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MATHEMATICAL LITERACY FOR DIGITAL ERA: REVIEW OF MATHEMATICAL THINKING AND COMPUTATIONAL THINKING FOR CURRICULUM DEVELOPMENT

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Abstract

In the Era of AI and Data Science under the fourth industrial revolution, Mathematical Capitalism (Ministry of Economy, Trade and Industry, Japan., 2019 [32]) becomes a key issue in Education up to higher education in this changing society. Mathematical Thinking provides the grand for Informatics as a school subject that develops computational thinking. This article confirmed mathematics as the key literacy subject in school and re-viewed it as universal literacy in the era through the confirmation of mathematical thinking and value are a necessary component of general competency in the era. This claim is illustrated by the historical meaning of mathematics, the current aims of mathematics

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education, and re-viewed the meaning of mathematics education for the digital society.

1 Introduction

In 2017, UNESCO Bangkok held the International Conference 'Cracking the code: girls' and women's education in science, technology, engineering and mathematics (STEM).' At the meeting, some speakers who were engaging in the region or national-level STEM education project for girls asserted the followings: Depending on the survey, Mathematics has been the barrier subject for girls to choose a career in science and engineering. Thus, in STEM education, it should be better to alternate mathematics with technology for enabling them to learn more science. Even though their assertions are not common in STEM education, it implies that mathematical and math educators around them had failed to implement their responsibility to develop authentic mathematics in Education which includes the ways of thinking and knowing in mathematics and developed ill-literate students for STEM education. On the curriculum in general, literacy is the word to explain and deduce the curriculum framework. Thus, depending on how we explain mathematical literacy, the objective of mathematics education itself changes. In this article, we confirm the historical meaning of Mathematics to explain historical literacy, and current issues on mathematical science, and define mathematical and computational thinking as general competency beyond subjects for general skills.

2 Mathematics Has Been the King of Subjects in the Curriculum

Mathematics has been a basic literacy subject. On the seven liberal arts in the middle age, the trivium of grammar, rhetoric, and logic was called as Latin subjects and the quadrivium of arithmetic, geometry, astronomy, and music were called mathematical subjects. Those subjects do not directly correspond to the current subjects' names. Mathematics originated as the integrated name of various subjects that should be learned since Ancients Greece. From the perspective of the current view, it might be strange that astronomy and music were parts of mathematics however they were explained with figural representation of geometry. Logic was a part of Latin subjects but became a part of mathematics in the 20^{th} century. Arithmetic which is represented by using geometric words is not likely the current meaning of arithmetic which enhances operations.

In Ancient Greece, ancient Mediterranean age, Mathematics was explained and written in the Greek Language with figural representation like Euclid's Elements and NOT represented by Algebraic expression. Learning Mathematics includes learning logical-exact reasoning in Greek. To study Academia, which was established by Plato to cultivate as a philosopher, geometry was the most necessary and basic subject.

Depending on Wikipedia, in Ancient Greek, it is $\mu\alpha\theta\eta\mu\alpha$ (mathema). It means "the subjects of instruction" (Heath, 1931 [7]) and on Wiktionary, its etymology is $\mu\alpha\nu\theta\alpha'\nu\omega$ (manthánō, "I learn". $\pm\mu\alpha-$ma, result noun suffix and as a noun, it means that (1) something that is learned: a lesson, (2) learning, knowledge, (3). (Often in the plural) the mathematical sciences, in particular, the quadrivium: arithmetic, geometry, astronomy and astrology, harmonics, and (4) a creed. Such Ancients Greek usage can be seen in the Commentary on Plato's Meno by Proclus in relation to figure: 'learning is nothing but the mind's remembering its own ideas. what learning ($\mu\alpha'\theta\eta\sigma\iota\varsigma$) is recollection of the eternal ideas on the soul' (Morrow, 1970, pp.37-38 [36]).

Figure 1. Claudii Ptolemæi, Johannes Wallis edited (1682). Harmonicorum libri tres. [33]

Ptolemy (AD.83-168) is known as the author of Almagest which treats Astronomical matters and Harmonikon (Harmonics) which treat the theory of music code. Those were part of mathematics ($\mu\alpha'\theta\eta\mu\alpha$) as well as arithmetic and geometry. The left side picture in Figure 1 shows Pythagorean codes by dividing the string by using the ratios of 1/2, 2/3, and 3/4 and the multiplication of them (such as p.86. [33]).

