subseteq H$, if for any sequence $\{x^k\} \subseteq D$ the following implication holds

$$\lim_{k\to\infty} \|Tx^k - x^k\| = 0 \Longrightarrow \lim_{k\to\infty} \rho(x^k, \mathrm{Fix}(T)) = 0. \tag{2.1}$$

- demiclosed if for any sequence $\{x^k\} \subset H$ it holds

$$(x^k \text{converges weakly to } \; x \text{ and } \|Tx^k - x^k\| \to 0) \Longrightarrow x \in \mathrm{Fix}(T). \tag{2.2}$$

- The family $\{\mathrm{Fix}(T_i) : i \in L\}$ is called boundedly regular, if for any bounded subset $D \subset H$ and for any $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for any $x \in D$

$$\max_{i\in L} \rho(x, C_i) \leq \delta \Longrightarrow \rho(x, C) \leq \varepsilon,$$

  where $\rho(x, C) = \inf_{y\in C} \|x - y\|$.

Clearly, a ρ-strongly quasi-nonexpansive mapping T with $\rho = 0$ is quasi-nonexpansive and a nonexpansive mapping T with $\mathrm{Fix}(T) \neq \emptyset$ is quasi - nonexpansive. In this case $\mathrm{Fix}(T)$ is closed and convex. A ρ-strongly quasi-nonexpansive mapping T with $\rho = 1$ is a cutter. A mapping T is ρ-strongly quasi-nonexpansive, if and only if $\lambda\langle p-x, Tx-x\rangle \geq \|Tx-x\|^2$ for all $x \in H$ and all $p \in \mathrm{Fix}(T)$, where $\lambda = 2/(\rho + 1)$. It is worth mentioning that in a general Hilbert space H (2.2) is only a necessary condition for implication (2.1) and even a firmly nonexpansive mapping may not have this property (see, [15]). A mapping $F : H \to H$ is said to be η-strongly monotone and γ-Lipschitz continuous, if it satisfies, respectively, the conditions $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$ and $\|Fx - Fy\| \leq \gamma\|x - y\|$ for all $x, y \in H$ with $\gamma \geq \eta > 0$. It is well known that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \; \forall x, y \in H.$$

Lemma 2.1 ([31]) *Let H be a real Hilbert space and let F be an η-strongly monotone and l-Lipschitz continuous mapping on H with some positive constants $l \geq \eta > 0$. Let $T^{\mu} = I - \mu F$ and let $T^{t,\mu} = I - t\mu F$. Then, for a fixed number $\mu \in (0, 2\eta/l^2)$ and any $t \in (0, 1)$, $I-\mu F$ and $I-t\mu F$ are all contractions with coefficients $1 - \tau$ and $1 - t\tau$, respectively, where $\tau = (1/2)\mu(2\eta - \mu l^2)$.*

Lemma 2.2 ([30]) *Let $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ be sequences of real numbers such that, for all $k \geq 0$, $a_{k+1} \leq (1 - b_k)a_k + b_k c_k$; $a_k \geq 0$; b_k satisfies a condition of type (t); and either $\sum_{k=1}^{\infty} b_k|c_k| < \infty$ or $\limsup_{k\to\infty} c_k \leq 0$. Then, $lim_{k\to\infty} a_k = 0$.*

Lemma 2.3 ([22]) *Let $\{a_k\}$ be a sequence of real numbers with a subsequence $\{l_k\}$ of $\{k\}$ such that $a_{l_k} < a_{l_k+1}$. Then, there exists a nondecreasing sequence $\{m_k\} \subseteq \{k\}$ such that $m_k \to \infty, a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$ for all (sufficiently large) numbers $k \geq 0$. In fact, $m_k = \max\{l \leq k : a_l \leq a_{l+1}\}$.*

Lemma 2.4 ([2,3]) *Let* $\{\varphi_k\} \subset [0, \infty)$ *and* $\{\delta_k\} \subset [0, \infty)$ *verify*
(i) $\varphi_{k+1} - \varphi_k \leq \theta_k(\varphi_k - \varphi_{k-1}) + \delta_k$,
(ii) $\sum_{k=1}^{\infty} \delta_k < \infty$,
and there holds condition (c1). *Then,* $\lim_{k\to\infty} \varphi_k$ *exists and* $\sum_{k=1}^{\infty}[\varphi_{k+1} - \varphi_k]_+ < \infty$, *where* $[t]_+ = \max\{t, 0\}$ *for any* $t \in \mathbb{R}$.

3. Main results

In order to find a point in C, we consider the following inertial block-iterative scheme with selective technique, a main scheme.
Main scheme:

St.0 Choose any two points $x^{-1}, x^0 \in H$ such that $x^{-1} \neq x^0$ and an integer $s \geq 1$. Set $k := 0$.

St.1 Calculate $u^k = x^k + \theta_k(x^k - x^{k-1})$, where θ_k satisfies (c1) and (c2).

St.2 For $t = 1, \cdots, s$, let L_t^k be an ordered subset of L such that $L = L_1^k \cup \cdots \cup L_s^k$ and define $y^{k,t}$ by the rule:

$$y^{k,0} := u^k,\ y^{k,t} = T_{i_{\max}(t)} y^{k,t-1},\ i_{\max}(t) = \arg\max_{i(t)\in L_t^k} p_{i(t)}(y^{k,t-1}), \quad (3.1)$$

where $p_i(x) = \|T_i x - x\|$ for any $x \in H$.

St.3 Compute $x^{k+1} = (1-\alpha_k)x^k + \alpha_k y^{k,s}$ with $\alpha_k \in [\varepsilon, 1]$. Then, set $x^{k-1} := x^k$, $x^k := x^{k+1}$ and $k := k+1$. Return to St.1.

