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ON THE COFINITENESS OF IN DIMENSION < 2 LOCAL COHOMOLOGY MODULES FOR A PAIR OF IDEALS

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Abstract

In this note, we prove the cofiniteness of local cohomology modules $H^i_{I,J}(N)$ with respect to a pair of ideals (I, J) for all $i < t$ and the finiteness of $(0 :_{H^t_{I,J}(N)} I)$ and $\operatorname{Ext}^1_R(R/I, H^t_{I,J}(N))$ provided that $\operatorname{Ext}^i_R(R/I, N)$ is finitely generated for all $i \leq t+1$ and $H^i_{I,J}(N)$ is in dimension < 2 for all $i < t$, where $t \geq 1$ is an integer (here, N is not necessarily finitely generated over R). This extends the results of Bahmanpour-Naghipour [5, Thm 2.6], Bahmanpour-Naghipour-Sedghi [4, Thm 2.8] and H-N [16, Thm 1.1].

1 Introduction

Throughout this note the ring R is commutative Noetherian. Let N be a finitely generated R-module and I an ideal of R. In [9], A. Grothendieck conjectured that if I is an ideal of R and N is a finitely generated R-module, then the R-module $(0 :_{H^j_I(N)} I)$ is finitely generated for all $j \geq 0$. R. Hartshorne provided

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a counter-example to this conjecture in [10]. He also defined an R-module K to be I-cofinite if $\mathrm{Supp}_R(K) \subseteq V(I)$ and $\mathrm{Ext}^j_R(R/I, K)$ is finitely generated for all $j \geq 0$ and he asked a question:
Question. For which rings R and ideals I are the modules $H^j_I(N)$ is I-cofinite for all $j \geq 0$ and all finitely generated modules N?
Hartshorne showed that if N is a finitely generated R-module and I a prime ideal with $\dim_R(R/I) = 1$, where R is a complete regular local ring, then $H^j_I(N)$ is I-cofinite (see [10, Coro 7.7]). K. I. Yoshida refined this result to more general situation that if N is a finitely generated module over a commutative Noetherian local ring R and I an ideal of R such that $\dim_R(R/I) = 1$, then the modules $H^j_I(N)$ are I-cofinite for all $j \geq 0$ (see [20, Thm 1.1]).
There are many mathematicians who continue to study this problem. In 2009, K. Bahmanpour-R. Naghipour have extended the result of Yoshida in [20] to the case of non-local ring; more precisely, they showed that "*If $t \geq 0$ is an integer such that* $\dim \mathrm{Supp}_R(H^j_I(N)) \leq 1$ *for all $j < t$ then $H^j_I(N)$ is I-cofinite for all $j < t$ and* $(0 :_{H^t_I(N)} I)$ *is finitely generated* (see [5, Thm 2.6])". In 2018, Bahmanpour-Naghipour-Sedghi (see [4, Thm 2.8]) improved this result by replacing the condition that $\dim \mathrm{Supp}_R(H^j_I(N)) \leq 1$ *for all $j < t$* with the condition that $H^j_I(N)$ *is weakly Laskerian for all $j < t$* (this notion is introduced by K. Divaani-Aazar and A. Mafi [8]: an R-module K is called to be weakly Laskerian if $\mathrm{Ass}_R(K/T)$ is a finite set for each submodule T of K).

There are some generalizations of the theory of local cohomology modules. The following generalization of local cohomology theory is given by R. Takahashi-Y. Yoshino-T. Yoshizawa in [17]: Let j be a non-negative integer, I, J ideals of R, and N an R-module. Then the j^{th} local cohomology functor $H^j_{I,J}(-)$ with respect to a pair of ideals (I, J) to be the j^{th} right derived functor of $\Gamma_{I,J}(-)$. They called $H^j_{I,J}(N)$ the j^{th} local cohomology module of N with respect to (I, J). These modules were studied further in many research papers such as: [17], [18], [19],.... It is clear that $H^j_{I,J}(N)$ is just the ordinary local cohomology module $H^j_I(N)$ when $J = 0$. The purpose of this paper is to investigate a similar question as above for the theory of local cohomology with respect to a pair of ideals. More precisely, the aim of this note is to extends the result of Bahmanpour-Naghipour in [5, Thm 2.6], Bahmanpour-Naghipour-Sedghi in [4, Thm 2.8], H-N in [16, Thm 1.1] as the following theorem.

