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THE GAME CHROMATIC NUMBER OF SOME JOIN GRAPHS

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Abstract

Two players, Alice and Bob, alternatively color vertices of a graph using a fixed set of colors with Alice staring first so that no two adjacent vertices receive the same color. Alice wins if all vertices are successfully colored and Bob wins if one of the players has no legal move before all vertices are completely colored. The *game chromatic number* of a graph G, denoted by $\chi_g(G)$, is the least number of colors such that Alice has a winning strategy. The *join graph* $G \vee H$ is the graph obtained from G and H by adding the edges between all vertices of G and all vertices of H. In this paper, we investigate the game chromatic number of some join graphs.

1 Introduction

Two players, Alice and Bob, alternatively color vertices of a graph using a fixed set of colors with Alice staring first so that no two adjacent vertices receive the same color. Alice wins if all vertices are successfully colored and Bob wins if, at some time before all vertices are completely colored, one of the players has no legal move. The *game chromatic number* of a graph G, denoted by $\chi_g(G)$, is the least number of colors such that Alice has a winning strategy.

The well-known game coloring was invented by Steven J. Bram and was published in 1981 by Martin Gardner[4]. Bodlaender[2] reinvented this game in 1991. Define $\chi_g(\mathcal{G}) = \max\{\chi_g(G)|G \in \mathcal{G}\}$. The game chromatic number of several classes of graphs are investigated. For example, $\chi_g(\mathcal{F}) = 4$ when $\mathcal{F}$ is the class of forests[3], $6 \leq \chi_g(\mathcal{OP}) \leq 7$ when $\mathcal{OP}$ is the class of outerplanar

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graphs[5],[6], $8 \leq \chi_g(\mathcal{P}) \leq 17$ when $\mathcal{P}$ is the class of planar graphs[6],[12], $\chi_g(\mathcal{KT}) = 3k + 2$ for $k \geq 2$ when $\mathcal{KT}$ is the class of k-trees[9],[11].

In 2007, Bartnicki et.al.[1] investigated the game chromatic number of the Cartesian product of two graphs. The *Cartesian product* $G \square H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. In the paper, the authors proved that $\chi_g(G \square H)$ is not bounded above in terms of $\chi_g(G)$ and $\chi_g(H)$. However, Zhu[10] found the the upper bound of $\chi_g(G \square H)$ in terms of the game chromatic number of G and the acyclic chromatic number of H. Furthermore, Bartnicki et.al. also gave the exact values of $\chi_g(P_2 \square P_n)$, $\chi_g(P_2 \square C_n)$ and $\chi_g(P_2 \square K_n)$. Later, Sia[8] found the exact value of $\chi_g(S_m \square P_n)$, $\chi_g(S_m \square C_n)$, $\chi_g(P_2 \square W_n)$ and $\chi_g(P_2 \square K_{m,n})$. In 2009, Raspaud and Wu[7] proved that $\chi_g(C_m \square C_n) \leq 5$ and $\chi_g(C_{2m} \square C_n) = 5$ for $m \geq 3$ and $n \geq 7$.

The *join graphs* $G \vee H$, is the graph obtained from G and H by adding the edges between all vertices of G and all vertices of H. In this paper, we investigated the game chromatic number of $G \vee H$.

2 Preliminaries

In this section, we will introduce some examples and remarks of the game chromatic number of graphs.

Example 2.1. Let n be a positive number. Then

$$\chi_g(P_n) = \begin{cases} 1 & ; n = 1 \\ 2 & ; n = 2, 3 \\ 3 & ; n \geq 4 \end{cases}$$

Denote the vertices of the fiber of P_n by $u_1, u_2, \ldots, u_n$.

Case 1. $n = 1$. Alice chooses a color to label u_1. Then her goal is achieved.

Case 2. $n = 2$. To label P_2, it requires at least two colors because u_1 and u_2 need different colors. Suppose that there are two available colors. First, Alice labels u_1 by color 1. By the game rule, Bob must use another color to label u_2; hence, Alice achieves her goal.