Remark
1. Clearly, u^k is a point in C if and only if $p_{i_{\max}(t)}(y^{k,t-1}) = 0$ for $t = 1, ..., s$. We will show that any sequence, generated by the main scheme, converges weakly to a point in C. For solving the stated problem, we replace $\{x^{k+1}\}$ in St.3 by a new x^{k+1}, defined by

$$x^{k+1} = (1-\alpha_k)(I - t_k\mu_k F)x^k + \alpha_k y^{k,s}, \quad (3.2)$$

where t_k satisfies (t),

$$\mu_k = \begin{cases} \frac{\langle Fx^k - Fx^{k-1}, x^k - x^{k-1}\rangle}{\|Fx^k - Fx^{k-1}\|^2}, & x^k \neq x^{k-1}, \\ \mu_{k-1}, & \text{otherwise}, \end{cases}$$

and condition (c2) is replaced by (c3).

We have the following results.

Theorem 3.1 *Any sequence* $\{x^k\}$*, generated by the main scheme, converges weakly to a point in* C*, as* $k \to \infty$.

Proof. Let p be any fixed point in C. Using the definition of $y^{k,t}$ and the properties of $T_{i_{\max(t)}}$, we get

$$\begin{aligned}\|y^{k,t}-p\|^2 &= \|T_{i_{\max(t)}}y^{k,t-1}-p\|^2\\ &\le \|y^{k,t-1}-p\|^2-\rho_{\min}\|T_{i_{\max(t)}}y^{k,t-1}-y^{k,t-1}\|^2,\end{aligned}$$

for all $t=1,2,...,s$. Summing the last inequalities with $t=1,...,s$ and noting $y^{k,0}=u^k$, we obtain

$$\|y^{k,s}-p\|^2 \le \|u^k-p\|^2-\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max(t)}}y^{k,t-1}-y^{k,t-1}\|^2. \tag{3.3}$$

On the other hand,

$$\begin{aligned}\|u^k-p\|^2 &= \|x^k-p+\theta_k(x^k-x^{k-1})\|^2\\ &= \|x^k-p\|^2+2\theta_k\langle x^k-p,x^k-x^{k-1}\rangle+\theta_k^2\|x^k-x^{k-1}\|^2\end{aligned} \tag{3.4}$$

From the well known property,

$$\langle u,v\rangle = -\frac{1}{2}\|u-v\|^2+\frac{1}{2}\|u\|^2+\frac{1}{2}\|v\|^2$$

for any two points $u,v\in H$, it follows that

$$\langle x^k-p,x^k-x^{k-1}\rangle = -\frac{1}{2}\|x^{k-1}-p\|^2+\frac{1}{2}\|x^k-p\|^2+\frac{1}{2}\|x^k-x^{k-1}\|^2.$$

This together with (3.3), the definition of x^{k+1} in St.3 and (3.4) implies that

$$\begin{aligned}\|x^{k+1}-p\|^2-\|x^k-p\|^2 \le\ &\theta_k(\|x^k-p\|^2-\|x^{k-1}-p\|^2)+2\theta_k\|x^k-x^{k-1}\|^2\\ &-\varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max(t)}}y^{k,t-1}-y^{k,t-1}\|^2.\end{aligned} \tag{3.5}$$

since $\theta_k^2\le\theta_k$. Therefore,

$$\|x^{k+1}-p\|^2-\|x^k-p\|^2 \le \theta_k(\|x^k-p\|^2-\|x^{k-1}-p\|^2)+2\theta_k\|x^k-x^{k-1}\|^2.$$

Using Lemma 2.4 with $\varphi_k=\|x^k-p\|^2$ and $\delta_k=\theta_k\|x^k-x^{k-1}\|^2$, we have the existence of $\lim_{k\to\infty}\|x^k-p\|$. Consequently, $\{x^k\}$ is bounded. Moreover, from (3.5) we can deduce that

$$\begin{aligned}\varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max(t)}}y^{k,t-1}-y^{k,t-1}\|^2 \le\ &\|x^k-p\|^2-\|x^{k+1}-p\|^2+2\theta_k\|x^k-x^{k-1}\|^2\\ &+\theta_k\big(\|x^k-p\|^2-\|x^{k-1}-p\|^2\big).\end{aligned}$$

Again, by using Lemma 2.4, we get $\sum_{k=1}^{\infty}[\|x^k - p\|^2 - \|x^{k-1} - p\|^2]_+ < \infty$. Hence,

$$\varepsilon\rho_{\min}\sum_{k=1}^{\infty}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 < \infty.$$

It means that

$$\lim_{k\to\infty}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 = 0,$$

which is equivalent to

$$\lim_{k\to\infty}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 = 0, \tag{3.6}$$

for $t = 1, ..., s$. Therefore, from (3.1) and (3.6) we have

$$\lim_{k\to\infty}\|y^{k,t} - y^{k,t-1}\| = 0. \tag{3.7}$$

Hence, $\lim_{k\to\infty}\|y^{k,t} - u^k\| = 0$, for $t = 1, ..., s$, as $y^{k,0} = u^k$. Further, from the definition of u^k with properties (c1) and (c2), we obtain

$$\|u^k - x^k\|^2 = \theta_k^2\|x^k - x^{k-1}\|^2 \le \theta_k\|x^k - x^{k-1}\|^2 \Longrightarrow 0,$$

as $k \to \infty$. Thus,

$$\lim_{k\to\infty}\|y^{k,t} - x^k\| = 0, \tag{3.8}$$

for $t = 1, ..., s$. Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{n_k}\} \subset \{x^k\}$ such that $\{x^{n_k}\}$ converges weakly to a point $\tilde{p} \in H$. Noting (3.7) and (3.8),

$$\lim_{k\to\infty}\|y^{n_k,t-1} - x^{n_k}\| = 0, \tag{3.9}$$

for $t = 1, ..., s$. As $L = L_1^{n_k} \cup \cdots \cup L_s^{n_k}$, for each $i \in L$ there exists at least an integer r_k such that $i \in L_{r_k}^{n_k} := \{i_1(r_k), ..., i, ..., i_{|L_{r_k}^{n_k}|}(r_k)\}$. Then, from the definition of $y^{k,t}$ in (3.1) and (3.6) with k and t replaced, respectively, by n_k and r_k, it is easy to verify that

$$0 \le \lim_{k\to\infty}\|T_i y^{n_k,r_k-1} - y^{n_k,r_k-1}\|^2 \le \lim_{k\to\infty}\|T_{i_{\max}(r_k)}y^{n_k,r_k-1} - y^{n_k,r_k-1}\|^2 = 0,$$