Theorem 1.1. *Let R be a Noetherian commutative ring, and I, J ideals of R. Let t be a positive integer, and N an R-module such that $\mathrm{Ext}^i_R(R/I, N)$ is finitely generated for all $i \leq t+1$. Assume that $H^i_{I,J}(N)$ is in dimension < 2 for all $i < t$. Then the following statements are true:*

(i) the R-module $H^i_{I,J}(N)$ is (I, J)-cofinite for all $i < t$, and

(ii) the R-modules $\mathrm{Hom}_R(R/I, H^t_{I,J}(N))$ and $\mathrm{Ext}^1_R(R/I, H^t_{I,J}(N))$ are finitely generated.

Here, we recall the notion of (I, J)-cofinite module which defined by A. Tehranian-A. P. E. Talemi in [18, Def 2.1] as follows: an R-module K is called (I, J)-*cofinite* if $\operatorname{Supp}_R(K) \subseteq W(I, J)$ and the R-modules $\operatorname{Ext}^j_R(R/I, K)$ is finitely generated for all $j \geq 0$, where

$$W(I, J) = \{p \mid p \in \operatorname{Spec} R, I^n \subseteq p + J \text{ for some integer } n \geq 0\}$$

was introduced by R. Takahashi-Y. Yoshino-T. Yoshizawa (see [17, Def 1.5]).

This note is divided into two sections. In Section 2, we first establish some auxiliary lemmas. The rest of Section 2 is devoted to prove Theorem 1.1 and its consequences.

2 Main result

We first recall the notion of in dimension $< n$ module defined by D. Asadollahi-R. Naghipour in [2].

Remark 2.1. Let n be a non-negative integer and K be an R-module. An R-module K is called *in dimension* $< n$ if there exists a finitely generated submodule T of K such that $\dim \operatorname{Supp}_R(K/T) < n$ (that is, we have $\dim(R/p) < n$ for all $p \in \operatorname{Supp}_R(K/T)$), see [2, Def 2.1]. It is clear that the class of in dimension $< n$ modules contains of class of finitely generated modules. Moreover, the class of in dimension $< n$ modules is a Serre subcategory, i.e, it is closed under taking submodules, quotients and extensions (cf. [14, Sect. 4] and [12, Coro 2.13]).

Before proving the main result in this section, we need the following lemma which is proved in [1, Thm 2.5] for the case $\dim_R(K) \leq 1$. We here use the hypothesis that $\dim \operatorname{Supp}_R(K) \leq 1$ instead of the condition $\dim_R(K) \leq 1$ as in [1] (because, in general the notion $\dim_R(K)$ may be regarded as $\dim_R(R/\operatorname{ann}_R(K))$ (see [11, p. 31]) which differs from the notion $\dim \operatorname{Supp}_R(K) = \sup\{\dim(R/p) \mid p \in \operatorname{Supp}_R(K)\}$). To avoid any possible misunderstanding here, we still give another proof of this result by another elementary arguments.

Lemma 2.2. *Let I be an ideal of R and let K be an R-module such that $1 \geq \dim \operatorname{Supp}_R(K)$. If the R-modules $\operatorname{Hom}_R(R/I, K)$ and $\operatorname{Ext}^1_R(R/I, K)$ are finitely generated, then $\operatorname{Ext}^j_R(R/I, K)$ is finitely generated for all $j \geq 0$.*

Proof. We have $\operatorname{Supp}_R(H^1_I(K)) \subseteq \operatorname{Max}(R) \cap V(I)$ (since $\dim \operatorname{Supp}_R(K) \leq 1$, if $\dim R/p = 1$ for some $p \in \operatorname{Supp}_R(H^1_I(K))$, then $p \in \min \operatorname{Supp}_R(K)$. Thus $\dim \operatorname{Supp}_{R_p}(K_p) = 0$, and hence $H^1_I(K)_p = 0$ by Grothendieck's vanishing theorem, a contradiction).

