East-West J. of Mathematics: Vol. 24, No 2 (2023) pp. 89-118
https://doi.org/10.36853/ewjm0396

EXISTENCE OF THREE WEAK SOLUTIONS FOR THE KIRCHHOFF-TYPE PROBLEM WITH MIXED BOUNDARY CONDITION IN A VARIABLE SOBOLEV SPACE

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Abstract

In this paper, we consider the Kirchhoff-type problem for a class of nonlinear operators containing $p(\cdot)$-Laplacian and mean curvature operator with mixed boundary conditions. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of at least three weak solutions according to hypotheses on given functions and values of parameters.

1 Introduction

In this paper, we consider the following mixed boundary value problem

$$\begin{cases} -M(\Phi(u))\mathrm{div}\,[\boldsymbol{a}(x,\boldsymbol{\nabla}u(x))] = \lambda f_0(x,u(x)) + \mu f_1(x,u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ M(\Phi(u))\boldsymbol{n}(x)\cdot\boldsymbol{a}(x,\boldsymbol{\nabla}u(x)) = \lambda g_0(x,u(x)) + \mu g_1(x,u(x)) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

Key words: $p(\cdot)$-Laplacian type equation, mean curvature operator, three weak solutions, mixed boundary value problem.
2010 AMS Mathematics Classification: 35H30, 35D05, 35J60, 35J70.

Here Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint open subsets of } \Gamma \text{ such that } \overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset, \tag{1.2}$$

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x, \boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. The function $M = M(s)$ defined in $[0, \infty)$ satisfies the following condition (M).

(M) $M : [0, \infty) \to [0, \infty)$ is a continuous and monotone increasing (i.e., non-decreasing) function, and there exist $0 < m_0 \leq m_1 < \infty$ and $l \geq 1$ such that

$$m_0 s^{l-1} \leq M(s) \leq m_1 s^{l-1} \text{ for all } s \geq 0.$$

Furthermore, the function $\Phi(u)$ is defined by

$$\Phi(u) = \int_\Omega A(x, \boldsymbol{\nabla} u(x)) dx, \tag{1.3}$$

where $A(x, \boldsymbol{\xi})$ is a function on $\Omega \times \mathbb{R}^N$ satisfying $\boldsymbol{a}(x, \boldsymbol{\xi}) = \boldsymbol{\nabla}_{\boldsymbol{\xi}} A(x, \boldsymbol{\xi})$.

Here the operator $u \mapsto \operatorname{div}[\boldsymbol{a}(x, \boldsymbol{\nabla} u(x)]$ is more general than the $p(\cdot)$-Laplace operator $\Delta_{p(x)} u(x) = \operatorname{div}[|\boldsymbol{\nabla} u(x)|^{p(x)-2} \boldsymbol{\nabla} u(x)]$ and the mean curvature operator $\operatorname{div}[(1+|\boldsymbol{\nabla} u(x)|^2)^{(p(x)-2)/2} \boldsymbol{\nabla} u(x)]$. This generality brings about difficulties and requires some conditions.

Thus we impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g_i : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ for $i = 0, 1$ are Carathéodory functions satisfying some structure conditions and λ, μ are real parameters.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [31]), in electrorheological fluids (Diening [11], Halsey [19], Mihăilescu and Rădulescu [24], Růžička [26]).

For physical motivation to the problem (1.1), we consider the case where $\Gamma = \Gamma_1$ and $p(x) = 2$. Then the equation

$$M(\|\boldsymbol{\nabla} u\|^2_{\boldsymbol{L}^2(\Omega)}) \Delta u(x) = f(x, u(x)) \tag{1.4}$$

is the Kirchhoff equation which arises in nonlinear vibration, namely

$$\begin{cases} u_{tt} - M(\|\boldsymbol{\nabla} u\|^2_{\boldsymbol{L}^2(\Omega)}) \Delta u = f(x, u) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \Gamma \times (0, T), \\ u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x). \end{cases} \tag{1.5}$$

Equation (1.4) is the stationary counterpart of (1.5). Such a hyperbolic equation is a general version of the Kirchhoff equation

$$\rho u_{tt} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

presented by Kirchhoff [21]. This equation extends the classical d'Alembert wave equation by considering the effect of the changes in the length of the strings during the vibrations, where L, h, E, ρ and ρ_0 are constants.

Over the last two decades, there are many articles on the existence of weak solutions for the Dirichlet boundary condition, that is, in the case $\Gamma_2 = \emptyset$ in (1.1), (for example, see Arosio and Pannizi [5], Cavalcante et al. [7], Corrêa and Figueiredo [9], D'Ancona and Spagnolo [10], He and Zou, [20], Yücedaĝ [28]).

However, since we find a few papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1) (for example, Aramaki [3, 4]). We are convinced of the reason for existence of this paper.

According to some assumptions on f_i, g_i $(i = 0, 1)$ and values of parameters, we derive the existence of at least three weak solutions for the problem (1.1) using the Ricceri theorem (cf. Ricceri [25, Theorem 2]). In the previous paper [4], we considered the similar problem for a class of operators containing $p(\cdot)$-Laplacian, but not containing the mean curvature operator. Thus this paper is an extension of [4].

The paper is organized as follows. Section 2 consists of four subsections. In Subsection 2.1, we recall some results on variable exponent Lebesgue-Sobolev spaces and trace. In Subsection 2.2, we consider some weighted variable exponent Lebesgue spaces. Subsection 2.3 is devoted to the Nemytskii operators and their properties. In Subsection 2.4, we introduce the Poincaré-type inequality by Ciarlet and Dinca [8]. According to this inequality, we can consider the mixed boundary value problem as (1.1). In Section 3, we give the setting of problem (1.1) rigorously and a main theorem (Theorems 3.7) on the existence of at least three weak solutions and its proof. Section 4 is devoted in the proof of the main theorem and furthermore, we obtain a corollary of the main theorem.

2 Preliminaries

Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Moreover, we assume that Γ satisfies (1.2).

In the present paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$.

Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \dots, a_N)$ and $\boldsymbol{b} = (b_1, \dots, b_N)$ in $\mathbb{R}^N$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^N a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*,B}$.