i.e.,

$$\lim_{k\to\infty}\|T_i y^{n_k,r_k-1} - y^{n_k,r_k-1}\| = 0. \tag{3.10}$$

By virtue of (3.9) with t replaced by r_k, (3.10), the property of the sequence $\{x^{n_k}\}$ and the demiclosed property of T_i, $\tilde{p} \in \mathrm{Fix}(T_i)$ for any $i \in L$, i.e., $\tilde{p} \in C$. Similarly, any weak cluster point of $\{x^k\}$ belongs to C. Then, by Corollary 3.3.3 in [14], the sequence $\{x^k\}$ converges weakly to a point in C. This completes the proof. □ Next, in order to prove the strong convergence of any sequence, generated by our scheme with new x^{k+1} in (3.2), we need the following Lemma.

Lemma 3.2 *For any sequence, generated by our scheme with new x^{k+1} in* (3.2) *and* (c3) *instead of* (c2)*, is bounded. Moreover, we still have*

$$\|x^{k+1}-p\|^2 \le (1-\alpha_k\gamma_k\beta_k)\|x^k-p\|^2 + 2\gamma_k\big(\langle Fp, p-x^k\rangle + \gamma_k\|Fp\|M_1\big) \\ + 2\theta_k\|x^k-x^{k-1}\|M_2 - \varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max(t)}}y^{k,t-1}-y^{k,t-1}\|^2, \tag{3.11}$$

for all $k \ge k^1$, a positive integer, where $\gamma_k = t_k\mu_k$, $\beta_k = (1/2)(2\eta - \gamma_k l^2)$, M_1 and M_2 are some positive constants.

Proof. Put

$$\eta_k = \frac{\langle Fx^k - Fx^{k-1}, x^k - x^{k-1}\rangle}{\|x^k-x^{k-1}\|^2} \quad \text{and} \quad l_k = \frac{\|Fx^k - Fx^{k-1}\|}{\|x^k-x^{k-1}\|}.$$

It is easy to see that

$$\eta \le \eta_k \le \frac{\|Fx^k - Fx^{k-1}\|}{\|x^k-x^{k-1}\|} = l_k \le l,$$

and hence,

$$\frac{\eta}{l^2} \le \mu_k = \frac{\eta_k}{l_k^2} \le \frac{1}{\eta_k} \le \frac{1}{\eta}. \tag{3.12}$$

Since $t_k \to 0$ as $k \to \infty$, from (3.12), we can confirm the existence of an integer k^1 such that $\gamma_k \in (0, \eta/l^2)$ and $\beta_k \ge \eta/2$ for all $k \ge k^1$. Thus, by using Lemma 2.1, (3.3) and (3.12), we obtain

$$\begin{aligned}\|(I-\gamma_kF)x^k - p\| &= \|(I-\gamma_kF)x^k - (I-\gamma_kF)p - \gamma_kFp\| \\ &\le (1-\gamma_k\beta_k)\|x^k-p\| + \gamma_k\|Fp\|\end{aligned}$$

and

$$\|y^{k,s}-p\| \le \|u^k-p\| \le \|x^k-p\| + t_k\frac{\theta_k}{t_k}\|x^k-x^{k-1}\|. \tag{3.13}$$

Let c be a positive constant such that $\frac{\theta_k}{t_k}\|x^k-x^{k-1}\| \le c$ for all $k \ge 0$. This constant exists due to (c3). Then,

$$\begin{aligned}\|x^{k+1}-p\| &\le (1-\alpha_k)\|(I-\gamma_kF)x^k-p\| + \alpha_k\|y^{k,s}-p\| \\ &\le (1-\alpha_k)\big((1-\gamma_k\beta_k)\|x^k-p\| + \gamma_k\|Fp\|\big) + \alpha_k\big(\|x^k-p\| + t_kc\big) \\ &\le (1-\alpha_k\gamma_k\beta_k)\|x^k-p\| + 2\alpha_k\gamma_k\beta_k\left(\frac{\|Fp\|}{\varepsilon\eta} + \frac{cl^2}{\eta^2}\right) \\ &\le r := \max\,\{\|x^{k^1}-p\|, 2(\|Fp\|/(\varepsilon\eta) + cl^2/\eta^2)\}\ \forall k \ge k^1.\end{aligned} \tag{3.14}$$

It means that $\{x^k\}$ is bounded. Consequently, $\|Fx^k\| \le M_1$ for all $k \ge 0$, where M_1 is some positive constant, that exists because $\{x^k\}$ is bounded and F is $\tilde{l}$-Lipschitz continuous. Further, by Lemma 2.1,

$$\begin{aligned}\|(I-\gamma_k F)x^k - p\|^2 &= \|(I-\gamma_k F)x^k - (I-\gamma_k F)p - \gamma_k Fp\|^2 \\ &\le (1-\gamma_k\beta_k)\|x^k-p\|^2 + 2\gamma_k\big(\langle Fp, p-x^k\rangle + \gamma_k\|Fp\|M_1\big),\end{aligned}$$

for all $k \ge k^1$. Clearly,

$$\begin{aligned}\|u^k-p\|^2 &\le \|x^k-p\|^2 + 2\theta_k\langle x^{k-1}-x^k, u^k-p\rangle \\ &\le \|x^k-p\|^2 + 2\theta_k\|x^k-x^{k-1}\|M_2,\end{aligned}$$

where, $M_2 = r + t_k c \le r + c$ for all $k \ge k^1$ due to (3.13), (3.14) and $t_k \in (0,1)$. Thus, (3.3) deduces

$$\|y^{k,s}-p\|^2 \le \|x^k-p\|^2 + 2\theta_k\|x^k-x^{k-1}\|M_2 - \rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1}-y^{k,t-1}\|^2. \tag{3.15}$$