Case 3. $n = 3$. To label P_3, it also requires at least two colors. Suppose that there are two available colors. First, Alice labels u_2 by color 1. Without loss of generality, Bob has to label u_1 by color 2. Finally, Alice labels u_3 by color 2; hence, Alice wins.

Case 4. $n \geq 4$. If there are three available colors, all vertices of P_n can be labeled because $\Delta(P_n) = 2 < 3$. Suppose that there are only two available colors. On the first move, if Alice labels v_i by color 1, then Bob can choose to label v_{i-2} or v_{i+2} by color 2. Then, v_{i-1} or v_{i+1} cannot be labeled. Hence, Bob wins.

Next, we will study about a relation between $\chi_g(H)$ and $\chi_g(G)$ when H is a subgraph of a graph G.

Remark 2.2. If H is a subgraph of a graph G such that $n(H) = n(G)$, then $\chi_g(H) \leq \chi_g(G)$.

Proof. Suppose that H is a subgraph of a graph G such that $n(H) = n(G)$. Let $m = \chi_g(G)$. For G, Alice has a winning strategy when there are m available colors. For H, Alice applies the same strategy with G to win this game. □

The condition $n(H) = n(G)$ in Remark 2.2 is essential because there exists a subgraph H of a graph G such that $\chi_g(H) > \chi_g(G)$ as shown in Remark 2.3.

Remark 2.3. There is a subgraph H of a graph G such that $\chi_g(H) > \chi_g(G)$.

Proof. Let H and G be graphs shown in Figure 2.1.

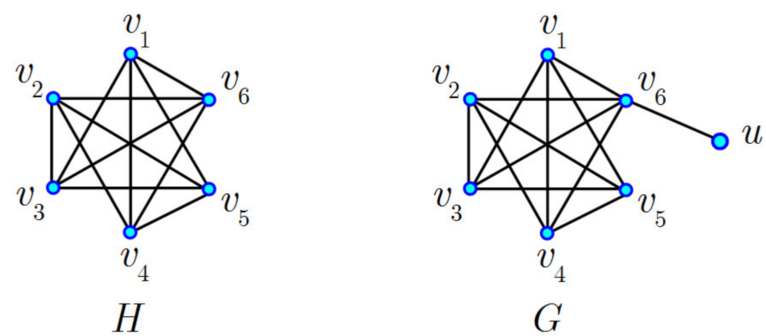

Figure 2.1: A subgraph H of a graph G such that $\chi_g(H) > \chi_g(G)$

It suffices to show that $\chi_g(H) > 4$ but $\chi_g(G) \leq 4$. Suppose that there are four available colors.

For H, without loss of generality, Alice labels v_1 by color 1, first. Then Bob labels v_2 by color 2. In the next move, Alice must use a new color because all remaining vertices are adjacent to both v_1 and v_2. Suppose that Alice labels v_3 by color 3. Then Bob labels v_4 by color 4. Hence, there is no available color v_5 and v_6. Therefore, Bob wins.

For G, Alice first labels u by color 1. Now, we focus on six remaining vertices with three available colors. Let $A_i = \{v_{2i-1}, v_{2i}\}$ for $i = 1, 2, 3$. Alice can force Bob to play first in each set. For $i = 1, 2, 3$, if Bob labels a vertex from A_i by any color, then Alice labels the other vertex by the same color. Hence, Alice wins. □

In fact, for any positive integer m, there exists a subgraph H of a graph H such that $n(H)+1 = n(G)$ and $e(H)+1 = E(G)$, but $\chi_g(H) \geq \chi_g(G)+n$, see Theorem 2.4.