2.1 Definitions of the Lebesgue and Sobolev spaces and their properties

In this subsection, we recall some well-known results on variable exponent Lebesgue and Sobolev spaces. See Fan and Zhang [16], Kováčik and Rácosník [22], Diening et al. [12] and references therein for more detail. Furthermore, we consider some new properties on variable exponent Lebesgue space. Define $C(\overline{\Omega}) = \{p; p \text{ is a continuous function on } \overline{\Omega}\}$, and for any $p \in C(\overline{\Omega})$, put

$$p^+ = \max_{x \in \overline{\Omega}} p(x) \text{ and } p^- = \min_{x \in \overline{\Omega}} p(x).$$

For any $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and for any measurable function u on Ω, a modular $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0; \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define a Sobolev space: for any integer $m \geq 0$,

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); \partial^\alpha u \in L^{p(\cdot)}(\Omega) \text{ for } |\alpha| \leq m\},$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$ and $\partial_i = \partial / \partial x_i$, endowed with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

Of course, $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$.

The following three propositions are well known (see Fan et al. [18], Fan and Zhao [17], Zhao et al. [30].

Proposition 2.1. Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$, and let $u, u_n \in L^{p(\cdot)}(\Omega)$ ($n = 1, 2, \ldots$). Then we have the following properties.

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ as $n \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ as $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. Let $p \in C_+(\overline{\Omega})$. where $C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}); p^- > 1\}$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have

$$\int_\Omega |u(x)v(x)|dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \leq 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, for any $p \in C_+(\overline{\Omega})$, $p'(\cdot)$ denote the conjugate exponent of $p(\cdot)$, that is, $p'(x) = p(x)/(p(x) - 1)$.

For $p \in C_+(\overline{\Omega})$, define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.3. Let Ω be a bounded domain of $\mathbb{R}^N$ with $C^{0,1}$-boundary and let $p \in C_+(\overline{\Omega})$ and $m \geq 0$ be an integer. Then we have the following properties.

(i) The spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

(ii) If $q(\cdot) \in C(\overline{\Omega})$ with $q^- \geq 1$ satisfies $q(x) \leq p(x)$ for all $x \in \Omega$, then $W^{m,p(\cdot)}(\Omega) \hookrightarrow W^{m,q(\cdot)}(\Omega)$, where $\hookrightarrow$ means that the embedding is continuous.

(iii) If $q(x) \in C(\overline{\Omega})$ with $q^- \geq 1$ satisfies that $q(x) < p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.

Next we consider the trace (cf. Fan [14]). Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and $p \in C(\overline{\Omega})$ with $p^- \geq 1$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}(\Omega)$, the trace $\gamma(u) = u\big|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1(\Gamma)$. We define

$$\begin{aligned}\mathrm{Tr}(W^{1,p(\cdot)}(\Omega)) &= (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma) \\ &= \{f; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}\end{aligned}$$

equipped with the norm

$$\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_\Gamma = f\}$$

for $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then we can see that $(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$ is a Banach space. In the later we also write $F\big|_{\Gamma}= g$ by $F = g$ on Γ. Moreover, for $i = 1, 2$, we denote

$$(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i) = \{f\big|_{\Gamma_i}; f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)\}$$

equipped with the norm

$$\|g\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}; f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f\big|_{\Gamma_i}= g\},$$

where the infimum can also be achieved, so for any $g \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F\big|_{\Gamma_i}= g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)} = \|g\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)}$.

For any $q \in C(\Gamma)$, we also define $q^+ = \max_{x\in\Gamma} q(x)$ and $q^- = \min_{x\in\Gamma} q(x)$. Let $q \in C_+(\Gamma) := \{q \in C(\Gamma); q^- > 1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma$. We define

$$L^{q(\cdot)}(\Gamma) = \Big\{u; u : \Gamma \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma \\ \text{satisfying } \int_\Gamma |u(x)|^{q(x)} d\sigma < \infty\Big\}$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma \left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma \le 1\right\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)} d\sigma.$$

Proposition 2.4. Let $q \in C(\Gamma)$ with $q^- \ge 1$, and let $u, u_n \in L^{q(\cdot)}(\Gamma)$. Then we have the following properties.

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{q(\cdot),\Gamma}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} > 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(iii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

(iv) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \to 0 \Longleftrightarrow \rho_{q(\cdot),\Gamma}(u_n) \to 0$.

(v) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \to \infty \Longleftrightarrow \rho_{q(\cdot),\Gamma}(u_n) \to \infty$.

The Hölder inequality also holds for functions on Γ.

Proposition 2.5. Let $q \in C_+(\Gamma)$. Then the following inequality holds.

$$\int_\Gamma |fg| d\sigma \le 2\|f\|_{L^{q(\cdot)}(\Gamma)}\|g\|_{L^{q'(\cdot)}(\Gamma)} \text{ for all } f \in L^{q(\cdot)}(\Gamma), g \in L^{q'(\cdot)}(\Gamma),$$

where $q'(x) = q(x)/(q(x) - 1)$ for $x \in \Gamma$.

Proposition 2.6. Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p \in C_+(\overline{\Omega})$. If $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C > 0$ such that

$$\|f\|_{L^{p(\cdot)}(\Gamma)} \leq C\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}.$$

In particular, If $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma_i)$ and $\|f\|_{L^{p(\cdot)}(\Gamma_i)} \leq C\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}$ for $i = 1, 2$.

For $p \in C_+(\overline{\Omega})$, define

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

The following proposition follows from Yao [27, Proposition 2.6].

Proposition 2.7. Let $p \in C_+(\overline{\Omega})$. Then if $q(x) \in C_+(\Gamma)$ satisfies $q(x) < p^\partial(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is well-defined and compact. In particular, the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Gamma)$ is compact and there exists a constant $C > 0$ such that

$$\|u\|_{L^{p(\cdot)}(\Gamma)} \leq C\|u\|_{W^{1,p(\cdot)}(\Omega)} \text{ for } u \in W^{1,p(\cdot)}(\Omega).$$

2.2 Weighted variable exponent Lebesgue spaces

Now we consider the weighted variable exponent Lebesgue space. Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and let $a(x)$ be a measurable function on Ω with $a(x) > 0$ a.e. $x \in \Omega$. We define a modular

$$\rho_{(p(\cdot),a(\cdot))}(u) = \int_\Omega a(x)|u(x)|^{p(x)}dx \text{ for any measurable function } u \text{ in } \Omega.$$

Then the weighted Lebesgue space is defined by

$$L^{p(\cdot)}_{a(\cdot)}(\Omega) = \left\{u; u \text{ is a measurable function on } \Omega \text{ satisfying } \rho_{(p(\cdot),a(\cdot))}(u) < \infty\right\}$$

equipped with the norm

$$\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \int_\Omega a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Then $L^{p(\cdot)}_{a(\cdot)}(\Omega)$ is a Banach space.