Finally, by using Lemma 2.1 and (3.15) we have

$$\begin{aligned}\|x^{k+1}-p\|^2 &\le (1-\alpha_k)\|(I-\gamma_k F)x^k-p\|^2 + \alpha_k\|y^{k,s}-p\|^2 \\ &\le (1-\alpha_k)\big[(1-\gamma_k\beta_k)\|x^k-p\|^2 + 2\gamma_k\big(\langle Fp,p-x^k\rangle + \gamma_k\|Fp\|M_1\big)\big] \\ &\quad + \alpha_k\big[\|x^k-p\|^2 + 2\theta_k\|x^k-x^{k-1}\|M_2 \\ &\quad - \varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1}-y^{k,t-1}\|^2\big] \\ &\le (1-\alpha_k\gamma_k\beta_k)\|x^k-p\|^2 + 2\gamma_k\big(\langle Fp,p-x^k\rangle + \gamma_k M_1\big) \\ &\quad + 2\theta_k\|x^k-x^{k-1}\|M_2 - \varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1}-y^{k,t-1}\|^2,\end{aligned}$$

that is (3.11). The proof is completed. □ Now, we are in the position to prove a strong convergence result.

Theorem 3.3 *Any sequence, generated by our block-iterative scheme with new* x^{k+1} *and* (c3) *instead of* (c2), *as* $k \to \infty$, *converges strongly to a point* $p_* \in H$, *solving* (1.1).

Proof. Obviously, from (3.12) and the definitions of γ_k, β_k in Lemma 3.2, we get

$$\theta_k\|x^k-x^{k-1}\| = t_k\frac{\theta_k}{t_k}\|x^k-x^{k-1}\| \le \gamma_k\beta_k(2l^2/\eta^2)\tau_k, \tag{3.16}$$

where, following condition (c3), $\tau_k = (\theta_k/t_k)\|x^k - x^{k-1}\| \to 0$ as $k \to \infty$. Then, from (3.11), $\beta_k \geq \eta/2$, $\alpha_k \in [\varepsilon, 1]$ and the last inequality, we have

$$\begin{aligned}\|x^{k+1}-p\|^2 \leq (1-\alpha_k\gamma_k\beta_k)\|x^k-p\|^2 + 4\gamma_k\beta_k\big[\langle Fp, p-x^k\rangle + \gamma_k\|Fp\|M_1\big]/\eta \\ + 2\gamma_k\beta_k(2l^2/\eta^2)\tau_k M_2 - \varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 \\ \leq (1-\varepsilon\gamma_k\beta_k)\|x^k-p\|^2 + \varepsilon\gamma_k\beta_k 4\big[\big(\langle Fp, p-x^k\rangle + \gamma_k\|Fp\|M_1\big)/\eta \\ + (l^2/\eta^2)\tau_k M_2\big]/\varepsilon - \varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2.\end{aligned} \tag{3.17}$$

Now, we need only to consider two cases.
Case 1. $\|x^{k+1}-p\| \leq \|x^k - p\|$ for all $k \geq k^1$.

Then there exists $\lim_{k\to\infty}\|x^k - p\|$ and it is easy to see from (3.17) that

$$\varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 \leq \|x^k-p\|^2 - \|x^{k+1}-p\|^2 + d_k\gamma_k, \tag{3.18}$$

where $d_k = 4\beta_k\|Fp\|\big[(r+\gamma_k M_1)/\eta) + \tau_k M_2 l^2/\eta^2$. Next, we prove that

$$\lim_{k\to\infty}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 = 0. \tag{3.19}$$

Clearly, if

$$\varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 \leq d_k\gamma_k$$

for all $k \geq k^1$, then (3.19) holds. If

$$0 \leq \varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 > d_k\gamma_k$$

for all $k \geq k^1$, then from (3.18) it follows that

$$\sum_{k=k^1}^{M}\big(\varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 - d_k\gamma_k\big) \leq \|x^{k^1}-p\|^2 - \|x^{M+1}-p\|^2,$$

for any positive integer M. Thus,

$$\sum_{k=k^1}^{\infty}\big(\varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1} - y^{k,t-1}\|^2 - d_k\gamma_k\big) \leq \|x^{k^1}-p\|^2,$$

Therefore,

$$\lim_{k\to\infty}\big(\varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{k,t-1}-y^{k,t-1}\|^2-d_k\gamma_k\big)=0,$$

and hence, we obtain (3.19), because $\gamma_k\to 0$. By the similar argument as in the proof of Theorem 3.1, we get (3.6)–(3.10) and that any weak cluster point of $\{x^k\}$ belongs to C. Therefore,

$$\limsup_{k\to\infty}\langle Fp_*,p_*-x^k\rangle\le 0. \tag{3.20}$$

Now, from (3.17), we obtain

$$\|x^{k+1}-p_*\|^2\le(1-b_k)\|x^k-p_*\|^2+b_kc_k, \tag{3.21}$$

where $b_k=\varepsilon\gamma_k\beta_k$ and

$$c_k=4\big[\big(\langle Fp,p-x^k\rangle+\gamma_k\|Fp\|M_1\big)/\eta+(l^2/\eta^2)\tau_kM_2\big]/\varepsilon.$$

Clearly, $\sum_{k=1}^{\infty}b_k\ge\sum_{k=1}^{\infty}\varepsilon t_k\eta^2/(2l^2)=\infty$ and from (3.20) with $\gamma_k,\tau_k\to 0$ it follows that $\limsup_{k\to\infty}c_k\le 0$. So, applying Lemma 2.3 to (3.21) with $a_k=\|x^k-p_*\|^2$, we get $\lim_{k\to\infty}\|x^k-p_*\|=0$.

Case 2. There exists a subsequence $\{l_k\}\subset\{k\}$ such that $\|x^{l_k}-p\|\le\|x^{l_k+1}-p\|$ for all $k\ge k^1$.