In the Era of East Roman Empire, AD 5C, Proclus wrote "The Pythagoreans considered all mathematical science to be divided into four parts: One half they marked off as concerned with quantity, the other half with magnitude;

and each of these they posited as twofold. A quantity can be considered in regard to its character by itself or in its relation to another quantity, magnitudes as either stationary or in motion. Arithmetic studies quantities as such, music the relations between quantities, geometry magnitude at rest, and spherics [astronomy] magnitude inherently moving (Figure 2) (Morrow, 1970, pp-29-30 [36]). 'In this translation, subjects of mathematics were translated by the word 'mathematical sciencen' because mathematical science means the subjects which use mathematical representations in current however, please note, it was represented by the figural representation of geometry. Proclus was Greek, not Christian, and known as the successors of Neoplatonism. His categorized explanation looks Aristotle's manner of explanation on the Form and Matter in Metaphysics. His Commentary for Euclid's Elements (geometry) were influenced to the latter age in Europe through Arabic.

In the process of the inter-religious conflict that Christianity spreads to the Mediterranean world, many manuscripts which were written hand on papyrus lost the opportunity to be copied and became ashes by unlearned Christian mobs. On the other hand, some of them that educated people, called Neoplatonists, paid attention to Greek studies and made it possible to learn them in Latin as well. A part of Aristoteles's writings as well as others such as Proclus was considered as meaning full writings for Christianity and influenced Europe in the middle age, 12^{th} century. The word 'science' was known by Aristoteles. In current science education, we usually refer to his writing to search for the meaning of science. However, the origin of mathematics was NOT the same as science because it meant the subjects, including science and others, to be learned.

Quantity		**Magnitude**	
Quantity itself	Quantity relation to another quantity	Stationary	Motion
Arithmetic	Music	Geometry	Sphere (Astronomy)

Figure 2. Proclus manner of the categorization of mathematics

In Europe, on this bases, various academic subjects emerged during the Renaissance period which included adaptations of Arabia and Greek subjects including mathematics via Crusaders and Reconquista. Margarita Philosophica by Reidch (1504) [38] was known as the book of Liberal Arts necessary for the University level at the Renaissance. On the left of Figure 3, the subjects in the lower parameter of the circle show liberal arts, and the subjects in the upper show the arts for the living. On the right, it shows the hierarchy of subjects: The third floor is Logic, Rhetoric, and Arithmetic, and fourth floor is Music, Geometry, and Astronomy, and the Top floor is Metaphysics. In the Era, as

Galileo want to be Philosopher, Philosophy was the top an academic subject. In Figure 4, by Reidch, there are the category of subjects: The part of circles shows the trivium and the quadrivium. At that time, the figural-geometrical represented subjects were still mathematics.

Figure 3. Margarita Philosophica by Reidch (1504) [38]: No page in this book

The meaning of mathematics was renovated through changing the key representation of mathematics from geometry to algebra. The change was emergent in the 16^{th} & 17^{th} century: It is known as Descartes work. In his book, the Rules for the Direction of the Mind (1701) [5], he claimed that we lost the intuition which Ancients had in mathematics, and we should reintegrate the subjects which were represented by quantity and magnitude (Figure 2 by Proclus) under Algebra which originated from Arabic, and then he renamed Universal Mathematics. His claim to re-integrate mathematics by Algebra directed mathematics to the current algebraic form of mathematics in our schools.

This revolution to renovate mathematics' meaning was emergent under the cultural commerce practice which used Arabic base ten place value system since the Era of Fibonacci in 1202 (Sigler, 2002) [39] and on the Era of scientific findings such as Copernicus and Galileo who used the data represented by Arabic numbers. However, it limited the meaning of mathematics based on Arithmetic and Algebraic language, and later, it had produced difficulty for teachers to teach the reasoning in their mother language in mathematics because Arithmetic and Algebraic language are the operations that produce the automated unique reasoning in their form. The uniqueness produced the teaching practice