We have the following proposition (cf. Fan [15, Proposition 2.5].

Proposition 2.8. Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$. For $u, u_n \in L^{p(\cdot)}_{a(\cdot)}(\Omega)$, we have the following.

(i) For $u \neq 0$, $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \lambda \Longleftrightarrow \rho_{(p(\cdot),a(\cdot))}\left(\frac{u}{\lambda}\right) = 1$.

(ii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{(p(\cdot),a(\cdot))}(u) < 1 (= 1, > 1)$.

(iii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(iv) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(v) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{(p(\cdot),a(\cdot))}(u_n - u) = 0$.

(vi) $\|u_n\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \to \infty$ as $n \to \infty \Longleftrightarrow \rho_{(p(\cdot),a(\cdot))}(u_n) \to \infty$ as $n \to \infty$.

The author of [15] also derived the following proposition (cf. [15, Theorem 2.1]).

Proposition 2.9. Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and $p \in C_+(\overline{\Omega})$. Moreover, let $a \in L^{\alpha(\cdot)}(\Omega)$ satisfy $a(x) > 0$ a.e. $x \in \Omega$ and $\alpha \in C_+(\overline{\Omega})$. If $q \in C(\overline{\Omega})$ satisfies

$$1 \leq q(x) < \frac{\alpha(x) - 1}{\alpha(x)} p^*(x) \text{ for all } x \in \overline{\Omega}.$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ is compact. Moreover, there exists a constant $c > 0$ such that

$$\int_\Omega a(x)|u(x)|^{q(x)}dx \leq c\|u\|^{q^+}_{W^{1,p(\cdot)}(\Omega)} \vee \|u\|^{q^-}_{W^{1,p(\cdot)}(\Omega)}.$$

Proof. Let $u \in W^{1,p(\cdot)}(\Omega)$. Set $h(x) = \alpha'(x)q(x)$. From the hypothesis, we have $h(x) < p^*(x)$ for all $x \in \overline{\Omega}$. By Proposition 2.3 (iii), the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ is compact. Since $|u(x)|^{q(x)} \in L^{\alpha'(\cdot)}(\Gamma)$, it follows from the Hölder inequality (Proposition 2.5) that

$$\int_\Omega a(x)|u(x)|^{q(x)}dx \leq 2\|a\|_{L^{\alpha(\cdot)}(\Omega)} \||u|^{q(\cdot)}\|_{L^{\alpha'(\cdot)}(\Omega)} < \infty.$$

Hence $W^{1,p(\cdot)}(\Omega) \subset L^{q(\cdot)}_{a(\cdot)}(\Omega)$. We show that the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ is compact. Let $u_n \to 0$ weakly in $W^{1,p(\cdot)}(\Omega)$. Then $u_n \to 0$ strongly in $L^{h(\cdot)}(\Omega)$, so $\||u_n|^{q(\cdot)}\|_{L^{\alpha'(\cdot)}(\Omega)} \to 0$. Hence

$$\int_\Omega a(x)|u_n(x)|^{q(x)}dx \leq 2\|a\|_{L^{\alpha}(\Omega)} \||u_n|^{q(\cdot)}\|_{L^{\alpha'(\cdot)}(\Omega)} \to 0.$$

This implies that $\|u_n\|_{L^{q(\cdot)}_{a(\cdot)}(\Omega)} \to 0$. Therefore, the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ is compact. By the Edmunds and Rákosník [13, Lemma 2.1], if $u \in$

$W^{1,p(\cdot)}(\Omega)$, then

$$\||u|^{q(\cdot)}\|_{L^{\alpha'(\cdot)}(\Omega)} \le \|u\|^{q^+}_{L^{h(\cdot)}(\Omega)} \vee \|u\|^{q^-}_{L^{h(\cdot)}(\Omega)}.$$

Since $\|u\|_{L^{h(\cdot)}(\Omega)} \le C\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for some constant $C > 0$, we obtain the estimate. □

Similarly, let $q \in C(\Gamma)$ with $q^- \ge 1$ and let $b(x)$ be a measurable function with respect to σ on Γ with $b(x) > 0$ σ-a.e. $x \in \Gamma$. We define a modular

$$\rho_{(q(\cdot),b(\cdot)),\Gamma}(u) = \int_\Gamma b(x)|u(x)|^{q(x)}d\sigma.$$

Then the weighted Lebesgue space on Γ is defined by

$$L^{q(\cdot)}_{b(\cdot)}(\Gamma) = \{u; u \text{ is a } \sigma\text{-measurable function on } \Gamma \text{ satisfying } \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < \infty\}$$

equipped with the norm

$$\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma b(x)\left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma \le 1\right\}.$$

Then $L^{q(\cdot)}_{b(\cdot)}(\Gamma)$ is a Banach space.

Then we have the following proposition.

Proposition 2.10. Let $q \in C(\Gamma)$ with $q^- \ge 1$. For $u, u_n \in L^{q(\cdot)}_{b(\cdot)}(\Gamma)$, we have the following.

(i) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} > 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \le \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \le \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \to \infty$ as $n \to \infty \Longleftrightarrow \rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n) \to \infty$ as $n \to \infty$.

The following proposition plays an important role in the present paper.

Proposition 2.11. Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p \in C_+(\overline{\Omega})$. Assume that $0 < b \in L^{\beta(\cdot)}(\Gamma)$, $\beta \in C_+(\Gamma)$. If $r \in C(\Gamma)$ satisfies

$$1 \le r(x) < \frac{\beta(x) - 1}{\beta(x)} p^\partial(x) \text{ for all } x \in \Gamma.$$

Then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ is compact.

2.3 The Nemytskii operators

Now we consider the Nemytskii operators.

Proposition 2.12. Let $q \in C(\overline{\Omega})$ with $q^- \geq 1$ and a be a measurable function with $a(x) > 0$ for a.e. $x \in \Omega$. Assume that

(F.1) A function $F(x,t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$.