Then, by Lemma 2.3, there exists a non-decreasing sequence $\{m_k\}\subseteq\{k\}$ such that $m_k\to\infty$,

$$\|x^{m_k}-p\|\le\|x^{m_k+1}-p\|\quad\text{and}\quad\|x^k-p\|\le\|x^{m_k+1}-p\| \tag{3.22}$$

for each $k\ge k^1$. Hence, from (3.16), (3.17) and the first inequality in (3.22), we have

$$\|x^{m_k}-p\|^2\le 4[\big(\langle Fp,p-x^k\rangle+\gamma_k\|Fp\|M_1\big)/\eta+(l^2/\eta^2)\tau_kM_2\big]/\varepsilon \tag{3.23}$$

and

$$\varepsilon\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max}(t)}y^{m_k,t-1}-y^{m_k,t-1}\|^2\le d_{m_k}\gamma_{m_k}.$$

Hence,

$$\lim_{k\to\infty}\|T_{i_{\max}(t)}y^{m_k,t-1}-y^{m_k,t-1}\|^2=0$$

for each $t=1,\cdots,s$. By the similar argument as in the proof of Theorem 3.1,

$$\lim_{k\to\infty}\|T_iy^{m_k,r_k-1}-y^{m_k,r_k-1}\|=0\quad\text{and}\quad\lim_{k\to\infty}\|x^{m_k}-y^{m_k,r_k-1}\|=0.$$

So, any weak cluster point of $\{x^{m_k}\}$ belongs to C. Consequently,

$$\limsup_{k\to\infty}\langle Fp_*, p_* - x^{m_k}\rangle \le 0.$$

Using (3.23) with p changed by p_*, the above $\limsup$ and $\gamma_{m_k}, \tau_{m_k} \to 0$, we obtain

$$\lim_{k\to\infty}\|x^{m_k} - p_*\| = 0. \tag{3.24}$$

Again, from (3.17) with k and p replaced, respectively, by m_k and p_*, we get

$$\|x^{m_k+1}-p_*\|^2 \le \|x^{m_k}-p_*\|^2+4\gamma_{m_k}\beta_{m_k}\big[\|Fp\|\big(r+\gamma_{m_k}M_1\big)/\eta+(l^2/\eta^2)\tau_{m_k}M_2\big].$$

From the last inequality, (3.24), $\gamma_{m_k}, \tau_{m_k} \to 0$ and $\beta_k \le \eta$ for all $k \ge k^1$, it follows that $\lim_{m\to\infty}\|x^{m_k+1} - p_*\|^2 = 0$. The last limit together with the second inequality in (3.22) implies that $\lim_{k\to\infty}\|x^k - p_*\| = 0$. This completes the proof. $\square$

Remarks

2. If the given T_i is quasi-nonexpansive, then the relaxation $\overline{T}_i = I + \beta(T - I)$ satisfies the condition

$$\|\overline{T}_i x - p\|^2 \le \|x - p\|^2 - \lambda\|T_i x - x\|^2 \ \forall x \in H, p \in \mathrm{Fix}(T_i)$$

for any fixed $\beta \in (0, 1]$ where $\lambda = \beta(1-\beta)$. It is easy to see that $\overline{T}_i$ is demiclosed if and only if T does. Moreover, we still have that $\mathrm{Fix}(T_i) = \mathrm{Fix}(\overline{T}_i)$ (see, [21]). Analyzing the proofs of the Theorems, we obtain that they are also valid, when we replace $T_{i(t)}$ in (3.1) by $\overline{T}_{i(t)}$.

3. Take $F = I - u$ for some fixed point $u \in H$. Clearly, F is η-strongly monotone with any $\eta \in (0, 1]$ and γ-Lipschitz continuous with any $\gamma \ge 1$. Clearly, in this case, $\mu_k = 1$ for all $k \ge 1$. Then, x^{k+1} in the methods listed above has the form,

$$x^{k+1} = (1 - \alpha_k)(1 - t_k)x^k + (1 - \alpha_k)t_k u + \alpha_k y^{k,s}. \tag{3.25}$$

Taking $u = 0$ in (3.25), we obtain an improvement modification for the selective inertial block-iterative scheme,

$$x^{k+1} = (1 - \alpha_k)(1 - t_k)x^k + \alpha_k y^{k,s}, \tag{3.26}$$

for all $k \ge 0$. Note that any sequence, generated by (3.26) converges strongly to a minimal-norm point of C, as $k \to \infty$.

4. Now, we consider the case when the expression x^{k+1} in (3.2) is replaced by

$$x^{k+1} = (1 - \alpha_k)y^{k,s} + \alpha_k(I - t_k\mu_k F)y^{k,s}, \tag{3.27}$$

with a new

$$\mu_k = \begin{cases} \frac{\langle Fy^{k,s} - Fy^{k-1,s}, y^{k,s} - y^{k-1,s}\rangle}{\|Fy^{k,s} - Fy^{k-1,s}\|^2}, & y^{k,s} \neq y^{k-1,s}, \\ \mu_{k-1}, & \text{otherwise}, \end{cases} \tag{3.28}$$

$y^{-1,s} = x^{-1}$ and $y^{0,s} = x^0$. Then, the inequalities in (3.12) are still true. Moreover, $\gamma_k \in (0, \eta/l^2)$ and $\beta_k \geq \eta/2$ for all $k \geq k^2$, some positive constant. Next, from (3.13) and $\frac{\theta_k}{t_k}\|x^k - x^{k-1}\| \leq c$, we get

$$\|y^{k,s} - p\| \leq \|x^k - p\| + t_k c \ \forall k \geq 0.$$

Consequently,

$$\begin{aligned} \|x^{k+1} - p\| &\leq (1-\alpha_k)\|y^{k,s} - p\| + \alpha_k\|(I - \gamma_k F)y^{k,s} - p\| \\ &\leq (1-\alpha_k)(\|x^k - p\| + t_k c) + \alpha_k\left[(1 - \gamma_k\beta_k)(\|x^k - p\| + t_k c) + \gamma_k\|Fp\|\right] \\ &\leq (1 - \alpha_k\gamma_k\beta_k)\|x^k - p\| + t_k c + \alpha_k\gamma_k\|Fp\| \\ &\leq (1 - \alpha_k\gamma_k\beta_k)\|x^k - p\| + \alpha_k\gamma_k\beta_k(c/(\alpha_k\mu_k\beta_k) + \|Fp\|/\beta_k) \\ &\leq r' = \max\{\|x^{k_1} - p\|, 2(cl^2/(\varepsilon\eta^2) + \|Fp\|/\eta)\}, \end{aligned}$$