The short exact sequence $0 \to \Gamma_I(K) \to K \to K/\Gamma_I(K) \to 0$ induces an exact sequence $(0 :_{K/\Gamma_I(K)} I) \to \operatorname{Ext}^1_R(R/I, \Gamma_I(K)) \to \operatorname{Ext}^1_R(R/I, K)$, where

$(0 :_{K/\Gamma_I(K)} I) = 0$. Hence, $\mathrm{Ext}^1_R(R/I, \Gamma_I(K))$ is finitely generated by the hypothesis. Note that $(0 :_{\Gamma_I(K)} I) = (0 :_K I)$ and $\mathrm{Supp}_R(\Gamma_I(K)) \subseteq V(I)$. Thus, we get by [6, Prop 2.6] that $\Gamma_I(K)$ is I-cofinite. Thus, by [7, Thm 2.1], we obtain that $(0 :_{H^1_I(K)} I)$ is finitely generated. We now obtain by [15, Lem 2.1] that $H^1_I(K)$ is I-cofinite and artinian. Hence, we get by [14, Coro 3.10] that $\mathrm{Ext}^j_R(R/I, K)$ is finitely generated for all $j \geq 0$. □

Remark 2.3. In [17], Takahashi-Yoshino-Yoshizawa defined the set

$$W(I, J) = \{p \mid p \in \mathrm{Spec}\, R, I^n \subseteq p + J \text{ for some integer } n \geq 0\}$$

for a pair of ideals I and J. Note that if $J = 0$, we have $W(I, 0) = V(I) = \{p \mid p \in \mathrm{Spec}\, R, I^n \subseteq p \text{ for some integer } n \geq 0\}$. Then, Tehranian-Talemi ([18, Def 2.1]) defined the notion of (I, J)-cofinite module which is a generalization of the notion I-cofinite. Note that if $J = 0$ then the notion $(I, 0)$-cofinite module coincides with the ordinary notion I-cofinite module introduced by Hartshorne in [10].

The following lemma shows a criterion for cofiniteness of a module with respect to a pair of ideals (I, J).

Lemma 2.4. *Let I, J be ideals of R, and let K be an in dimension < 2 R-module such that $\mathrm{Supp}_R(K) \subseteq W(I, J)$. If the R-modules $(0 :_K I)$ and $\mathrm{Ext}^1_R(R/I, K)$ are finitely generated, then K is (I, J)-cofinite.*

Proof. By definition there is a finitely generated submodule T of K such that $\dim \mathrm{Supp}_R(K/T) \leq 1$ and $\mathrm{Supp}_R(K/T) \subseteq W(I, J)$. The short exact sequence $0 \to T \to K \to K/T \to 0$ induces the following exact sequence

$$\begin{aligned} 0 \to (0 :_T I) \to (0 :_K I) \to (0 :_{K/T} I) \to \mathrm{Ext}^1_R(R/I, T) \\ \to \mathrm{Ext}^1_R(R/I, K) \to \mathrm{Ext}^1_R(R/I, K/T) \to \mathrm{Ext}^2_R(R/I, T). \end{aligned}$$

It follows that $(0 :_{K/T} I)$ and $\mathrm{Ext}^1_R(R/I, K/T)$ are finitely generated. Therefore we get by Lemma 2.2 that R-module $\mathrm{Ext}^j_R(R/I, K/T)$ is finitely generated for all $j \geq 0$. It yields from the first exact sequence that $\mathrm{Ext}^j_R(R/I, K)$ is finitely generated for all $j \geq 0$. Hence, the R-module K is (I, J)-cofinite. □

Remark 2.5. Let K be an R-module, and I, J ideals of R.

(i) We obtain by [17, Coro 1.13] that $K/\Gamma_{I,J}(K)$ is an (I, J)-torsion-free-module, that is, $\Gamma_{I,J}(K/\Gamma_{I,J}(K)) = 0$.

(ii) Since $(0 :_{K/\Gamma_{I,J}(K)} I)$ is a submodule of $\Gamma_{I,J}(K/\Gamma_{I,J}(K))$, we have by the statement (i) that $(0 :_{K/\Gamma_{I,J}(K)} I) = 0$.

We are ready to prove the main result in this note.