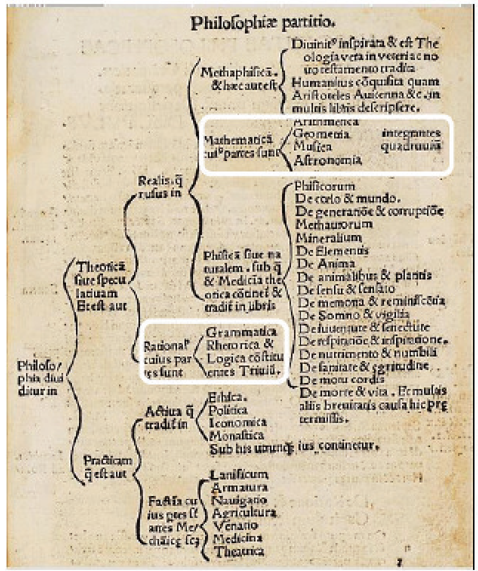

Figure 4. Category of Subjects by Reidch (1504) [38]: No page in this book

just as teaching skills which are limited within mathematics.

As a consequence, the mathematics used by the subjects to learn exact reasoning was lost and misunderstanding image of the school mathematics which someone believe to be alternated calculator or computer has appeared, even though STEM in this era is not possible without algebraic representations. This limited-narrow view of mathematics is NOT the current aims of mathematics education in the world. In the following two chapters, we would like to clarify the aims.

3 Current Aims of Mathematics Education for Human Character Formation

The aims of mathematics education as for public education which develops human resources are explained by the following four perspectives: Material Discipline based on a kind of language for daily life up to the language for mathematical science, Formal Discipline based on the habit of thinking and the attitude on life in relation to Human Character Formation such as mathematical thinking, Cultivation for understanding others saying metaphorically such as mathematical conclusion might be correct and sharable even though we do understand it exactly, these three perspectives are preparation for the future life and society on public education, and Enjoy Current Learning Activity in a living classroom, school, and community which is necessary to cultivate human character for the future (see such as Nakajima, 1981 [28]: Isoda & Kageyama, 2021 [20]).

Here, Cultivation is based on Material and Formal disciplines which also can be seen based on the perspective of Aristotle. By using mathematical representations, Mathematics is a unique subject to represent other ideas in our mind exactly the same as others. Such benefits are shared in Mathematical Sciences. However, even though mathematical sciences, if the premise of the mathematical model changes the conclusion also changes: It is not the discussion within the model such as if the parameters on the model change the resulting change simultaneously. For example, in the historical discussions of world warming, there were two types of skepticism to global warming. The first type is politics which tried to deny it (see such Oregon Petition on Wikipedia [42]). The second type is the scientific discussions in relation to the difference in the premise of the mathematical models. Both reasonings can be also explained by mathematical logic, meta mathematical view: Any propositions are true if the assumption is wrong but the reasoning itself is concluded wrong because the assumption itself is denied. In the case of nonmathematical sciences, others can only understand others, hypothetically (see the discussion of Hermeneutics such as Isoda, 2015a [15]). For example, in the case of music, even though we shared the score, it will change depending on players skill, instrument, and conductors interpretations, as well as the mindset of players, audience, the situation of hall, and so on at the moment. Thus, a good music for a person is not sure good or not for others.

The exactly sharable nature of mathematics strengthens the following two perspectives for Human Character formation in Mathematics Education (Isoda, 2015b [16]; Mangao et al., 2017 [31]): One perspective is a competitive attitude. In the case of pure mathematicians, the first finder who produced the proof to conjecture is the winner and proved it in another way is also the winner and others are followers. Mathematicians are very much patient thinkers for seeking unknown or unestablished theories to solve unsolved problems. Another perspective is a sympathetic attitude. In mathematicians, the academic paper is only accepted if other mathematicians can re-present it exactly the same in the review process. Mathematicians seminars on a specific topic are also enjoyable because they are also the opportunity to re-present finding of each other. Both attitudes are the authentic mathematical activity in mathematics classrooms proposed by Inprasitha (2004 [8]; 2006 [9]). The Open Approach proposed by Nohda (1983 [**], in English 2000 [**]), known as a theory for teaching through problem-solving are based on both perspectives under the various solutions for the problem: It is open for mathematics because of the acceptableness of various solutions and extensions of the problem and mathematical ideas, for students because of the acceptance of various their ideas which were previously learned by students, and for teachers because of the designing the lesson with the presumption of students discussion based on they already learned and implementing it with flexible assessment for appropriate questioning depending on the students discussion. In this Approach, the teachers role is questioning for enabling students to produce problem (or problematic) for students and to represent others ideas in each other for appreciation. Various solutions are produced under competitiveness by students, but it is the opportunity for students to learn from others enjoyably under their sympathetic attitude: These are the product of both of them (Lee, 2000 [26]; Isoda, 2002 [11]; Isoda et al, 2006 [13]). Successful classroom, students usually show teachers their twinkling eyes by enjoining their learning of mathematics by and for themselves. It is an authentic mathematical activity in the classroom which is proposed by Nohda (1983 [29], in English 2000 [30]) and Inprasitha (2004 [8]; 2006 [9]) and the Approach also cultivates both perspectives on students attitudes