a positive constant, for all $k \geq k^2$. It means that $\{x^k\}$ is bounded. Hence, $\{y^{k,s}\}$ is also bounded. Then, $\|Fy^{k,s}\| \leq M_1'$, some positive constants, and

$$\begin{aligned} \|x^{k+1} - p\|^2 &\leq (1-\alpha_k)\|y^{k,s} - p\|^2 + \alpha_k\|(I - \gamma_k F)y^{k,s} - p\|^2 \\ &\leq (1-\alpha_k)\|y^{k,s} - p\|^2 + \alpha_k[(1 - \gamma_k\beta_k)\|y^{k,s} - p\|^2 \\ &\quad + 2\gamma_k(\langle Fp, p - y^{k,s}\rangle + \gamma_k\langle Fp, Fy^{k,s}\rangle)] \\ &\leq (1 - \alpha_k\gamma_k\beta_k)\|y^{k,s} - p\|^2 + 2\gamma_k(\langle Fp, p - y^{k,s}\rangle + \gamma_k\|Fp\|M_1'). \end{aligned}$$

This together with (3.15) and (3.16) implies that

$$\begin{aligned} \|x^{k+1} - p\|^2 &\leq (1 - \alpha_k\gamma_k\beta_k)\|x^k - p\|^2 + \alpha_k\gamma_k\beta_k(2l^2/(\varepsilon\eta^2))\tau_k \\ &\quad - (1 - \alpha_k\gamma_k\beta_k)\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max(t)}}y^{k,t-1} - y^{k,t-1}\|^2 \\ &\quad + 4\alpha_k\gamma_k\beta_k(\langle Fp, p - y^{k,s}\rangle + \gamma_k\|Fp\|M_1')/(\varepsilon\eta) \\ &\leq (1 - \alpha_k\gamma_k\beta_k)\|x^k - p\|^2 + 2\alpha_k\gamma_k\beta_k[(l^2/\eta)\tau_k + 2\langle Fp, p - x^k\rangle \\ &\quad + \|Fp\|(2\|y^{k,s} - x^k\| + \gamma_k M_1')]/(\varepsilon\eta) \\ &\quad - (1 - \alpha_k\gamma_k\beta_k)\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max(t)}}y^{k,t-1} - y^{k,t-1}\|^2. \end{aligned}$$

Since $\gamma_k \to 0$, $\alpha_k > \varepsilon$ and $\beta_k \geq \eta/2$ for all $k \geq k^1$, there exists a positive integer $\tilde{k} \geq k^1$ such that, for all $k \geq \tilde{k}$, $1 - \alpha_k\gamma_k\beta_k \geq \tilde{c}$, a positive constant.

Therefore, the last inequality yields

$$\begin{aligned}\|x^{k+1}-p\|^2 \le (1-\alpha_k\gamma_k\beta_k)\|x^k-p\|^2 &+ 2\alpha_k\gamma_k\beta_k[(l^2/\eta)\tau_k + 2\langle Fp, p-x^k\rangle \\ &+ \|Fp\|(2\|y^{k,s}-x^k\| + \gamma_k M_1')]/(\varepsilon\eta) \\ &- \tilde{c}\rho_{\min}\sum_{t=1}^{s}\|T_{i_{\max(t)}}y^{k,t-1} - y^{k,t-1}\|^2,\end{aligned}$$

where $\lim_{k\to\infty}\|y^{k,s}-x^k\| = 0$ due to (3.8). Repeating step by step the proof of Theorem 3.3, we obtain the following result.

Theorem 3.4 *Any sequence* $\{x^k\}$*, generated by the main scheme with new* x^{k+1}*, defined by* (3.27) *and* (3.28)*, as* $k\to\infty$*, converges strongly to the point* p_**, solving* (1.1).

4. Taking $\alpha_k = 1$ for all $k \ge 0$, (3.27) has the form,

$$x^{k+1} = (I - t_k\mu_k F)y^{k,s}, \tag{3.29}$$

that together with $\theta_k = 0$ is the steepest descent block iterative method, studied in [7] with $\mu_k = \mu \in (0, 2\eta/l^2)$ and $y^{k,s} = x^{k,s}$. In the case that $F = I - u$, we have the following Halpern's selective inertial block-iterative scheme, that is the our main inertial iterative scheme with x^{k+1} in (3.29) replaced by

$$x^{k+1} = t_k u + (1-t_k)y^{k,s}, \tag{3.30}$$

since, in this case, $\mu_k = 1$ for all $k \ge 0$.

4. Applications and computational experiments

We consider a networked system consisting of an operator, who manages the system, and a finite number $m-1$ of participating users. In the system the manage operator can be seen as an user m. We suppose that each i-user has its own private objective function f_i on $\mathbb{E}^n$, an n-dimensional Euclidian space, and own capacity constraint, depicted by a nonempty closed convex C_i in $\mathbb{E}^n$. Moreover, the following is assumed.

- T_i is a mapping on $\mathbb{E}^n$ with $\mathrm{Fix}(T_i) = C_i$ for each $i \in L$ and $\cap_{i\in L}\mathrm{Fix}(T_i) \neq \emptyset$.
- f_i is a concave and Fréchet differentiable function on $\mathbb{E}^n$ such that $-\bigtriangledown f_i$ is η_i-strongly monotone and l_i-Lipschitz continuous.
- User $i \in L$ can use its own private C_i and f_i.
- The operator can communicate with all users.