Theorem 2.4. *For any positive integer m, there is a subgraph H of a graph G such that $\chi_g(H) \geq \chi_g(G) + m$.*

Proof. Let m be a positive integer. We will construct a graph G and its subgraph H such that $\chi_g(H) \geq 2m+3$ but $\chi_g(G) \leq m+3$. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_{2m+4}, u\}$ such that each pair of $v_1, v_2, \ldots, v_{2m+4}$ is adjacent except v_{2i-1} and v_{2i} for all i, and u is a vertex with degree 1 which is adjacent to v_{2m+4}. Let H be the subgraph of G obtained from G by deleting u. Let $A_i = \{v_{2i-1}, v_{2i}\}$ for $i = 1, 2, \ldots, m+2$.

Suppose that there are only $2m+2$ available colors for coloring H. Bob can force Alice to first label a vertex in each A_i. For each A_i, when Alice labels a vertex by a color, Bob uses a different color to label the other vertex. Then each move requires a new color. After $2m+2$ moves, there is no available color for two remaining vertices because two vertices. Hence, Bob wins.

Suppose that there are only $2m+3$ available colors for coloring G. Alice first labels u by color 1. Now, we focus on $2m+4$ remaining vertices with $m+2$ available colors. Alice can force Bob to play first in each set A_i. For $i = 1, 2, \ldots, m+2$, if Bob labels a vertex from A_i by any color, then Alice labels the other vertex by the same color. Then $m+2$ colors is enough for $2m+4$ vertices; hence, Alice wins. □

3 The join of graphs

In this section, we start investigating relations between join of some graphs. First, we study a relation between $\chi_g(G)$ and $\chi_g(G \vee K_1)$. It is not true that $\chi_g(G \vee K_1) = \chi_g(G)+1$. There exists a graph G such that $\chi_g(G \vee P_1) = \chi_g(G)$. The graph is P_4. Example 2.1 shows that $\chi_g(P_4) = 3$ and Example 3.1 confirms that $\chi_g(P_4 \vee P_1) = 3$, as well.

Example 3.1. Let P_4 and P_1 be paths with four and one vertices, respectively. Then $\chi_g(P_4 \vee P_1) = 3$.

Denote the vertices of two fibers P_n and P_2 by u_1, u_2, u_3, u_4 and v, respectively. Since P_4 requires two colors and P_1 requires one more color, $P_4 \vee P_1$ need at least three colors. Suppose that there are three available colors. We will prove that Alice has a winning strategy.

On the first move, Alice labels v by color 1. Let $A_1 = \{u_1, u_3\}$ and $A_2 = \{u_2, u_4\}$. For $i = 1, 2$, Alice can force Bob to first label a vertex in A_i. If Bob labels a vertex in A_i then Alice labels the other vertex by the same color. Hence, Alice wins.

Example 3.2 shows that there exists a graph G such that $\chi_g(G \vee P_1) = \chi_g(G) + 2$. The graph is $P_4 \vee P_1$.

Example 3.2. Let P_4 and P_2 be paths with four and two vertices, respectively. Then $\chi_g(P_4 \vee P_2) = 5$.

Denote the vertices of two fibers P_4 and P_2 by u_1, u_2, u_3, u_4 and v_1, v_2, respectively. Bob can force Alice to first label a vertex of P_4, say u_i Then Bob labels u_{i-2} or u_{i+2}. Hence, u_{i-1} or u_{i+1} requires the third color. Since P_2 need two more colors, $P_4 \vee P_2$ need at least five colors.

To prove that $\chi_g(P_4 \vee P_2) \leq 5$, suppose that there are five available colors. We will prove that Alice has a winning strategy.

Notice that $P_4 \vee P_2$ has six vertices but there are five available colors. Hence, Alice immediately wins when two vertices are labeled by the same color. On the first move, Alice labels u_2 by color 1. On the second move, we divide Bob's choice into three cases. If bob does not label u_4, then Alice labels u_4 by color 1. If Bob labels u_4 by color 2, then Alice labels u_1 by color 2. If Bob labels u_4 by color 1, then Alice can label any vertex with any color. The remaining vertices can be labeled. Hence, Alice wins.