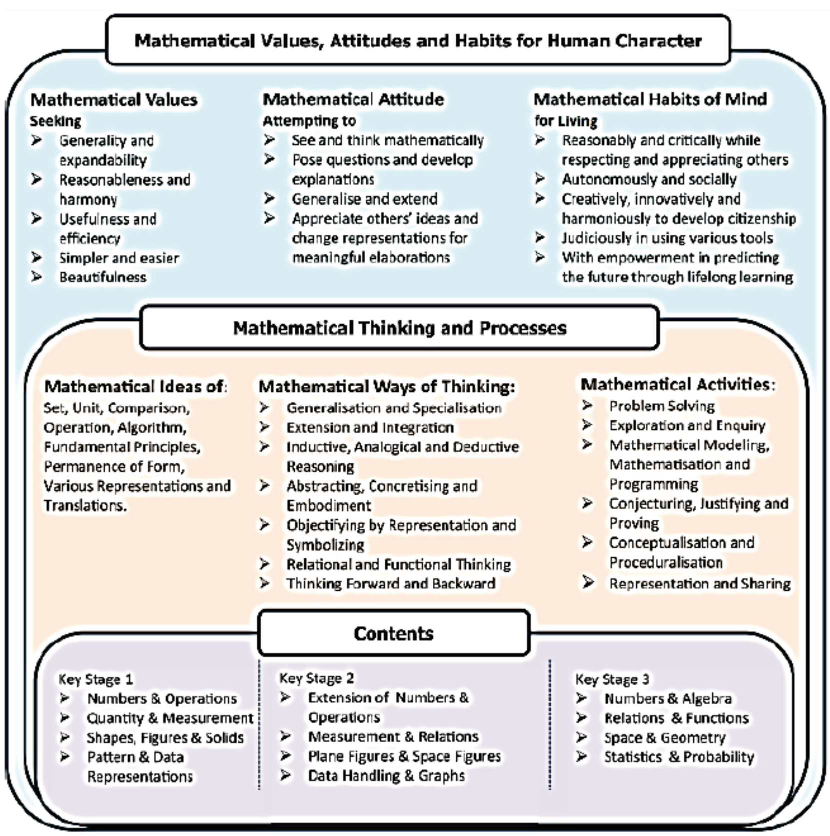

Figure 5. SEA-BES: CCRLS for Mathematics Framework (Teh, Isoda& Gang, 2021, p.2 [40])

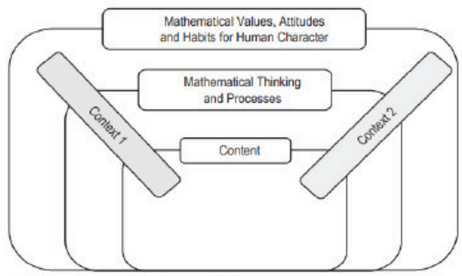

Figure 6. Context 1 is mathematical modeling and Context 2 is the mathematization of mathematics (Isoda et al., 2022: p.3 [35])

On Education in general, the necessity of the definition of Competency by OECD (2005) [33] also mentioned both perspectives through the wording to success in this competitive society and to build welfare for the society. Student Agency on OECD Education 2030 (2019) [34] enhanced the students who have learning skills for challenging problems. These perspectives on mathematics and the Approach are well aligned with the international reform issues in Education by OECD.

