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PICARD OPERATORS IN STRONG b-TVS CONE METRIC SPACES

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Abstract

Let X be a topological space. A mapping $T : X \to X$ is called a Picard operator if T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converge to $\bar{x}$. In this paper, we give new results concerning the existence of Picard operators in strong b-TVS cone metric space. Our result is an extension of Sh. Rezapuor and R. Hamlbarani [19].

1 Introduction

The fixed point theorems have various application in chemistry, biology, computer sciences, differential equations, existence of invariant subspaces of linear operators and much more. Because of this, many scientists work on developing new fixed point theorems. See, for example, the book [14]. For contractions, the existence and the uniqueness of a fixed point are proved by the famous theorem of S. Banach [1].

Theorem 1.1. [1] Let (X, d) be a complete metric space and T be a self-mapping on X satisfying

$$d(Tx, Ty) \leqslant sd(x, y),$$

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for all $x, y \in X$ and $s \in [0, 1)$. Then T is a Picard operator.

The Banach Theorem is an abstract formulation of Picard iterative process, and it became a classical tool in nonlinear analysis. Moreover, it has many generalizations; see [2, 3, 13, 14, 15, 16, 20] and others. On the other hand, Connell [4] showed that Theorem 1.1 cannot characterize the completeness of X which means the notion of contractions is too strong from this point of view. In 1968, R. Kannan [11] proved the following theorem.

Theorem 1.2. [11] Let (X, d) be a complete metric space and T be a self-mapping on X satisfying

$$d(Tx, Ty) \leqslant s\big(d(x, Tx) + d(y, Ty)\big),$$

for all $x, y \in X$ and $s \in [0, \frac{1}{2})$. Then T is a Picard operator.

A mapping that satisfies the assumption of Theorem 1.2 is called a Kannan map. Example 2 in [12], R. Kannan showed a particular case of a discontinuous Kannan mapping, which is a property different Banach contraction principle. Another important application of the Kannan mapping is to be able to describe the completeness of the metric space in terms of the fixed point property of the mapping. This was proved by P. V. Subramanyam [21] in 1975, That is, a metric space (X, d) is complete if and only if every Kannan mapping on (X, d) has a fixed point. Note that the Banach contraction mapping class does not have this property, see [4]. Therefore, the mapping class in Theorem 1.2 immediately attracted the interest of many mathematicians, such as L. S. Dube and S. P. Singh [5], J. Górnicki [6, 7], G. Hiranmoy et al. [8] and others. Recently, non-convex analysis has found some applications in optimization theory, and so there have been some investigations about non-convex analysis, especially ordered normed spaces, normal cones and Topical functions (for example [17, 18]). In there efforts an order is introduced by substituting an ordered normed space for the real numbers. L.-G. Huang and X. Zhang [10] reviewed cone metric space in 2007, Later, the authors extended Theorem 1.1 and Theorem 1.2. In 2008 Sh. Rezapuor and R. Hamlbarani [19] by providing non-normal cones and omitting the assumption of normality in some results of [10], and obtain generalizations of the results. Next, In 2022, D.T. Hieu et al. used this approach in [9]; They defined strong b-TVS cone metric spaces.

Let E always be a real Hausdorff locally convex topological vector spaces with its zero vector θ and C is subset of E. We say that C is a cone in E if

(i) C is closed, nonempty and $C \neq \{\theta\}$,
(ii) $ax + by \in C$ for all $x, y \in C$ and non-negative real numbers a, b,
(iii) $C \cap (-C) = \{\theta\}$.

For a given cone C in E, we can define a partial ordering $\preceq$ with respect to C by $x \preceq y$ if and only if $y - x \in C$, while $x \ll y$ will stand for $y - x \in \operatorname{int} C$, where $\operatorname{int} C$ denotes the interior of C, $x \prec y$ if and only if $x \preceq y$ and $x \neq y$. In this paper, we always suppose E be a real Hausdorff locally convex topological vector spaces, C be a cone in E with $\operatorname{int} C \neq \emptyset$ and $\preceq$ is partial ordering with respect to C.

Definition 1.3. [9] Let X be a nonempty set and $K \geq 1$. The mapping $d : X \times X \to E$ is called a strong b-cone metric on X if

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \preceq d(x, z) + Kd(z, y)$ for all $x, y, z \in X$.