Recall that if G is P_4 then $\chi_g(G \vee P_1) = \chi_g(G)$ and, if G is $P_4 \vee P_1$ then $\chi_g(G \vee P_1) = \chi_g(G) + 2$.

Example 3.3 and Example 3.4 show that there exist a subgraph H of a graph G such that $\chi_g(H) > \chi_g(G)$.

Example 3.3. Let P_3 be a path with three vertices. Then $\chi_g(P_3 \vee P_3) = 5$.

Denote the vertices of the two fibers of P_3 by u_1, u_2, u_3 and v_1, v_2, v_3, respectively. It suffices to show that Bob wins if four colors are available and Alice wins if five colors are available.

Suppose that there are four available colors. Let $A = \{u_1, u_3, v_1, v_3\}$. Bob can force Alice to first label vertices in A because the number of vertices outside A is even. Without loss of generality, Alice labels u_1 by color 1. Then Bob label u_3 by color 2. However, v_2, u_1, u_2 need three more different colors. Therefore, Bob wins.

Suppose that there are five available colors. Notice that u_1, u_3, v_1, v_3 have degree four; hence, they can always be labeled. First, Alice labels v_2 by color 1. If Bob labels u_2 in the second move, then the proof is done; otherwise, Alice labels u_2 in the third move.

Example 3.4. Let P_4 and P_3 be paths with four vertices and three vertices, respectively. Then $\chi_g(P_4 \vee P_3) = 4$.

Denote the vertices of the two fibers of P_4 and P_3 by u_1, u_2, u_3, u_4 and v_1, v_2, v_3, respectively. Notice that P_4 need at least two colors and P_3 need two more colors. Then $P_4 \vee P_3$ need at least four colors. Suppose that there are four available colors. We will prove that Alice has a winning strategy. First, Alice label v_2 by color 1. Let $A_1 = \{u_1, u_3\}, A_2 = \{u_2, u_4\}$ and $A_3 = \{v_1, v_3\}$. For $= 1, 2, 3$, Alice can force Bob to plays first in each A_i. When Bob labels a vertex in A_i, Alice labels the other vertex by the same color. Hence, Alice wins.

Next, we study a relation between $\chi_g(G_1 \vee G_2)$ in terms of $\chi_g(G_1)$ and $\chi_g(G_2)$.

Remark 3.5. If $n(G_1)$ and $n(G_2)$ are even, then $\chi_g(G_1 \vee G_2) \geq \chi_g(G_1) + \chi_g(G_2)$.

Proof. Let G_1 and G_2 be graphs such that $n(G_1)$ and $n(G_2)$ are even. Let $m = \chi_g(G_1)$ and $n = \chi_g(G_2)$. Notice that Bob can force Alice to first label in both graphs because $n(G_1)$ and $n(G_2)$ are even.

Assume that there are $m+n-1$ available colors. Without loss of generality, suppose that there are at most $m-1$ available colors for G_1. Then Bob can apply a winning strategy when there are $m-1$ colors for G_1. □

The condition $n(G_1)$ and $n_(G_2)$ in Remark 3.5 is necessary. Without the condition, there exist graphs G_1 and G_2 such that $\chi_g(G_1 \vee G_2) < \chi_g(G_1) + \chi_g(G_2)$ as shown in Remark 3.6.

Remark 3.6. There exist graphs G_1 and G_2 such that $\chi_g(G_1 \vee G_2) < \chi_g(G_1) + \chi_g(G_2)$.

Proof. Let $G_1 = P_4$ and $G_2 = K_1$. Then $G_1 \vee G_2 = P_4 \vee K_1$. Recall that $\chi_g(P_4) = 3$, $\chi_g(K_1) = 1$ and $\chi_g(P_4 \vee K_1) = 3$. Hence, $\chi_g(G_1 \vee G_2) < \chi_g(G_1) + \chi_g(G_2)$. □

Remark 3.7. There exist graphs G_1 and G_2 such that $\chi_g(G_1 \vee G_2) > \chi_g(G_1) + \chi_g(G_2)$.