In the case of Southeast Asia, SEAMEO Basic Education Standards were prepared for mathematics and science education (Mangao et al., 2017 [31]; Isoda et al., 2022 [24]). The mathematics framework for it (see Figure 5) was proposed based on Isoda and Katagiri (2012) [14] and Isoda (2015a) [15] with synchronization of OECD (2005) and slightly revised a few words by Teh et al. (2021) [40] and Isoda et al. (2022) [24]. The images of mathematics teaching are usually limited to Content. however the framework described as Mathematical Thinking and Processes (in short MT) and Mathematical Values, Attitudes and Habits for Human Character (in short MV) are the content of mathematics education under Isoda and Katagiri (Thai edition is in printing by Inprasitha, [in press]) as for general competency. In the mathematics classrooms, it can be learned through the appropriate contexts (Figure 6) which originated as authentic activities and provides the opportunities for reflection and appreciation beyond just acquisition of content. For this, it is necessary to re-interpret as possible approaches in the classroom practice on curriculum by every teacher.

Context 1 is known as mathematical modeling under curriculum which is originally known as applied mathematicians activity such as mathematical scientist activity. Context 2 is mathematization based on problem (task) sequence under curriculum which is originally known as pure mathematicians' activity (Isoda et al 2019 [18]). Both contexts in the classroom are explained as mathematical problem-solving processes under pseudo real situations and pseudo-mathematical problem situations produced by teachers and students in the classroom. In the design of classroom practice to develop MT and MV (in short MTV), teachers have to set the opportunity for reflection and application. Because MT is the process, students need the opportunity to reflect to learn MT how they did, and it is also the opportunity to recognize and appreciate MA. For reflection and appreciation in the classroom, a sympathetic attitude for understanding others is necessary: It is a part of the theory of teaching through problem-solving such as the Open Approach. The appropriate task sequence such as Task 2 which extends the idea of Task 1 (Isoda & Katagiri, 2012 [14]; Isoda et al, 2019 [18]) provides the opportunity for reflection and appreciation. The wordings of MTV are written for the objective in the curriculum: Teachers are necessary to reinterpret these words into concrete activities and words that will be used for children: See Isoda et al (2021a, 2021b, 2022) [21][22][23]. For example, the idea of set on MT means conditions to recognize the set on teaching in the classroom. In the counting activity, students say "five" teacher needs to re-word it as "five apples" because apple denominate the object of counting and unit quantity in counting. MV is realized on the occasion of comparison. Ordered blocks in number sequence are much more beautiful than non-ordered blocks. Counting marbles by ten heaped groups is much easier to see at a glance than just one whole heap.

Figure 5 as the ben diagram illustrates that MT and MV are general competencies because it is possible to teach through non-mathematical content such as logical-dialectical reasoning and discussion like "if your saying (the conclusion) is true what will happen" which direct to analytic reasoning began from the conclusion as for Heuristic (Pappus of Alexandria by Alexander, Jones 1986 [25]: Polya, 1945 [35]), the proof by contradiction and contrapositive: These need to consider the law of the excluded middle. In mathematics education, Fawcett (1938) [6] taught reflective thinking in relation to teaching definition and proof by using the nonmathematical examples in our life (such as p.31-32) as well as mathematical content. The politics type on the skepticism to global warming is also in this category and it is a part of STEM education to develop critical thinkers based on MT and MV. To see MT and MV in Figure 5 as general competency beyond the traditional mathematical content is necessary to review mathematics as original meaning. In the next chapter, we would like to discuss computational thinking is NOT just using the calculator and how deeply related to mathematical thinking.

4 Computational Thinking from the Perspective of Mathematical Thinking

Industrial revolution 4.0 has been ongoing by using AI and Data Science. It produces a digital society under Mathematical Capitalism (see the Ministry of Economy, Trade, and Industry, Japan, METI, 2019) [32]. To develop Computational Thinking (in short CT), or Programming Thinking in traditional word, is a global movement that is necessary to develop the necessary human capital via Education. In the APEC Human Resource Development Working Group, InMside Project which was led by Isoda, Araya, and Inprasitha (2021c) [18] reported the recommendation of curriculum reform for APEC economies based on Computational Thinking (CT) and Statistical Thinking for the digital society. Here we apply related discussions from Chapter 2 for CT on the report:

CT defined by Jeannette Wing (2006) [43]: "CT is the thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer" human or machine "can effectively carry out". In other words, CT is understood as a problem-solving paradigm of mathematizing a problem in such a way that the computer can execute it (Toh, 2020) [41]. Indeed, the problem-solving process with CT includes analyzing the process and breaking down several parts and representing or alternating all or some parts by computer or algorithm and fixe to alternate the original solving process to a more convenient and simple process with a computer. Here, it can be seen as a mathematical problem-solving process if "computer or algorithm" alternate "mathematics". Programing language is a kind of mathematical language. The exceptional difference between these two is computers implement limited process with time, memory, and bag but mathematics does not necessarily consider these on the proven mathematical system. Mathematical representation of distinguished mathematical language (theory) such as geometry and algebra provide us with different intuition. For example, functional equations, tables, and graphs of functions provides us with a lot of information when compare with just using only one representation, independently (see Isoda, 2015b [16]: in English 2018 [17]). Even in the graph, if we change the scale for the yaxis from real number to logarithm, the intuition changes. On mechanism, each representation on the functional equation, and mechanical model, graph representation provides us with different intuition and helps to develop fruitful meanings if we use them all (Isoda & Matsuzaki, 2003 [12]). That ls why we should develop students who are able to use various representations and translate amongst representations in mathematics.

In another wording for problem-solving with the computer on CT, Kano (2020) [27] explained four basic strategies of CT: Decomposition, pattern recognition, algorithms, and abstraction. The problem has to be considered analyzed as a sequence of simpler problems. All or some of them will be done object for computer. Pattern Recognition detects and describes powerful patterns that can contain the mechanisms to find a solution to the problem. The algorithm is the basis of algorithmic thinking which occurs when someone observes repeated patterns in problems and then generalizes a set of rules for dealing with such situations. Abstraction means extracting some hidden characteristics that are critical for the problem and simplifying them to be able to manage them without losing their power. All of these are not operating isolated from each other, they have to be interconnected and successfully integrated. The strategies for problem-solving work in Mathematics.

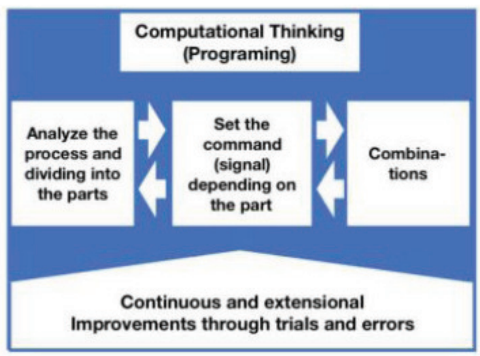

Figure 7. Four basic strategies for Programming (Kano, 2020) [26]

Araya et al. explained Problem Solving by the computer in Informatics in Figure 8. Even though Machine Learning is done by the system but also needs coding for the operation of the interface. Based on Figure 8, the framework for computational thinking on the recommendation of curriculum reform is summarized in Figure 9 (see also Araya & Isoda, 2021 [4]). The upper part of the triangle in Figure 9 based on Figure 8 is also applicable to higher education. Inprasihta (2022) [10] leads to establishing compulsory courses on the liberal arts at Khon Kaen University

CT can be developed without a computer as well as with a computer. Araya (2019) [1] illustrated it on the painting book activity for Kinder Garden and Lower Primary School level, For example (Figure 10):

From the perspective to develop MTV, Araya's work in Figure 10 shows the task sequence from Task 1 to Task 2 to develop MTV. On the first task, children explain to others where each of them painted and how. There are activities to explain it exactly verbally, they have to explain each painted ball clearly and discuss such as S1: I painted first bottle two balls, second two balls, S2: I also do that but not the same as your painting. T: Good, is it possible to explain how different both of their paintings are? S3: S1 painted from the top, number 1 ball and number 2 ball in the first bottle and S2 painted from the bottom.'