Proof of Theorem 1.1. Set $L = \Gamma_{I,J}(N)$ and $\overline{N} = N/\Gamma_{I,J}(N)$. We have an exact sequence

$$0 \to L \to N \to \overline{N} \to 0. \tag{1}$$

Take E to be the injective hull of R-module $\overline{N}$. We get by [17, Prop 1.10] that

$$\operatorname{Ass}_R(\Gamma_{I,J}(E)) = \operatorname{Ass}_R(E) \cap W(I,J),$$
$$\operatorname{Ass}_R(\Gamma_{I,J}(\overline{N})) = \operatorname{Ass}_R(\overline{N}) \cap W(I,J).$$

So $\operatorname{Ass}_R(\Gamma_{I,J}(E)) = \operatorname{Ass}_R(\Gamma_{I,J}(\overline{N})) = \emptyset$ since $\Gamma_{I,J}(\overline{N}) = 0$ and $\operatorname{Ass}_R(E) = \operatorname{Ass}_R(\overline{N})$. This implies that $H^0_{I,J}(E) = \Gamma_{I,J}(E) = 0$, and so $(0 :_E I) = 0$ since $(0 :_E I) \subseteq \Gamma_{I,J}(E)$. We also obtain $(0 :_{\overline{N}} I) = 0$ since $(0 :_{\overline{N}} I)$ is a submodule of $\Gamma_{I,J}(\overline{N})$.

We now prove the theorem by induction on $t \geq 1$. We first consider the case $t = 1$. From the exact sequence (1), we get the following exact sequences

$$0 \to (0 :_L I) \to (0 :_N I) \to (0 :_{\overline{N}} I) = 0,$$
$$0 = (0 :_{\overline{N}} I) \to \operatorname{Ext}^1_R(R/I, L) \to \operatorname{Ext}^1_R(R/I, N).$$

Note that $(0 :_N I)$ and $\operatorname{Ext}^1_R(R/I, N)$ are finitely generated by the hypothesis. Thus, we obtain by the above exact sequences that $(0 :_L I)$ and $\operatorname{Ext}^1_R(R/I, L)$ are finitely generated. Moreover, since L is in dimension < 2 and $\operatorname{Supp}_R(L) \subseteq W(I,J)$ by the hypothesis, we obtain by Lemma 2.4 that the R-module L is (I,J)-cofinite, that is, $H^0_{I,J}(N)$ is (I,J)-cofinite. Therefore statement (i) is true for the case $t = 1$.

We obtain by [17, Coro 1.13] that $H^1_{I,J}(N) \cong H^1_{I,J}(\overline{N})$. Hence, we get isomorphisms $(0 :_{H^1_{I,J}(N)} I) \cong (0 :_{H^1_{I,J}(\overline{N})} I)$ and $\operatorname{Ext}^1_R(R/I, H^1_{I,J}(N)) \cong \operatorname{Ext}^1_R(R/I, H^1_{I,J}(\overline{N}))$. We next show that the R-modules $(0 :_{H^1_{I,J}(\overline{N})} I)$ and $\operatorname{Ext}^1_R(R/I, H^1_{I,J}(\overline{N}))$ are finitely generated (and thus the statement (ii) is true for the case $t = 1$). Consider the short exact sequence

$$0 \to \overline{N} \to E \to P \to 0, \tag{2}$$

where E is the injective hull of R-module $\overline{N}$, and $P = E/\overline{N}$. This induces the following exact sequence

$$0 = H^0_{I,J}(E) \to H^0_{I,J}(P) \to H^1_{I,J}(\overline{N}) \to H^1_{I,J}(E) = 0.$$

It implies that $H^1_{I,J}(\overline{N}) \cong H^0_{I,J}(P)$. Hence $(0 :_{H^1_{I,J}(\overline{N})} I) \cong (0 :_{H^0_{I,J}(P)} I)$ and $\operatorname{Ext}^1_R(R/I, H^1_{I,J}(\overline{N})) \cong \operatorname{Ext}^1_R(R/I, H^0_{I,J}(P))$. Note that

$$(0 :_{H^0_{I,J}(P)} I) = (0 :_{\Gamma_{I,J}(P)} I) = (0 :_P I).$$

On the other hand, by (2), we get the following exact sequence

$$\begin{aligned} 0 = (0 :_E I) \to (0 :_P I) \to \operatorname{Ext}^1_R(R/I, \overline{N}) \to \operatorname{Ext}^1_R(R/I, E) = 0 \\ \to \operatorname{Ext}^1_R(R/I, P) \to \operatorname{Ext}^2_R(R/I, \overline{N}) \to \operatorname{Ext}^2_R(R/I, E) = 0. \end{aligned}$$