Then (X, E, C, K, d) is called a strong b-TVS cone metric space.

Definition 1.4. [9] Let (X, E, C, K, d) is a strong b-TVS cone metric space. Let $\{x_n\}$ be a sequence in X. We say that

(i) x is the limit of $\{x_n\}$ if for every $e \in E$ with $\theta \ll e$ there is n_0 such that $d(x_n, x) \ll e$ for all $n \geq n_0$. We denote this by $x_n \to x$ or $\lim_{n\to\infty} x_n$.

(ii) $\{x_n\}$ is Cauchy sequence if for every $e \in E$ with $\theta \ll e$ there is n_0 such that $d(x_n, x_m) \ll e$ for all $n, m \geq n_0$.

Lemma 1.5. [9] Let (X, E, C, K, d) is a strong b-TVS cone metric space and $\{x_n\}$ be a sequence in X. Then we have:

(i) If $\{x_n\}$ converges to $x \in X$ then $\{x_n\}$ is a Cauchy sequence.

(ii) If $\{x_n\}$ converges to $x \in X$ and $\{x_n\}$ converges to $y \in X$, then $x = y$.

2 Main results

In this section, we prove the following theorems, which are generalizations of Sh. Rezapuor and R. Hamlbarani [19].

Theorem 2.1. Let (X, E, C, K, d) be a complete strong b-TVS cone metric space and the mapping $T : X \to X$ satisfy the contractive condition

$$d(Tx, Ty) \preceq sd(x, y),$$

for all $x, y \in X$, where $s \in [0, 1)$ is a constant. Then T is a Picard operator.

Proof. Choose $x_0 \in X$, and define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geqslant 0$. Set $d_n = d(x_n, x_{n+1})$ for all $n \geqslant 0$. By hypothesis we have

$$\begin{aligned} d_{n+1} &= d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \\ &\preceq sd(x_n, x_{n+1}) = sd_n, \end{aligned}$$

for all $n \geqslant 0$. Hence

$$d_{n+1} \preceq sd_n,\ n = 0, 1, ...$$

It follows that

$$d_n \preceq s^n d_0,\ n = 1, 2, ...$$

So for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\preceq Kd(x_n, x_{n+1}) + ... + Kd(x_{m-2}, x_{m-1}) + d(x_{m-1}, x_m) \\ &= Kd_n + Kd_{n+1} + ... + Kd_{m-2} + d_{m-1} \\ &\preceq K(s^n d_0 + s^{n+1} d_0 + ... + s^{m-2} d_0) + s^{m-1} d_0 \\ &= K(1 + s + s^2 + ... + s^{m-n-2}) s^n d_0 + s^{m-1} d_0 \\ &\preceq \frac{K}{1-s} s^n d_0 + s^{m-1} d_0. \end{aligned}$$

Set $A_{(n,m)} := \frac{K}{1-s} s^n d_0 + s^{m-1} d_0$ for $n, m \geq 1$. Then

$$A_{(n,m)} - d(x_n, x_m) \in C \text{ with } m, n \geqslant 1.$$

with e is an arbitrary element of E and $\theta \ll e$, then there is a neighborhood U of the θ in E such that $e - U \subset \operatorname{int} C$. Since $\lim_{m,n\to\infty} A_{(n,m)} = \theta$, there is n_0 such that

$$A_{(n,m)} \in U \text{ for all } m, n \geqslant n_0.$$

This implies

$$e - A_{(n,m)} \in e - U \subset \operatorname{int} C \text{ for all } m, n \geqslant n_0.$$

Thus,

$$\begin{aligned} e - d(x_n, x_m) &= (e - A_{(n,m)}) + (A_{(n,m)} - d(x_n, x_m)) \\ &\in \operatorname{int} C + C = \operatorname{int} C \text{ for all } m, n \geqslant n_0. \end{aligned}$$