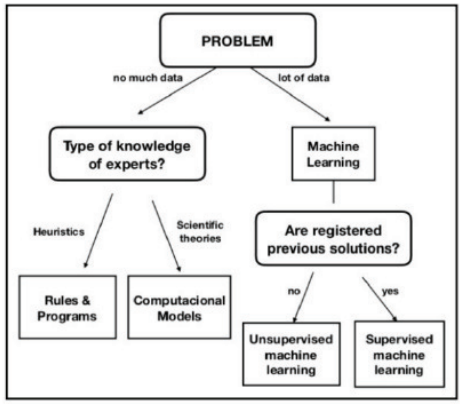

Figure 8. Various ways of problem-solving through Computer Programming [4]

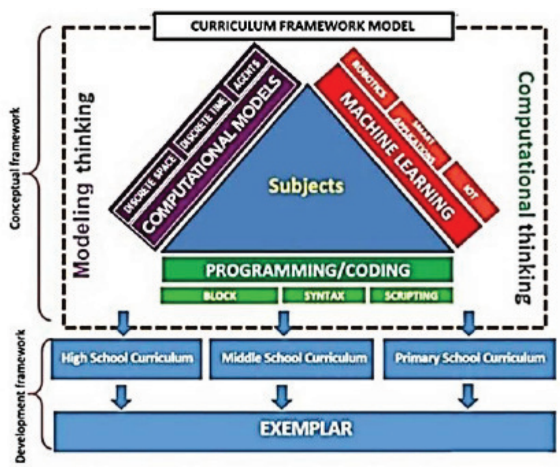

Figure 9. Curriculum Design Framework to Develop Computational Thinking [4]

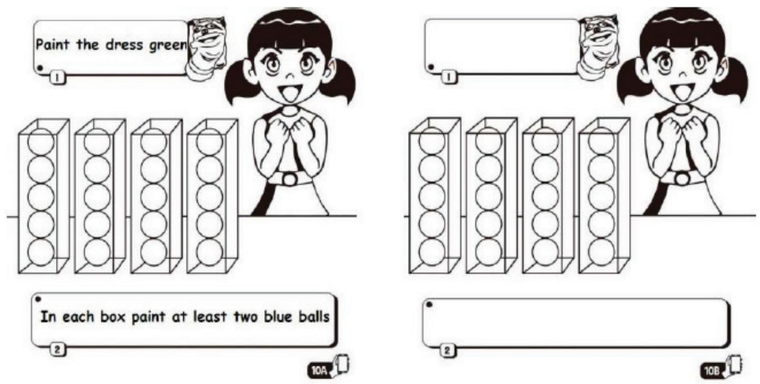

Figure 10. Activity to learn Computational Thinking/ Cording

From the perspective of MT, this dialog includes ordering with an ordered numbers through comparison to distinguish the position of the object. It will be the basis for coordinating and further problems in Mathematics. From the perspective of MV, it includes explaining the beautifulness of the painting by using the order of natural numbers. From the perspective of CT, another goal of these non-text activities is for the student to write their own instructions for a variation or innovation. She writes them down for a third student to do. This corresponds in computer science to writing pseudocode. This could also correspond to writing precise instructions but in natural language, so that an AI program then translates them into computational code in a specified computational language, such as chatGPT does.

Another example is the Robot Programming Activity by Isoda and Inprasitha at the Institute for Research and Development in Teaching Profession for ASEAN (IRDTP) for teacher training of CT and MT by using ArtecRobo with Scratch. Figure 11 illustrates the activity of designing various representations and translations among them. Translation between different representations is also MT in Figure 5. Figure 12 is the explanation of MT which was used in the designing process done by one group. Figure 13 illustrates how works what they learned on Task 1 for the next design on Task 2.

Because of the difference of intuitions on each mathematical representation on every theoretical difference, we cannot alternate the content of mathematics to informatics content. Translation of different representations enables us to reason with various intuition and produce further meanings. Thus, it does NOT deny the traditional content. What mathematics educators need to consider is how we can re-design our curriculum for more meaningful and useful in the limitation of the curriculum. It is STEM (or might be METS) education which across the curriculum