It implies $(0 :_P I) \cong \operatorname{Ext}^1_R(R/I, \overline{N})$ and $\operatorname{Ext}^1_R(R/I, P) \cong \operatorname{Ext}^2_R(R/I, \overline{N})$. Moreover, by (1), we have the following exact sequence

$$\begin{aligned} \operatorname{Ext}^1_R(R/I,N) \to \operatorname{Ext}^1_R(R/I, \overline{N}) \to \operatorname{Ext}^2_R(R/I, L) \\ \to \operatorname{Ext}^2_R(R/I, N) \to \operatorname{Ext}^2_R(R/I, \overline{N}) \to \operatorname{Ext}^3_R(R/I, L) \end{aligned}$$

in which $\operatorname{Ext}^1_R(R/I, N)$ and $\operatorname{Ext}^2_R(R/I, N)$ are finitely generated by the hypothesis of the case $t = 1$. Note that $\operatorname{Ext}^2_R(R/I, L)$ and $\operatorname{Ext}^3_R(R/I, L)$ are finitely generated by the (I, J)-cofiniteness of L as mentioned above. Thus, $\operatorname{Ext}^1_R(R/I, \overline{N})$ and $\operatorname{Ext}^2_R(R/I, \overline{N})$ are finitely generated, and hence the R-modules $(0 :_P I)$ and $\operatorname{Ext}^1_R(R/I, P)$ are finitely generated. By the finiteness of the R-modules $(0 :_P I)$, we obtain that $(0 :_{H^1_{I,J}(\overline{N})} I)$ is finitely generated.

On the other hand, the short exact sequence

$$0 \to H^0_{I,J}(P) \to P \to P/H^0_{I,J}(P) \to 0$$

induces the following exact sequence

$$(0 :_{P/H^0_{I,J}(P)} I) \to \operatorname{Ext}^1_R(R/I, H^0_{I,J}(P)) \to \operatorname{Ext}^1_R(R/I, P),$$

where $(0 :_{P/H^0_{I,J}(P)} I) = 0$ (by Remark 2.5(ii)) and $\operatorname{Ext}^1_R(R/I, P)$ is finitely generated (by the above paragraph). Hence, $\operatorname{Ext}^1_R(R/I, H^0_{I,J}(P))$ is finitely generated, and so $\operatorname{Ext}^1_R(R/I, H^1_{I,J}(\overline{N}))$ is finitely generated. Thus, the statement (ii) of theorem is true for the case of $t = 1$.

We now assume that $t > 1$ and the theorem is true for $t - 1$. Hence, the R-modules $H^i_{I,J}(N)$ is (I, J)-cofinite for all $i < t - 1$, and the R-modules $(0 :_{H^{t-1}_{I,J}(N)} I)$ and $\operatorname{Ext}^1_R(R/I, H^{t-1}_{I,J}(N))$ are finitely generated. The rest of this proof devotes to Claim that $H^{t-1}_{I,J}(N)$ is (I, J)-cofinite, and the R-modules $(0 :_{H^t_{I,J}(N)} I)$ and $\operatorname{Ext}^1_R(R/I, H^t_{I,J}(N))$ are finitely generated.

By again the exact sequence (2), we obtain the following exact sequence

$$H^i_{I,J}(E) \to H^i_{I,J}(P) \to H^{i+1}_{I,J}(\overline{N}) \to H^{i+1}_{I,J}(E), \text{ and}$$

$$\begin{aligned} \operatorname{Ext}^i_R(R/I, E) \to \operatorname{Ext}^i_R(R/I, P) \to \operatorname{Ext}^{i+1}_R(R/I, \overline{N}) \\ \to \operatorname{Ext}^{i+1}_R(R/I, E), \end{aligned}$$

where $H^i_{I,J}(E) = 0$ for all $i \geq 0$. Note that $(0 :_E I) = 0$ and E is injective, so $\operatorname{Ext}^i_R(R/I, E) = 0$ for all $i \geq 0$. From the above exact sequence, we get by [17, Coro 1.13] that