The considered problem is formulated as finding a point p_* in $\mathbb{E}^n$ such that

$$f(p_*) = \max_{p \in C} f(p), f(x) = \sum_{i \in L} f_i(x), C = \cap_{i \in L} \mathrm{Fix}(T_i). \tag{4.1}$$

Problem (4.1) is closely related to network recourse allocation [5,28] which is a central issue in modern communication networks. The main objective of the problem is to share the available resources among users in the network so as to maximize the sum of their utilities subject to the feasible regions for allocating the resources. This problem is equivalent to the following one, $\tilde{f}(p_*) = \inf_{p \in C} \tilde{f}(p)$, where $\tilde{f} = -\sum_{i \in L} f_i(x)$ is convex and Fréchet differentiable with η-strongly monotone and l-Lipschitz continuous $\bigtriangledown \tilde{f}$ where $\tilde{l} = \max_{i \in L} l_i$ and $\eta = \min_{i \in L} \eta_i$. To solve (4.1), when $- \bigtriangledown f_i$ is η_i-strongly monotone and l_i-Lipschitzian and T_i is quasi-nonexpansive, Iiduka [18] introduced a parallel optimization algorithm, at each iteration steep of which the value μ is chosen in dependence of η_i and l_i. It is easy to see that the considered problem can be solved by any method, generated by one of our schemes, where $F = \bigtriangledown \tilde{f}$ and C is given in (4.1).

For computations, we consider the case, when $\tilde{f}$ is a differentiable convex function, the derivative of which, $Fx := \tilde{f}'(x)$, is η-strongly monotone and $\tilde{l}$-Lipschitz continuous and T_i is the subgradient projection relative to a convex function g_i, defined by

$$P_{g_i}x := \begin{cases} x - \frac{[g_i(x)]_+}{\|\eta_i(x)\|^2}\eta_i(x), & \text{if } \eta_i(x) \neq 0, \\ x, & \text{otherwise,} \end{cases}$$

where $\eta_i(x) \in \partial g_i(x) := \{z \in \mathbb{E}^n : g_i(y) - g_i(x) \geq \langle z, y - x\rangle, \text{ for all } y \in \mathbb{E}^n\}$, called a subgradient of g_i. We know in [14] that $T_i := P_{g_i}$ is a demiclosed cutter and $\mathrm{Fix}(T_i) = \{z \in \mathbb{E}^n : g_i(z) \leq 0\}$. We take a function $\tilde{f}(x) = \|x - u\|^2/2$ and $g_i(x) = \|x - P_{Q_i}x\|^2/2$ for $i \in L$, where P_{Q_i} is the metric projection of $\mathbb{E}^n$ onto the set $Q_i = \{x \in \mathbb{E}^n : \|x - a^i\|^2 \leq 1\}$ and a^i are points in $\mathbb{E}^n$ such that $\cap_{i \in L} Q_i \neq \emptyset$. Clearly, $\partial g_i(x) = g_i'(x) = x - P_{Q_i}x$ and

$$P_{Q_i}x = \begin{cases} a^i + \frac{1}{\|x - a^i\|}(x - a^i), & \text{if } \|x - a^i\| > 1, \\ x, & \text{otherwise.} \end{cases}$$

Therefore,

$$P_{g_i}x := \begin{cases} (x + P_{Q_i}x)/2, & \text{if } \eta_i(x) \neq 0, \\ x, & \text{otherwise.} \end{cases}$$

The numerical results, calculated with $n = 5, m = 6$,

$$a^1 = (0; -1/2; 0; 0; 0),\ a^2 = (0; 1/2; 0; 0; 0),\ a^3 = (1/2; 0; 0; 0; 0),$$
$$a^4 = (-1/2; 0; 0; 0; 0), a^5 = (0; 0; 1/2; 0; 0),\ a^6 = (0; 0; -1/2; 0; 0),$$

$L = L_1^k \cup L_2^k$, where $L_1^k = \{1, 2, 3\}$ and $L_2^k = \{4, 5, 6\}$. Taking $u = (0; 0; 0; 0; 0)$, we get the unique solution $p_* = (0; 0; 0; 0; 0)$. Numerical results, obtained without the inertial term ($\theta_k = 0$), are calculated by the formula

$$\begin{aligned} x^{k,1} &= T_{i_{\max}(1)} x^{k,0},\ i_{\max}(1) = \arg\max_{i \in L_1^k} \|T_i x^{k,0} - x^{k,0}\|, \\ x^{k,2} &= T_{i_{\max}(2)} x^{k,1},\ i_{\max}(2) = \arg\max_{i \in L_2^k} \|T_i x^{k,1} - x^{k,1}\|, \\ x^{k+1} &= (1 - t_k) x^{k,2}, \end{aligned} \tag{4.2}$$

following from (3.30), with the started point $x^0 = (2; 2.5; 3; 3.5; 1)$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}	x_4^{k+1}	x_5^{k+1}
10	0.0557779218	0.0506246428	0.0630224743	0.0976113631	0.0278889609
20	0.0292170066	0.0265176728	0.0330117724	0.0511297616	0.0146085033
30	0.0197921658	0.0179635848	0.0223628135	0.0346362901	0.0098960829
40	0.0149648083	0.0135822227	0.0169084687	0.0261884145	0.0074824041
50	0.0120305321	0.0109190418	0.0135930827	0.0210534313	0.0060152661

Table 1. Numerical results by method (4.2)

The following numerical table is obtained by using our iterative scheme

$$\begin{aligned} u^k &= x^k + \theta_k (x^k - x^{k-1}), y^{k,0} = u^k, \\ y^{k,1} &= T_{i_{\max}(1)} y^{k,0},\ i_{\max}(1) = \arg\max_{i \in L_1^k} \|T_i y^{k,0} - y^{k,0}\|, \\ y^{k,2} &= T_{i_{\max}(2)} y^{k,1},\ i_{\max}(2) = \arg\max_{i \in L_2^k} \|T_i y^{k,1} - y^{k,1}\|, \\ x^{k+1} &= (1 - t_k) y^{k,2}, \end{aligned} \tag{4.3}$$

with

$$\theta_k = \begin{cases} \min\{\varepsilon_k / \|x^k - x^{k-1}\|, \theta\}, & \text{if} x^k \neq x^{k-1} \leq 0, \\ \theta_{k-1}, & \text{otherwise}, \end{cases} \tag{4.4}$$

where $\varepsilon_k = 1/(k+1)^2$, $\theta = 0.1$ and $x^{-1} = (1; 3.5; 3; 2.5; 2)$, and the same values for other parameters. We get the numerical results in Table 2.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}	x_4^{k+1}	x_5^{k+1}
10	0.0431792242	0.0339803589	0.0444127522	0.0737654623	0.0179932518
20	0.0208410698	0.0164011057	0.0214364492	0.0356039544	0.0086847001
30	0.0134755486	0.0106047292	0.0138605131	0.0230210267	0.0056154075
40	0.0098608472	0.0077601006	0.0101425482	0.0168458319	0.0041091222
50	0.0077297385	0.0060830016	0.0079505588	0.0132051408	0.0032210660

Table 2. Numerical results by method (4.3) and (4.4)

The numerical results in Tables 1 and 2 show the effectiveness of the introduced schemes. Moreover, they also show that the results calculated by using the inertial term u^k is better than that without the same term.