Proof. Let $G_1 = P_4 \vee K_1$ and $G_2 = K_1$. Then $G_1 \vee G_2 = P_4 \vee K_2$. Recall that $\chi_g(P_4 \vee K_1) = 3$, $\chi_g(K_1) = 1$ and $\chi_g(P_4 \vee K_2) = 5$. Hence, $\chi_g(G_1 \vee G_2) > \chi_g(G_1) + \chi_g(G_2)$. □

Theorem 3.8. *For a positive integer m, there exists a graph G such that $\chi_g(G) \geq \chi_g(G \vee K_1) + m$.*

Proof. Let m be a positive integer and K_1 be the graph with one vertex, say u. Let G be the graph with $V(G) = \{v_1, v_2, \ldots, v_{2m+4}\}$ and all vertices are adjacent except v_{2i-1} and v_{2i} for all i. Consider $G \vee \{u\}$.

It suffices to show that $\chi_g(G) > 2m+2$ but $\chi_g(G \vee K_1) \leq m+3$. Let $A_i = \{v_{2i-1}, v_{2i}\}$ for $i = 1, 2, \ldots, m+2$.

For G, Bob can force Alice to first label a vertex in each A_i. When Alice labels a vertex in A_i, Bob uses a different color to label the other vertex. After $2m+2$ moves, there is no available color for remaining vertices because two vertices in different sets A_i must be labeled by different colors. Hence, Bob wins.

For $G \vee K_1$, Alice first labels u by color 1. Now, we focus on $2m+4$ remaining vertices with $m+2$ available colors. Alice can force Bob to play first in each A_i. When Bob labels a vertex from A_i, Alice labels the other vertex by the same color. Hence, $m+2$ colors are enough for $2m+4$ vertices. That is, Alice wins. □

4 Exact Values

In this section, we study the join of a path and another graph G.

Theorem 4.1. *If P_n is a path with n vertices, then*

$$\chi_g(P_n \vee P_1) = \begin{cases} 2 & ; n = 1 \\ 3 & ; n = 2, 3, 4 \\ 4 & ; n \geq 5 \end{cases}$$

Proof. Denote the vertices of the two fibers of P_n and P_1 by $u_1, u_2, \ldots u_n$ and v, respectively. It is easy to check that $\chi_g(P_1 \vee P_1) = 2$ and $\chi_g(P_2 \vee P_1) = 3$. Then suppose that $n \geq 3$.

Case 1. $n = 3, 4$. Since K_3 is a subgraph of $P_n \vee P_1$, we obtain $\chi_g(P_n \vee P_1) \geq 3$. Next, suppose that there are three available colors. We will show that Alice has a winning strategy. On the first move, Alice labels v by color 1. On the second move, if Bob labels u_i for some i, then Alice labels either u_{i-2} or u_{i+2} by the same color. Hence, all vertices of P_n can be labeled by using two colors. Therefore, Alice has a winning strategy.

Case 2. $n \geq 5$. Suppose that there are four available colors. On the first move, Alice labels v by color 1. The remaining vertices are P_n and $\Delta(P_n) = 2 < 3$, all remaining vertices can be labeled by three remaining colors. Hence, Alice has a winning strategy.

Suppose that there are three available colors and we will prove that Bob has a winning strategy. On the first move, we divide into two cases.

Case 1. Alice labels u_i for some i. Then Bob labels either u_{i-2} or u_{i+2} by another color. Hence u_{i-1} or u_{i+1} requires the third color. That is, there is no available color for v.