$$H^i_{I,J}(P) \cong H^{i+1}_{I,J}(\overline{N}) \cong H^{i+1}_{I,J}(N) \text{ for all } i \geq 0 \tag{3}$$

and $\operatorname{Ext}^i_R(R/I, P) \cong \operatorname{Ext}^{i+1}_R(R/I, \overline{N})$ for all $i \geq 0$. Thus, the R-module $H^i_{I,J}(P)$ is in dimension < 2 for all $i < t-1$ by the hypothesis of the R-modules $H^{i+1}_{I,J}(N)$. Moreover, we get again by (1) that the following sequence

$$\operatorname{Ext}^{i+1}_R(R/I, N) \to \operatorname{Ext}^{i+1}_R(R/I, \overline{N}) \to \operatorname{Ext}^{i+2}_R(R/I, L)$$

is exact for all i. Note that $\operatorname{Ext}^{i+2}_R(R/I, L)$ is finitely generated for all i by the (I, J)-cofiniteness of L as mentioned; and $\operatorname{Ext}^{i+1}_R(R/I, N)$ is finitely generated for all $i+1 \leq t+1$ by the hypothesis. Thus, $\operatorname{Ext}^{i+1}_R(R/I, \overline{N})$ is finitely generated for all $i+1 \leq t+1$. Hence, $\operatorname{Ext}^i_R(R/I, P)$ is finitely generated for all $i \leq t = (t-1)+1$.

Thus, the above arguments ensure that the R-module P satisfies all conditions of the theorem for the case $t-1$. Hence, we get by the inductive assumption that $H^i_{I,J}(P)$ is (I, J)-cofinite for all $i < t-1$, and the R-modules $(0 :_{H^{t-1}_{I,J}(P)} I)$ and $\operatorname{Ext}^1_R(R/I, H^{t-1}_{I,J}(P))$ are finitely generated. Therefore, we get by the isomorphisms (3) that $H^{t-1}_{I,J}(N)$ is (I, J)-cofinite, and the R-modules $(0 :_{H^t_{I,J}(N)} I)$ and $\operatorname{Ext}^1_R(R/I, H^t_{I,J}(N))$ are finitely generated, and the Claim is proved. Hence the proof of the theorem is finished. □

Since $H^i_{I,0}(N) \cong H^i_I(N)$ and $W(I, 0) = V(I)$, if we replace $J = 0$ in Theorem 1.1, we then get the following result for the case of usual local cohomology modules immediately.

Corollary 2.6. [16, Thm 1.1] *Let R be a Noetherian commutative ring, and I an ideal of R. Let t be a positive integer, and N an R-module such that $\operatorname{Ext}^i_R(R/I, N)$ is finitely generated for all $i \leq t+1$. Assume that $H^i_I(N)$ is in dimension < 2 for all $i < t$. Then the following statements are true:*

(i) the R-module $H^i_I(N)$ is I-cofinite for all $i < t$ and

(ii) the R-modules $\operatorname{Hom}_R(R/I, H^t_I(N))$ and $\operatorname{Ext}^1_R(R/I, H^t_I(N))$ are finitely generated.

Corollary 2.7. (covers [16, Coro 2.3]) *Let t be a positive integer, and N an R-module such that $\operatorname{Ext}^i_R(R/I, N)$ is finitely generated for all $i \leq t+1$. If $H^i_{I,J}(N)$ is in dimension < 2 for all $i < t$, then the R-modules $\operatorname{Hom}_R(R/I, H^t_{I,J}(N)/K)$ and $\operatorname{Ext}^1_R(R/I, H^t_{I,J}(N)/K)$ are finitely generated, where K is any in dimension < 1 submodule of $H^t_{I,J}(N)$.*

Proof. Let K be an in dimension < 1 submodule of $H^t_{I,J}(N)$. We first show that K is (I,J)-cofinite. Note that $(0 :_{H^t_{I,J}(N)} I)$ and $\operatorname{Ext}^1_R(R/I, H^t_{I,J}(N))$ are finitely generated by Theorem 1.1 (*). Hence, since K is a submodule of $H^t_{I,J}(N)$, we obtain that $(0 :_K I)$ also is a submodule of $(0 :_{H^t_{I,J}(N)} I)$. Hence, the R-module $(0 :_K I)$ is finitely generated.