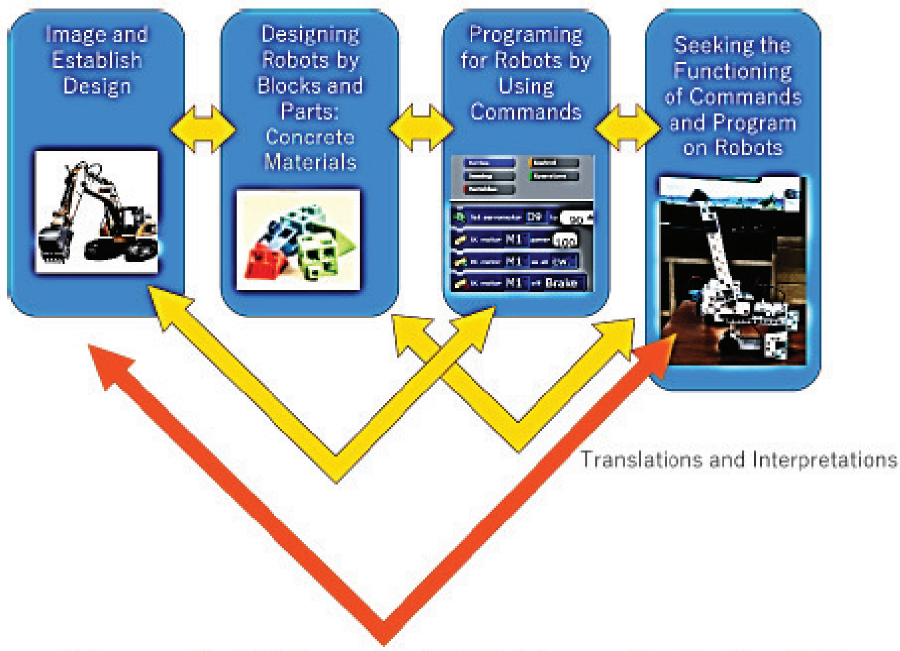

Figure 11. Mathematical Thinking on Designing Robot

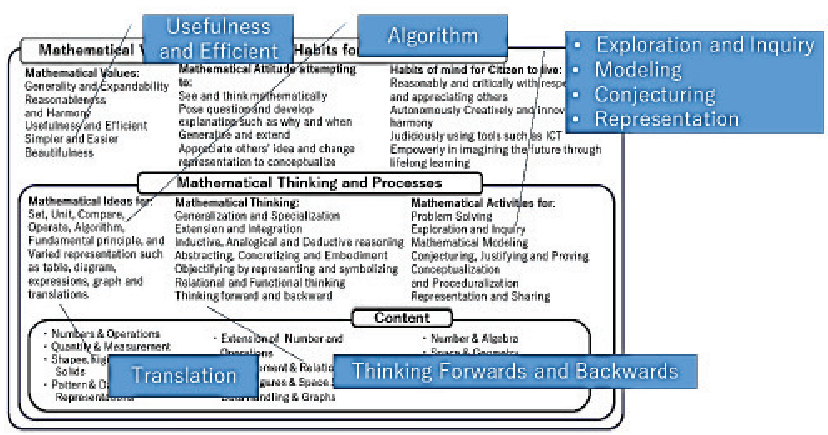

Figure 12. Used Mathematical Thinking explained by participant: a case

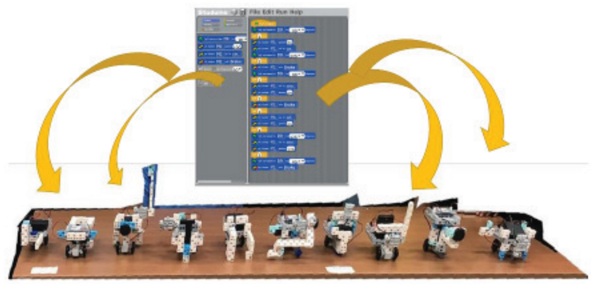

Figure 13. Adaptation of the program to the others: Program functions in a kind of form in mathematics

5 Conclusion

In this article, we confirmed the historical meaning of Mathematics in Liberal Arts, and current issues in mathematical science, and define mathematical and computational thinking as general competency beyond subjects for general skills. Mathematics was the name of several subjects which represented figural geometry with the mother language and functioned to learn logical reasoning: it was learned as liberal arts. This meaning had changed though algebraic representation became major. In academic subjects, it is the basis for Mathematical Sciences. On the other hand, it is a narrow view of mathematics education if teachers only make an effort for acquiring the skills in mathematical language. Indeed, the current mathematics curriculum framework includes Mathematical Thinking (MT) and Values (MTV) for human character formation. MTV is a general competency that can be also cultivated by the untraditional content of mathematics. This paper, it is illustrated in relation to Computational Thinking (CT). What mathematics educators need to consider is how we can re-design our curriculum for more meaningful and useful in the limitation of the curriculum. For this consideration, educators need to explore other subjects and how their practice is significant to enculturate MTV.

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