(F.2) The growth condition holds: there exist $c \in L^{q_1(\cdot)}(\Omega)$ with $c(x) \geq 0$ a.e. $x \in \Omega$, $q_1 \in C(\overline{\Omega})$ with $q_1^- \geq 1$, and a constant $c_1 > 0$ such that

$$|F(x,t)| \leq c(x) + c_1 a(x)^{1/q_1(x)} |t|^{q(x)/q_1(x)}.$$

Then the Nemytskii operator $N_F : L^{q(\cdot)}_{a(\cdot)}(\Omega) \ni u \mapsto F(x,u(x)) \in L^{q_1(\cdot)}(\Omega)$ is continuous and there exists a constant $C > 0$ such that

$$\rho_{q_1(\cdot)}(N_F(u)) \leq C(\rho_{q_1(\cdot)}(u) + \rho_{(q(\cdot),a(\cdot))}(u)) \text{ for all } u \in L^{q(\cdot)}_{a(\cdot)}(\Omega).$$

In particular, if $q_1(x) \equiv 1$, then $N_F : L^{q(\cdot)}_{a(\cdot)}(\Omega) \to L^1(\Omega)$ is continuous.

Proof. The map $\Omega \ni x \mapsto F(x,u(x))$ is clearly measurable in Ω from (F.1) and the estimate easily follows from (F.2). We show the continuity of N_F. Let $u_n \to u$ in $L^{q(\cdot)}_{a(\cdot)}(\Omega)$. Then $a^{1/q(\cdot)} u_n \to a^{1/q(\cdot)} u$ in $L^{q(\cdot)}(\Omega)$. Then there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $g \in L^{q(\cdot)}(\Omega)$ such that $a(x)^{1/q(x)} u_{n'}(x) \to a(x)^{q(\cdot)} u(x)$ a.e. $x \in \Omega$ and $a(x)^{1/q(x)} |u_{n'}(x)| \leq g(x)$ a.e. $x \in \Omega$ (cf. [3, Proposition A.1]). Since $a(x) > 0$, $u_{n'}(x) \to u(x)$ a.e. $x \in \Omega$. Thus $F(x,u_{n'}(x)) \to F(x,u(x))$ a.e. $x \in \Omega$, so $|F(x,u_n(x)) - F(x,u(x))|^{q_1(x)} \to 0$ a.e. in Ω as $n \to \infty$. On the other hand, it follows from (F.2) that

$$\begin{aligned}
&|F(x,u_{n'}(x)) - F(x,u(x))|^{q_1(x)} \\
&\qquad \leq C_1(c(x)^{q_1(x)} + c_1 a(x)|u_{n'}(x)|^{q(x)} + c_1 a(x)|u(x)|^{q(x)}) \\
&\qquad\qquad \leq C_2(c(x)^{q_1(x)} + c_1 g(x)^{q(x)})
\end{aligned}$$

for some constants $C_1, C_2 > 0$. The last term is an integrable function in Ω independent of n'. Hence by the Lebesgue dominated convergence theorem, $N_F(u_{n'}) \to N_F(u)$ in $L^{q_1(\cdot)}(\Omega)$. By the convergent principle (Zeidler [29, Proposition 10.13], for full sequence $\{u_n\}$, $N_F(u_n) \to N_F(u)$ in $L^{q_1(\cdot)}(\Omega)$. □

Remark 2.13. This proposition is an extension of [3, Proposition 2.12].

Similarly we have the following proposition.

Proposition 2.14. Let $r \in C(\overline{\Gamma_2})$ with $r^- \geq 1$ and b be a σ-measurable function with $b(x) > 0$ for σ-a.e. $x \in \Gamma_2$. Assume that

(G.1) A function $G(x,t)$ is a Carathéodory function on $\Gamma_2 \times \mathbb{R}$.

(G.2) The growth condition holds: there exist $d \in L^{r_1(\cdot)}(\Gamma_2)$ with $d(x) \geq 0$ σ-a.e. $x \in \Gamma_2$, $r_1 \in C(\overline{\Gamma_2})$ with $r_1 \geq 1$, and a constant $d_1 > 0$ such that

$$|G(x,t)| \leq d(x) + d_1 b(x)^{1/r_1(x)} |t|^{r(x)/r_1(x)}.$$

Then the Nemytskii operator $N_G : L^{r(\cdot)}_{b(\cdot)}(\Gamma_2) \ni u \mapsto G(x,u(x)) \in L^{r_1(\cdot)}(\Gamma_2)$ is continuous and there exists a constant $C > 0$ such that

$$\rho_{r_1(\cdot),\Gamma_2}(N_G(u)) \leq C(\rho_{r_1(\cdot),\Gamma_2}(d) + d_1 \rho_{(r(\cdot),b(\cdot)),\Gamma_2}(u) \text{ for all } u \in L^{r(\cdot)}_{b(\cdot)}(\Gamma_2).$$

In particular, if $r_1(x) \equiv 1$, then $N_G : L^{r(\cdot)}(\Gamma_2) \to L^1(\Gamma_2)$ is continuous.

2.4 The Poincaré-type inequality

In this subsection, we state an important proposition, so, that is why we can consider the mixed boundary value problem.

Define a space by

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.1}$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We show the following Poincaré-type inequality (cf. [8]).

Proposition 2.15. Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and let $p \in C_+(\overline{\Omega})$. Then there exists a constant $C = C(\Omega, N, p) > 0$ such that

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \text{ for all } u \in X.$$

In particular, $\|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for $u \in X$.

For the direct proof, see [4, Lemma 2.5].

Thus we can define the norm on X so that

$$\|v\|_X = \|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \text{ for } v \in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Proposition 2.15.

3 Assumptions and a main theorem

In this section, we state the assumptions and main theorems.

Let $p \in C_+(\overline{\Omega})$ be fixed and let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying that for a.e. $x \in \Omega$, the function $A(x,\cdot) : \mathbb{R}^N \ni \boldsymbol{\xi} \mapsto A(x,\boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi} \in \mathbb{R}^N$, the function $A(\cdot,\boldsymbol{\xi}) : \Omega \ni x \mapsto A(x,\boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x,\mathbf{0}) = 0$ and put $\boldsymbol{a}(x,\boldsymbol{\xi}) = \boldsymbol{\nabla}_{\boldsymbol{\xi}} A(x,\boldsymbol{\xi})$. Then $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function. Assume that there exist constants $c, k_0, k_1, \gamma_0 > 0$ and nonnegative functions $h_0 \in L^{p'(\cdot)}(\Omega)$ and $h_1 \in L^1_{\mathrm{loc}}(\Omega)$ with $h_1(x) \geq 1$ a.e. $x \in \Omega$ such that the following conditions hold.