By the hypothesis of K, there exists a finitely generated submodule T of K such that $\dim \operatorname{Supp}_R(K/T) \leq 0$. The short exact sequence

$$0 \to T \to K \to K/T \to 0 \tag{4}$$

induces the exact sequence $(0 :_K I) \to (0 :_{K/T} I) \to \operatorname{Ext}^1_R(R/I, T)$. Thus, the R-module $(0 :_{K/T} I)$ is finitely generated. On the other hand, we have $\operatorname{Supp}_R(0 :_{K/T} I) \subseteq \operatorname{Supp}_R(K/T) \subseteq \operatorname{Max}(R)$. Thus $(0 :_{K/T} I)$ is an R-module of finite length. On the other hand, we have K/T is the union of its artinian submodules since $\operatorname{Supp}_R(K/T) \subseteq \operatorname{Max}(R)$. We then have by [13, Thm 5.3] that Koszul cohomology module $H^i(a_1, ..., a_s; K/T)$ is finitely generated for all i where $(a_1, ..., a_s) = I$. Thus, we get by [14, Thm 2.1] that $\operatorname{Ext}^i_R(R/I, K/T)$ is finitely generated for all i. Moreover we have $\operatorname{Supp}_R(K/T) \subseteq W(I,J)$. Hence we obtain that K/T is (I,J)-cofinite. Thus, we get by again the sequence (4) that K is (I,J)-cofinite.

From the short exact sequence $0 \to K \to H^t_I(N) \to H^t_I(N)/K \to 0$ we get the following exact sequence

$$\begin{aligned}&(0 :_{H^t_I(N)} I) \to (0 :_{H^t_I(N)/K} I) \to \operatorname{Ext}^1_R(R/I, K)\\ &\quad \to \operatorname{Ext}^1_R(R/I, H^t_I(N)) \to \operatorname{Ext}^1_R(R/I, H^t_I(N)/K) \to \operatorname{Ext}^2_R(R/I, K).\end{aligned}$$

Therefore, the R-modules $\operatorname{Hom}_R(R/I, H^t_I(N)/K)$ and $\operatorname{Ext}^1_R(R/I, H^t_I(N)/K)$ are finitely generated by the cofiniteness of K with respect to a pair of ideals (I,J) and by the fact (*). □

Remark 2.8. Let N be a finitely generated R-module. So $\operatorname{Ext}^j_R(R/I, N)$ is finitely generated for all $j \geq 0$. It is clear that $H^j_I(N)$ is in dimension < 2 for all $j < t$ when $\dim \operatorname{Supp}_R(H^j_I(N)) \leq 1$ for all $j < t$. Hence, we get by Corollary 2.6 the following corollary.

Corollary 2.9. (covers [5, Thm 2.6]) *Let N be a finitely generated R-module, and t a positive integer such that* $\dim \operatorname{Supp}_R(H^i_I(N)) \leq 1$ *for all $i < t$. Then, $H^i_I(N)$ is I-cofinite for all $i < t$, and the modules* $\operatorname{Hom}_R(R/I, H^t_I(N))$ *and* $\operatorname{Ext}^1_R(R/I, H^t_I(N))$ *are finitely generated.*

Remark 2.10. Note that if an R-module K is weakly Laskerian (this notion is introduced by K. Divaani-Aazar and M. Mafi in [8]), then there exists a finitely generated submodule T of K such that $\operatorname{Supp}_R(K/T)$ is a finite set (see [3, Thm 3.3]), and hence $\dim \operatorname{Supp}_R(K/T) \leq 1$. Moreover, if an R-module K'

is minimax (this kind of minimax module is defined by H. Zoschinger in [21]), then K' is in dimension < 1. Thus, as an consequence of Theorem 1.1 and Corollary 2.7, we get the following corollary.

Corollary 2.11. [4, Thm 2.8] *Let t be a positive integer, and N a finitely generated R-module. Assume that $H^i_I(N)$ is weakly Laskerian for all $i < t$. Then $H^i_I(N)$ is I-cofinite for all $i < t$, and $\operatorname{Hom}_R(R/I, H^t_I(N)/K)$ and $\operatorname{Ext}^1_R(R/I, H^t_I(N)/K)$ are finitely generated for any minimax submodule K of $H^t_I(N)$.*

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