This means that

$$d(x_n, x_m) \ll e \text{ for all } m, n \geqslant n_0.$$

Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x}$. Since $\lim_{n\to\infty} x_n = \bar{x}$ then for each $k \geqslant 1$, there is a natural number n_k such that

$$d(x_n, \bar{x}) \ll \frac{e}{2ks} \text{ and } d(x_{n+1}, \bar{x}) \ll \frac{e}{2kK},$$

for all $n \geq n_k$. Hence, we have

$$\begin{aligned} d(T\bar{x}, \bar{x}) &\preceq d(T\bar{x}, Tx_n) + Kd(Tx_n, \bar{x}) \\ &\preceq sd(\bar{x}, x_n) + Kd(x_{n+1}, \bar{x}) \ll \frac{e}{k}, \end{aligned}$$

for all $n \geqslant n_k$. Hence, $\frac{e}{k} - d(T\bar{x}, \bar{x}) \in C$ for all $k \geqslant 1$. By $\frac{e}{k} \to 0$ as $k \to \infty$ and C is closed, $-d(T\bar{x}, \bar{x}) \in C$. This implies $d(T\bar{x}, \bar{x}) = \theta$, means that $T\bar{x} = \bar{x}$. So $\bar{x}$ is a fixed point of T.

Now if $\bar{y}$ is another fixed point of T, then

$$d(\bar{x},\bar{y}) = d(T\bar{x},T\bar{y}) \preceq sd(\bar{x},\bar{y}).$$

Hence $d(\bar{x},\bar{y}) = \theta$ and $\bar{x} = \bar{y}$. Therefore, the fixed point of T is unique. So, T is Picard operators. □

Remark 2.2. Obviously, if $K = 1$, E is Banach space then Theorem 2.1 reduces to Theorem 2.3 in [19]. Furthermore, the following example shows that Theorem 2.1 is applicable, but the Theorem 2.3 in [19] is not.

Example 2.3. Let $X = \{0,1,2\}, E = \mathbb{R}^2$ and

$$C = \{(x,y) \in E : x \geqslant 0, y \geqslant 0\}.$$

Function $d : X \times X \to E$ define by

$$d(x,x) = \theta \ \text{ with } x \in X, \qquad d(x,y) = d(y,x) \ \text{ for all } x,y \in X$$

and

$$d(0,1) = (4,4), \quad d(0,2) = d(1,2) = (1,1).$$

Let $T : X \to X$ by $T1 = T2 = T0 = 0$. Easy to see, $(X,E,C,K = 3,d)$ is a complete strong b-TVS cone metric space but the triangle inequality is not satisfied. Indeed, we have that

$$d(0,2) + d(2,1) = (2,2) \preceq (4,4) = d(0,1). \tag{2.1}$$

Therefore, Theorem 2.3 in [19] cannot be applied. However, we can easily check all the assumptions of Theorem 2.1 are satisfied and T is a Picard operator.

Theorem 2.4. Let (X,E,C,K,d) be a complete strong b-TVS cone metric space and T be a self-mapping on X satisfying

$$d(Tx,Ty) \preceq s\big(d(x,Tx) + d(y,Ty)\big),$$

for all $x,y \in X$ and $s \in [0,\frac{1}{2})$. Then T is a Picard operator.

Proof. Let $x_0 \in X$ be a fixed. Consider sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geqslant 0$. Set $d_n = d(x_n,x_{n+1})$ for all $n \geqslant 0$. By hypothesis, for any $n \geqslant 0$ we have

$$\begin{aligned} d_{n+1} &= d(x_{n+1},x_{n+2}) = d(Tx_n,Tx_{n+1}) \\ &\preceq s\big(d(x_n,Tx_n) + d(x_{n+1},Tx_{n+1})\big) \\ &= s\big(d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})\big) \\ &= s\big(d_n + d_{n+1}\big). \end{aligned}$$

This implies

$$d_{n+1} \preceq \frac{s}{1-s} d_n = h d_n \text{ for all } n \geqslant 0, \text{ where } h = \frac{s}{1-s} \in [0,1).$$

Hence

$$d_n \preceq h^n d_0 \text{ for all } n \geqslant 1.$$

For $m, n \geqslant 1$, we have

$$\begin{aligned} d(x_m, x_n) &= d(Tx_{m-1}, Tx_{n-1}) \\ &\preceq s\big(d(x_{m-1}, Tx_{m-1}) + d(x_{n-1}, Tx_{n-1})\big) \\ &= s\big(d(x_{m-1}, x_m) + d(x_{n-1}, x_n)\big) \\ &\preceq s\big(d_{m-1} + d_{n-1}\big). \end{aligned}$$