(A.1) $|\boldsymbol{a}(x, \boldsymbol{\xi})| \leq c(h_0(x) + h_1(x)|\boldsymbol{\xi}|^{p(x)-1})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A.2) A is $p(\cdot)$-uniformly convex, that is,

$$A\left(x, \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right) + k_1 h_1(x)|\boldsymbol{\xi} - \boldsymbol{\eta}|^{p(x)} \leq \frac{1}{2}A(x, \boldsymbol{\xi}) + \frac{1}{2}A(x, \boldsymbol{\eta}) \quad \text{for all } \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N \text{ and a.e. } x \in \Omega.$$

(A.3) $k_0 h_1(x)|\boldsymbol{\xi}|^{p(x)} \leq \boldsymbol{a}(x, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \leq p(x)A(x, \boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A.4) $(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) > 0$ for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$ with $\boldsymbol{\xi} \neq \boldsymbol{\eta}$ and a.e. $x \in \Omega$.

(A.5) $A(x, -\boldsymbol{\xi}) = A(x, \boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Remark 3.1. (i) The condition (A.1) is more general than that of Mashiyev et al [23] who considered the case $h_1(x) \equiv 1$. In our case, to overcome this we have to consider the space Y defined by (3.1) later as a basic space rather than the space X defined by (2.2).

(ii) (A.3) implies that A is $p(\cdot)$-sub-homogeneous, that is,

$$A(x, s\boldsymbol{\xi}) \leq A(x, \boldsymbol{\xi})s^{p(x)} \text{ for any } \boldsymbol{\xi} \in \mathbb{R}^N, \text{ a.e. } x \in \Omega \text{ and } s > 1.$$

For the proof, see Aramaki [2, (4.14)].

Example 3.2. (i) $A(x, \boldsymbol{\xi}) = \frac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)}$ with $p^- \geq 2$, $h \in L^1_{\mathrm{loc}}(\Omega)$ satisfying $h(x) \geq 1$ a.e. $x \in \Omega$. Then $\boldsymbol{a}(x, \boldsymbol{\xi}) = \frac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)-2}\boldsymbol{\xi}$.

(ii) $A(x, \boldsymbol{\xi}) = \frac{h(x)}{p(x)}((1 + |\boldsymbol{\xi}|^2)^{p(x)/2} - 1)$ with $p^- \geq 2$, $h \in L^{p'(\cdot)}(\Omega)$ satisfying $h(x) \geq 1$ a.e. $x \in \Omega$. Then $\boldsymbol{a}(x, \boldsymbol{\xi}) = h(x)(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}\boldsymbol{\xi}$.

Then $A(x, \boldsymbol{\xi})$ and $\boldsymbol{a}(x, \boldsymbol{\xi})$ of (i), (ii) satisfy (A1)-(A5).

Remark 3.3. (i) When $h(x) \equiv 1$, (i) corresponds to the $p(\cdot)$-Laplacian and (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

(ii) The condition (A.1) is more general than that of [23] who considered the case $h_1(x) \equiv 1$.

For the function $h_1 \in L^1_{\mathrm{loc}}(\Omega)$ with $h_1(x) \geq 1$ a.e. $x \in \Omega$, we define a modular

$$\rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} v) = \int_\Omega h_1(x)|\boldsymbol{\nabla} v(x)|^{p(x)}dx \text{ for } v \in X,$$

where the space X is defined by (2.1). Define our basic space

$$Y = \{v \in X; \rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} v) < \infty\} \tag{3.1}$$

equipped with the norm

$$\|v\|_Y = \inf\left\{\lambda > 0; \rho_{(p(\cdot),h_1(\cdot))}\left(\frac{\boldsymbol{\nabla} v}{\lambda}\right) \le 1\right\}.$$

Proposition 3.4. The space $(Y, \|\cdot\|_Y)$ is a separable and reflexive Banach space.

Proof. The author of [2, Lemma 2.12] showed that the space Y is a reflexive Banach space. We show the separability of Y. We note that $u \in Y$ if and only if $h_1^{1/p(\cdot)}\boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(\Omega)$, and $\|u\|_Y = \|h_1^{1/p(\cdot)}\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$. Thus the operator $T : Y \ni u \mapsto h_1^{1/p(\cdot)}\boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(\Omega)$ is linear and isometric. Since $\boldsymbol{L}^{p(\cdot)}(\Omega)$ is separable, TY is also separable (cf. Brezis [6, Proposition III.2.2]), so Y is separable. $\square$

We note that $C_0^\infty(\Omega) \subset Y$. Since $h_1(x) \ge 1$ a.e. $x \in \Omega$, it follows that

$$\rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} v) = \rho_{p(\cdot)}(h_1^{1/p(\cdot)}\boldsymbol{\nabla} v) \ge \rho_{p(\cdot)}(\boldsymbol{\nabla} v) \text{ for } v \in Y$$

and

$$\|v\|_Y = \|h_1^{1/p(\cdot)}\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \ge \|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} = \|v\|_X \text{ for } v \in Y. \tag{3.2}$$

From (3.2) and Proposition 2.1, we have the following proposition.

Proposition 3.5. Let $p \in C_+(\overline{\Omega})$ and let $u, u_n \in Y$ $(n = 1, 2, \ldots)$. Then the following properties hold.

(i) The embedding $Y \hookrightarrow X$ is continuous and $\|u\|_X \le \|u\|_Y$.
(ii) $\|u\|_Y > 1 (= 1, < 1) \Longleftrightarrow \rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} u) > 1 (= 1, < 1)$.
(iii) $\|u\|_Y > 1 \Longrightarrow \|u\|_Y^{p^-} \le \rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} u) \le \|u\|_Y^{p^+}$.
(iv) $\|u\|_Y < 1 \Longrightarrow \|u\|_Y^{p^+} \le \rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} u) \le \|u\|_Y^{p^-}$.
(v) $\lim_{n\to\infty} \|u_n - u\|_Y = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} u_n - \boldsymbol{\nabla} u) = 0$.
(vi) $\|u_n\|_Y \to \infty$ as $n \to \infty \Longleftrightarrow \rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} u_n) \to \infty$ as $n \to \infty$.