5. Conclusion

In this paper, for finding a point in the set common fixed points of a finite family of demiclosed strongly quasi-nonexpansive mappings in Hilbert spaces, we suggested a selective inertial block-iterative schemes with weak convergence. Then, based on a specific combination of the scheme with the steepest-descent method, we propose new schemes, for solving a variational inequality problem over the set of common fixed points of a finite family of SQNE mappings in Hilbert spaces. The strong convergence of the latter scheme is proved without the approximately shrinking and boundedly regular assumptions on the mappings and their fixed point sets, respectively, that are usually required recently in literature. An application to a networked system and computational experiments are given for illustration and comparison.

Acknowledgements
This work was supported by the Vietnam National Foundation for Science and Technology Development under Grant No. 101.02-2021.11.

Compliance with ethical standards

Conflict interest The authors declare that they have no conflict of interest.

References

[1] A. Aleyner, S. Reich, Block-iterative algorithms for solving convex feasibility problems in Hilbert and in Banach spaces, J. Math. Anal. Appl. 343 (2008) 427-435.

[2] F. Alvarez, On the minimizing property of a second order dissipative dynamical system in Hilbert spaces, SIAM J. Control Optim. 39 (2000) 1102-1119.

[3] F. Alvarez, Weak convergence of a relaxed and inertial projection-proximal point algorithm for maximal monotone operators in Hilbert spaces, SIAM J. Optim. 14(3) (2004) 773-782.

[4] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, 2011.

[5] D.P. Bertsekas, R. Gallager, Data network, Prentice Hall Upper Sadler River, N.J. 1987.

[6] R.I. Bot, E.R. Csetnek, C. Hendrich, Inertial Douglas-Rachford splitting for monotone inclusion problem, Appl. Math. Comput. 256 (2015) 472-487.

[7] Ng. Buong, L.T.T. Duong, An explicit iterative algorithm for a class of variational inequalities, J. Optim. Theory Appl. 151 (2011) 513-521.

[8] Ng. Buong, Steepest-descent block-iterative methods for a finite family of quasi-nonexpansive mappings, J. Industrial and Management Optimization, doi:10.3934/jimo2021133

[9] A. Cegielski, S. Reich, R, Zalas, Regular sequences of quasi-nonexpansive operators and their applications, SIAM J. Optim. 28 (2018) 1503-1532.

[10] A. Cegielski, F. Al-Musallam, Strong convergence of a hybrid steepest descent method for the split common fixed point problem, Optimization. (2016) 1147038.

[11] A. Cegielski, General method for solving the split common fixed point problem, J. Optim. Theory Appl. 165 (2015) 385-304.

[12] A. Cegielski, R. Zalas, Properties of a class approximately shrinking operators and their applications, Fixed Point Theory, 15(2) (2014) 399-426.

[13] A. Cegielski, R. Zalas, Methods for variational inequality problem over the intersection of fixed point sets of quasi-nonexpansive operators, Numer. Funct. Anal. Optim. 34(3) (2013) 255-283.

[14] A. Cegielski, Iterative Methods for Fixed Point Problems in Hilbert Spaces, Lecture Notes in Mathematics, 2057, Springer, Heidelberg, 2012.

[15] A. Gibali, S. Reich, R. Zalas, Outer approximation methods for solving variational inequalities in Hilbert spaces, Optimization. 66 (2017) 417-437.

[16] B. Halpern, Fixed points of nonexpanding mapps, Bull. Am. Math. Soc. 73 (1967) 957-961.

[17] S. He, H. Tian, Selective projection methods for solving a class of variational inequalities, Numer. Algor 80 (2019) 617-634.

[18] H. Iiduka, Parallel optimization algorithm for smooth convex optimization over fixed point sets of quasi-nonexpansive mappings, J. Oper. Research Soc. Japan 58(4) (2015) 330-352.

[19] S. Ishikawa, Fixed points by new iteration method, Proc. Amer. Math. Soc. 44 (1974) 147-150.

[20] M.A. Krasnoselskii, Two remarks on the method of successive approximations, Uspekhi Math. Nauk 10 (1955) 123-127.

[21] P.E. Maingé, The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, Comput. Math. Appl. 59(1) (2009) 74-79.

[22] F.E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Var. Anal. 16 (2008) 899-912.

[23] P.E. Maingé, Convergence theorems for inertial KM-type algorithms, J. Comput. Math. Appl. **219** (2008), 223-236.

[24] P.E. Maingé, Inertial iterative process for fixed points of certain quasi-nonexpansive mappings, Set-Valued Anal. 15 (2007) 67-79.

[25] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506-510.

[26] A. Moudafi, Viscosity approximation methods for fixed-point problems, J. Math. Anal. Appl. 241 (2000) 46-55.

[27] Y. Shehu, New inertial iterative method for nonexpansive mappings, J. Nigerian Math. Soc. 38(3) (2019) 533-546.

[28] R. Srikant, The Mathematics of Internet Congestion Control, Birkhauser, Boston, M.A. 2004.

[29] B. Tan, Zh. Zhou, S. Li, Strong convergence of modified Mann algorithms for nonexpansive mappings, arXiv:2001.07346v1 [Math. OC] 21 Jan 2020

[30] H.K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116 (2003) 659-678.

[31] I. Yamada, The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications* (D.Butnariu, Y. Censor and S. Reich, Eds). North-Holland, Amsterdam, (2001) 473-504.