Set $B_{(m,n)} := s(d_{m-1} + d_{n-1})$ for all $m, n \geqslant 1$. Then

$$B_{(m,n)} - d(x_m, x_n) \in C \text{ for all } m, n \geq 1.$$

For $e \in E, \theta \ll e$ arbitrary, then there is a neighborhood U of θ in E such that $e - U \subset \operatorname{int} C$. Sence $\lim\limits_{m,n\to\infty} B_{(n,m)} = \theta$, there is n_0 such that

$$B_{(n,m)} \in U \text{ for all } m, n \geqslant n_0.$$

This implies

$$e - B_{(n,m)} \in e - U \subset \operatorname{int} C \text{ for all } m, n \geqslant n_0.$$

Hence

$$\begin{aligned} e - d(x_n, x_m) &= (e - B_{(m,n)}) + (B_{(m,n)} - d(x_m, x_n)) \\ &\in \operatorname{int} C + C = \operatorname{int} C \text{ for all } m, n \geqslant n_0. \end{aligned}$$

This means

$$d(x_m, x_n) \ll e \text{ for all } m, n \geqslant n_0.$$

Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $\bar{x} \in X$ such that $\lim\limits_{n\to\infty} x_n = \bar{x}$. Since $\lim\limits_{n\to\infty} x_n = \bar{x}$ then for any $k \geqslant 1$, there is a natural number n_k such that

$$d(x_n, x_{n+1}) \ll \frac{e(1-s)}{2ks} \text{ and } d(x_{n+1}, \bar{x}) \ll \frac{e(1-s)}{2kK},$$

for all $n \geqslant n_k$. Hence, we have

$$\begin{aligned} d(T\bar{x}, \bar{x}) &\preceq d(T\bar{x}, Tx_n) + K d(Tx_n, \bar{x}) \\ &\preceq s\big(d(\bar{x}, T\bar{x}) + d(x_n, Tx_n)\big) + K d(Tx_n, \bar{x}), \end{aligned}$$

for all $n \geqslant 0$. Hence,

$$\begin{aligned} d(T\bar{x}, \bar{x}) &\preceq \frac{1}{1-s}\big(sd(x_n, x_{n+1}) + Kd(x_{n+1}, \bar{x})\big) \\ &\preceq \frac{e}{2k} + \frac{e}{2k} = \frac{e}{k}, \end{aligned}$$

for all $n \geqslant n_k$. So, $\frac{e}{k} - d(T\bar{x}, \bar{x}) \in C$ for all $k \geqslant 1$. Since $\frac{e}{k} \to 0$ as $k \to \infty$ and C is closed $-d(T\bar{x}, \bar{x}) \in C$. This implies $d(T\bar{x}, \bar{x}) = \theta$. Thus, $T\bar{x} = \bar{x}$. Therefore, $\bar{x}$ is a fixed point of T.

Now if $\bar{y}$ is another fixed point of T, then

$$d(\bar{x}, \bar{y}) = d(T\bar{x}, T\bar{y}) \preceq s\big(d(\bar{x}, T\bar{x}) + d(\bar{y}, T\bar{y})\big) = \theta.$$

Hence $d(\bar{x}, \bar{y}) = \theta$ and $\bar{x} = \bar{y}$. Therefore, the fixed point of T is unique. So, T is Picard operators. $\square$

Remark 2.5. Note that, if $K = 1$, E is Banach space then Theorem 2.4 reduces to Theorem 2.6 in [19]. Furthermore, the following example shows that Theorem 2.4 is applicable, but the Theorem 2.6 in [19] is not.

Example 2.6. Let $X = \{0, 2, 3\}, E = \mathbb{R}^2$ and

$$C = \{(x, y) \in E : x \geqslant 0, y \geqslant 0\}.$$

Function $d : X \times X \to E$ define by

$$d(x, x) = \theta = (0, 0) \text{ with } x \in X,$$

$$d(3, 0) = d(0, 3) = (3, 3),\ d(2, 0) = d(0, 2) = d(2, 3) = d(3, 2) = (1, 1).$$

Let $T : X \to X$ by $T0 = T2 = T3 = 0$. Then $(X, E, C, K = 2, d)$ is a complete strong b-TVS cone metric space but the triangle inequality is not satisfied. Indeed, we have that

$$d(0, 2) + d(2, 3) = (2, 2) \preceq (3, 3) = d(0, 3).$$

Therefore, Theorem 2.6 in [19] cannot be applied. However, we can easily check all the assumptions of Theorem 2.4 are satisfied and T is a Picard operator.

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