For $i = 0, 1$, we assume the following $(\mathrm{f_i})$ and $(\mathrm{g_i})$.

$(\mathrm{f_i})$ A function $f_i(x, t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ and there exist $1 \le a_i \in L^{\alpha_i(\cdot)}(\Omega)$ with $\alpha_i \in C_+(\overline{\Omega})$ and $q_i \in C(\overline{\Omega})$ such that

$$1 \le q_i(x) < \frac{\alpha_i(x) - 1}{\alpha_i(x)} p^*(x) \text{ for all } x \in \overline{\Omega}$$

satisfying

$$|f_i(x, t)| \le c_i(1 + a_i(x)|t|^{q_i(x)-1}) \text{ for a.e.} x \in \Omega \text{ and } t \in \mathbb{R}$$

for some constant $c_i > 0$.

($\mathrm{g_i}$) A function $g_i(x,t)$ is a Carathéodory function on $\Gamma_2 \times \mathbb{R}$ and there exist $1 \leq b_i \in L^{\beta_i(\cdot)}(\Gamma_2)$ with $\beta_i \in C_+(\overline{\Gamma_2})$ and $r_i \in C(\overline{\Gamma_2})$ such that

$$1 \leq r_i(x) < \frac{\beta_i(x)-1}{\beta_i(x)} p^{\partial}(x) \text{ for all } x \in \overline{\Gamma_2}$$

satisfying

$$|g_i(x,t)| \leq d_i(1 + b_i(x)|t|^{r_i(x)-1}) \text{ for a.e.} x \in \Gamma_2 \text{ and } t \in \mathbb{R}$$

for some constant $d_i > 0$.

We introduce the notion of a weak solution for the problem (1.1).

Definition 3.6. We say $u \in Y$ is a weak solution of (1.1), if

$$\begin{aligned} M(\Phi(u)) &\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x) dx \\ &= \lambda \left(\int_\Omega f_0(x, u(x)) v(x) dx + \int_{\Gamma_2} g_0(x, u(x)) v(x) d\sigma \right) \\ + \mu &\left(\int_\Omega f_1(x, u(x)) v(x) dx + \int_{\Gamma_2} g_1(x, u(x)) v(x) d\sigma \right) \text{ for all } v \in Y. \quad (3.3) \end{aligned}$$

For the Carathéodory functions f_i, g_i and the function M in (M), define

$$\widehat{M}(t) = \int_0^t M(s)ds, F_i(x,t) = \int_0^t f_i(x,s)ds \text{ and } G_i(x,t) = \int_0^t g_i(x,s)ds. \quad (3.4)$$

We obtain the following main theorem.

Theorem 3.7. Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ satisfying (1.2) and let $p \in C_+(\overline{\Omega})$. Assume that (A.1)-(A.5) hold and functions f_0 and g_0 satisfy ($\mathrm{f_0}$) and ($\mathrm{g_0}$), respectively. Suppose that

$$lp^+ < \min\left\{ \frac{\alpha_0^- - 1}{\alpha_0^-} \frac{Np^-}{N-p^-}, \frac{\beta_0^- - 1}{\beta_0^-} \frac{(N-1)p^-}{N-p^-} \right\} \text{ if } p^- < N. \quad (3.5)$$

Moreover, suppose that

$$\max\left\{ \limsup_{t \to 0} ess\ sup_{x \in \Omega} \frac{F_0(x,t)}{a_0(x)|t|^{lp^+}}, \limsup_{t \to 0} ess\ sup_{x \in \Gamma_2} \frac{G_0(x,t)}{b_0(x)|t|^{lp^+}} \right\} \leq 0, \quad (3.6)$$

$$\max\left\{ \limsup_{|t| \to \infty} ess\ sup_{x \in \Omega} \frac{F_0(x,t)}{a_0(x)|t|^{lp^-}}, \limsup_{|t| \to \infty} ess\ sup_{x \in \Gamma_2} \frac{G_0(x,t)}{b_0(x)|t|^{lp^-}} \right\} \leq 0, \quad (3.7)$$

and there exists $\delta_1 > 0$ such that

$$F_0(x,t) > 0 \text{ for a.e. } x \in \Omega \text{ and } 0 < t \le \delta_1. \tag{3.8}$$

Set

$$\theta = \inf\Bigg\{ \frac{\widehat{M}(\Phi(u))\Phi(u)}{\int_\Omega F_0(x,u(x))dx + \int_{\Gamma_2} G_0(x,u(x))d\sigma}; u \in Y \text{ with} \\ \int_\Omega F_0(x,u(x))dx + \int_{\Gamma_2} G_0(x,u(x))d\sigma > 0 \Bigg\}. \tag{3.9}$$

Then for each compact interval $[a,b] \subset (\theta,\infty)$, there exists $r > 0$ with the following properties: for every $\lambda \in [a,b]$ and any functions f_1 and g_1 satisfying (f_1) and (g_1), respectively, there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$, problem (1.1) has at least three weak solutions whose norms are less that r.

Remark 3.8. If we choose $\varphi \in C_0^\infty(\Omega)(\subset Y)$ such that $0 \le \varphi(x) \le \delta_1$ and $\varphi \not\equiv 0$, then from (3.8)

$$\int_\Omega F_0(x,\varphi(x))dx + \int_{\Gamma_2} G_0(x,\varphi(x))d\sigma = \int_\Omega F_0(x,\varphi(x))dx > 0.$$

So θ is well-defined and $\theta \ge 0$.

4 Proof of Theorem 3.7

In order to prove Theorem 3.7, we apply the following Ricceri theorem of [25, Theorem 2].