Case 2. Alice labels v by color 1. On the second move, Bob labels u_3 by color 2. On the forth move, Bob can choose to label u_1 or u_5 by color 3. Hence, there is no available color for u_2 or u_4. Therefore, Bob wins. □

Lemma 4.2. *Let G be a graph. Then $\chi_g(G \vee K_{2n}) \geq \chi_g(G) + 2n$.*

Proof. Let G be a graph such that $\chi_g(G) = m$. Suppose that there are only $m + 2n - 1$ available colors for $G \vee K_{2n}$. It suffices to prove that Bob has a winning strategy.

Notice that if G uses more than $m-1$ colors, then there is not enough color for K_{2n}; hence, Bob wins. Suppose that there are $m - 1$ available colors for G.

Bob can force Alice to be the first to label a vertex of G because the number of vertices of K_{2n} is even. Whenever Alice labels a vertex of K_{2n}, Bob labels a vertex of K_{2n}, as well. Hence, Bob can apply a winning strategy when labeling G for $m-1$ colors. That is, Bob has a winning strategy when labeling $G \vee K_{2n}$ for $m + 2n - 1$ colors. □

Remark 4.3. Let G be a graph. Then $\chi_g(G) \leq \Delta(G) + 1$

Theorem 4.4. *Let m, n be positive integer. Then*

$$\chi_g(P_n \vee K_m) = \begin{cases} m+1 & \text{if } n = 1 \\ m+2 & \text{if } n = 2, 3, \text{ or } n = 4 \text{ and } m \text{ is odd} \\ m+3 & \text{if } n = 4 \text{ and } m \text{ is even, or } n \geq 5 \end{cases}$$

Proof. Denote the vertices of the fibers of P_n by $u_1, u_2, \dots u_n$ and the vertices of K_m by $v_1, v_2, \dots, v_m$

By Example 2.1, Theorem 4.1 and Lemma 4.2, we obtain that

$$\chi_g(P_n \vee K_m) \geq \begin{cases} m+1 & \text{if } n = 1 \\ m+2 & \text{if } n = 2, 3, \text{ or } n = 4 \text{ and } m \text{ is odd} \\ m+3 & \text{if } n = 4 \text{ and } m \text{ is even, or } n \geq 5 \end{cases}$$

By Remark 4.3, we obtain that

$$\chi_g(P_n \vee K_m) \leq \begin{cases} m+1 & \text{if } n = 1 \\ m+2 & \text{if } n = 2 \\ m+3 & \text{if } n = 4 \text{ and } m \text{ is even, or } n \geq 5 \end{cases}$$

Hence, two cases are remaining.

Case 1. $n = 3$. First, Alice labels u_2 by color 1. Let $A = \{u_1, u_3\}$. Notice that if both vertices in A are labeled by the same colors, then $P_3 \vee K_m$ can be labeled by $m + 2$ colors. Whenever Bob labels a vertex in A, Alice will label the other vertex by the same color. Bob can force Alice to first label a vertex from A only if m is odd and they label all vertices of K_m. That is, u_1 and u_3 have only one available color; hence, they have no choice but label the last color on both vertices. Hence, Alice wins. That is, $\chi_g(P_3 \vee K_m) \leq m + 2$.

Case 2. $n = 4$ and m is odd. Notice that if all vertices of $V(P_4)$ are labeled by only two colors, then $P_4 \vee K_m$ can be labeled by $m + 2$ colors. Let $A_1 = \{u_1, u_3\}$ and $A_2 = \{u_2, u_4\}$. Alice can force Bob to first label vertices in each A_i because m is odd. When Bob labels a vertex in A_i, Alice labels the other vertex by the same color. Hence, Alice wins. □

5 Open problems

In the previous section, the game chromatic number of $P_n \vee K_m$ is investigated. Let G be an m-vertex graph and S_m be an independent set of size m. The upper bound of the game chromatic number of $P_n \vee G$ is obtained. The game chromatic number of $P_n \vee S_m$ is not yet investigated. If the game chromatic number of $P_n \vee S_m$ is studied, then the lower bound of the game chromatic number of $P_n \vee G$ is also obtained.

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