Theorem 4.1. Let $(B, \|\cdot\|_B)$ be a separable, reflexive and real Banach space. Assume that a functional $\Psi : B \to \mathbb{R}$ is coercive, that is, $\Psi(u) \to \infty$ as $\|u\|_B \to \infty$, sequentially weakly lower semi-continuous, of C^1-functional belonging to $\mathcal{W}_B$, that is, if $u_n \to u$ weakly in B and $\liminf_{n\to\infty} \Psi(u_n) \le \Psi(u)$, then the sequence $\{u_n\}$ has a subsequence converging to u strongly in B, bounded on every bounded subset of B and the derivative $\Psi' : B \to B^*$ admits a continuous inverse $(\Psi')^{-1} : B^* \to B$. Moreover, assume that $J : B \to \mathbb{R}$ is a C^1-functional with compact derivative, and assume that Φ has a strictly local minimum $u_0 \in B$ with $\Psi(u_0) = J(u_0) = 0$. Finally, put

$$\alpha = \max\left\{0, \limsup_{\|u\|_B\to\infty} \frac{J(u)}{\Psi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Psi(u)}\right\}, \tag{4.1}$$

$$\beta = \sup_{u\in\Psi^{-1}((0,\infty))} \frac{J(u)}{\Psi(u)}, \tag{4.2}$$

and assume that $\alpha < \beta$. Then for each compact interval $[a, b] \subset (1/\beta, 1/\alpha)$ (with the conventions $1/0 = \infty, 1/\infty = 0$), there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1-functional $K : B \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation $\Psi'(u) = \lambda J'(u) + \mu K'(u)$ has at least three solutions in B whose norms are less that r.

Proof of Theorem 3.7

We apply Theorem 4.1 with $(B, \|\cdot\|_B) = (Y, \|\cdot\|_Y)$. The proof of Theorem 3.7 consists of some propositions and lemmas. First we note that Y is a separable, reflexive and real Banach space by Proposition 3.4.

Define functionals on Y by

$$\Psi(u) = \widehat{M}(\Phi(u)), \tag{4.3}$$

where $\Phi(u)$ is defined by (1.3) and $\widehat{M}(t) = \int_0^t M(s)ds$,

$$J(u) = \int_\Omega F_0(x, u(x))dx + \int_{\Gamma_2} G_0(x, u(x))d\sigma, \tag{4.4}$$

and

$$K(u) = \int_\Omega F_1(x, u(x))dx + \int_{\Gamma_2} G_1(x, u(x))d\sigma \tag{4.5}$$

for $u \in Y$.

It easily follows from (M) that

$$\frac{m_0}{l} t^l \le \widehat{M}(t) \le \frac{m_1}{l} t^l \text{ for all } t \ge 0,$$

and $\widehat{M}(t)$ is of C^1-class and a convex and strictly monotone increasing function on $[0, \infty)$.

Lemma 4.2. (i) We have

$$\frac{k_0}{p^+}\rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} u) \le \Phi(u) \le c(2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_Y + \rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} u)) \text{ for } u \in Y,$$

where c and k_0 are the constants of (A.1) and (A.3).

(ii) We have $\Phi\left(\frac{u+v}{2}\right) + k_1\rho_{(p(\cdot),h_1(\cdot))}(\boldsymbol{\nabla} u - \boldsymbol{\nabla} v) \le \frac{1}{2}\Phi(u) + \frac{1}{2}\Phi(v)$ for all $u, v \in Y$, in particular, $\Phi((1-\tau)u + \tau v) \le (1-\tau)\Phi(u) + \tau\Phi(v)$ for all $u, v \in Y$ and $\tau \in [0, 1]$.

(iii) $|F_i(x, t)| \le c_i(|t| + a_0(x)|t|^{q_i(x)})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$, where c_i is the constant of (f_i).

(iv) $|G_i(x, t)| \le d_i(|t| + b_i(x)|t|^{r_i(x)})$ for all $t \in \mathbb{R}$ and σ-a.e. $x \in \Gamma_2$, where d_i is the constant of (g_i).

Proof. (i) Since $A(x,\mathbf{0})=0$, it follows from (A.1) and (A.3) that

$$\begin{aligned}\frac{k_0}{p^+}h_1(x)|\boldsymbol{\xi}|^{p(x)}\le A(x,\boldsymbol{\xi}) &= |A(x,\boldsymbol{\xi})-A(x,\mathbf{0})|\\ &= \left|\int_0^1\frac{d}{d\tau}A(x,\tau\boldsymbol{\xi})d\tau\right|\\ &= \left|\int_0^1\boldsymbol{a}(x,\tau\boldsymbol{\xi})\cdot\boldsymbol{\xi}d\tau\right|\\ &\le c\int_0^1(h_0(x)|\boldsymbol{\xi}|+h_1(x)|\tau\boldsymbol{\xi}|^{p(x)-1}|\boldsymbol{\xi}|)d\tau\\ &\le c(h_0(x)|\boldsymbol{\xi}|+h_1(x)|\boldsymbol{\xi}|^{p(x)}).\end{aligned}$$

Hence

$$\frac{k_0}{p^+}\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}dx\le\Phi(u)\le c\left(\int_\Omega(h_0(x)|\nabla u(x)|+h_1(x)|\nabla u(x)|^{p(x)})dx\right)$$

for any $u\in Y$. By the Hölder inequality (Proposition 2.2) and Proposition 3.5 (i),

$$\begin{aligned}\int_\Omega(h_0(x)|\nabla u(x)|dx &\le 2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|\nabla u\|_{L^{p(\cdot)}(\Omega)}\\ &\le 2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_X\le 2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_Y.\end{aligned}$$

(ii), (iii) and (iv) easily follows from (A.2), (f_0) and (g_0). □

The functional Ψ defined by (4.3) is a continuous modular on a real Banach space Y in the sense of [12, Definition 2.1.11], that is, Ψ has the following properties (a)-(e).

(a) $\Psi(0)=0$. This easily follows from $A(x,\mathbf{0})=0$ and the definition of $\widehat{M}$.

(b) $\Psi(-u)=\Psi(u)$ for every $u\in Y$. This follows from (A.5).

(c) Ψ is convex. Indeed, since $\widehat{M}$ is convex and strictly monotone increasing, and Φ is convex, for any $u,v\in Y$ and $\tau\in[0,1]$,

$$\begin{aligned}\Psi((1-\tau)u+\tau v) &= \widehat{M}(\Phi((1-\tau)u+\tau v))\\ &\le\widehat{M}((1-\tau)\Phi(u)+\tau\Phi(v))\le(1-\tau)\Psi(u)+\tau\Psi(v).\end{aligned}$$

(d) The function $[0,\infty)\ni\lambda\mapsto\Psi(\lambda u)$ is continuous for every $u\in Y$. Indeed, let $[0,\infty)\ni\lambda_n\to\lambda_0$ as $n\to\infty$. Here we can assume that $0\le\lambda_n\le\lambda_0+1$ for all $n\in\mathbb{N}$. From Lemma 4.2 (i), we have

$$|A(x,\lambda_n\nabla u(x))|\le c(\lambda_0+1)h_0(x)|\nabla u(x)|+c(\lambda_0+1)^{p^+}h_1(x)|\nabla u(x)|^{p(x)}.$$

Since $h_0 \in L^{p'(\cdot)}(\Omega)$ and $|\boldsymbol{\nabla} u(\cdot)| \in L^{p(\cdot)}(\Omega)$ and $u \in Y$, the right-hand side in the above inequality is an integrable function independent of n. Clearly, we see that $A(x, \lambda_n \boldsymbol{\nabla} u(x)) \to A(x, \lambda_0 \boldsymbol{\nabla} u(x))$ as $n \to \infty$ for a.e. $x \in \Omega$. By the Lebesgue dominated convergent theorem, we see that $\Phi(\lambda_n u) \to \Phi(\lambda_0 u)$ as $n \to \infty$, so $\Psi(\lambda_n u) \to \Psi(\lambda_0 u)$.

(e) $\Psi(u) = 0$ implies $u = 0$. Indeed, if $\Psi(u) = 0$, then $\Phi(u) = 0$. Hence it follows from (A.3) and the Poincaré-type inequality (Proposition 2.15) that $u = 0$. .

Thus we can define a modular space

$$Y_\Psi = \{u \in Y; \lim_{\tau \to 0} \Psi(\tau u) = 0\} = \{u \in Y; \Psi(\tau u) < \infty \text{ for some } \tau > 0\}$$

and the Luxemburg norm

$$\|u\|_\Psi = \inf \left\{ \tau > 0; \Psi\left(\frac{u}{\tau}\right) \le 1 \right\} \text{ for } u \in Y_\Phi.$$

Then $(Y_\Psi, \|\cdot\|_\Psi)$ is a normed linear space over $\mathbb{R}$ from [12, Theorem 2.1.7]. Clearly we see that $Y_\Psi = Y$.

Lemma 4.3. There exist positive constants c_3 and C_3 such that $c_3\|u\|_Y \le \|u\|_\Psi \le C_3\|u\|_Y$ for all $u \in Y$.

Proof. By Lemma 4.2 (i), for $u \in Y$,

$$\begin{aligned}\Phi(u) \le c \int_\Omega (h_0(x)|\boldsymbol{\nabla} u(x)| + h_1(x)|\boldsymbol{\nabla} u(x)|^{p(x)}) dx \\ \le c(\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_Y + \|u\|_Y^{p^+} \vee \|u\|_Y^{p^-}).\end{aligned}$$

Hence

$$\Psi(u) \le \frac{m_1}{l}\left(c(\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_Y + \|u\|_Y^{p^+} \vee \|u\|_Y^{p^-})\right)^l. \tag{4.6}$$

Here and from now on, we denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for any real numbers a and b.

On the other hand, we have

$$\Phi(u) \ge \frac{k_0}{p^+} \int_\Omega h_1(x)|\boldsymbol{\nabla} u(x)|^{p(x)} dx \ge \frac{k_0}{p^+}(\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-}),$$

so

$$\Psi(u) = \widehat{M}(\Phi(u)) \ge \frac{m_0}{l}\left(\frac{k_0}{p^+}(\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-})\right)^l, \tag{4.7}$$

Hence $u_n \to 0$ in Y if and only if $\Psi(u_n) \to 0$, so if and only if $\|u_n\|_\Psi \to 0$ from [12, Lemma 2.1.9]. □

Lemma 4.4. If $u_n \to u$ weakly in Y and $\Psi(u_n) \to \Psi(u)$ as $n \to \infty$, then we have $\Psi\left(\frac{u_n - u}{2}\right) \to 0$ as $n \to \infty$. In particular, $u_n \to u$ strongly in Y as $n \to \infty$.

Proof. Let $u_n \to u$ weakly in Y and $\Psi(u_n) \to \Psi(u)$ as $n \to \infty$. Then if we use [12, Lemma 2.4.17] (cf. Aramaki [1, Lemma 20]), then we can show that $\Psi\left(\frac{u_n - u}{2}\right) \to 0$ as $n \to \infty$, so $u_n \to u$ strongly in Y using (4.7). □ We check that the assumptions of Theorem 4.1 hold.

• Ψ is coercive, that is, $\Psi(u) \to \infty$ if $\|u\|_Y \to \infty$. This easily follows from (4.7).

• Ψ is sequentially weakly lower semi-continuous. This follows from [2, Proposition 4.4 (iii)] and the fact that $\widehat{M}$ is monotone increasing and continuous.

• $\Psi \in C^1(Y, \mathbb{R})$. This follows from [2, Proposition 4.1] and $\widehat{M} \in C^1([0,\infty))$.

• $\Psi \in \mathcal{W}_Y$. Indeed, let $u_n \to u$ weakly in Y and $\liminf_{n\to\infty} \Psi(u_n) \le \Psi(u)$. Since Ψ is sequentially weakly lower semi-continuous, $\Psi(u) \le \liminf_{n\to\infty} \Psi(u_n)$, so $\liminf_{n\to\infty} \Psi(u_n) = \Psi(u)$. Hence there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\lim_{n'\to\infty} \Psi(u_{n'}) = \Psi(u)$. By Lemma 4.4, $u_{n'} \to u$ strongly in Y.

• Ψ is bounded on every bounded subset of Y. This easily follows from (4.6).

• $\Psi' : Y \to Y^*$ admits a continuous inverse $(\Psi')^{-1} : Y^* \to Y$. This follows from the following proposition.

Proposition 4.5. (i) Ψ' is strictly monotone in Y, that is,

$$\langle \Psi'(u) - \Psi'(v), u - v\rangle_{Y^*,Y} > 0 \text{ for all } u, v \in Y \text{ with } u \neq v.$$

Moreover, Ψ' is bounded on every bounded subset of Y and coercive in the sense that

$$\lim_{\|u\|_Y \to \infty} \frac{\langle \Psi'(u), u\rangle_{Y^*,Y}}{\|u\|_Y} = \infty.$$

(ii) Ψ' is of (S_+)-type, that is, if $u_n \to u$ weakly in Y and

$$\limsup_{n\to\infty} \langle \Psi'(u_n), u_n - u\rangle_{Y^*.Y} \le 0,$$

then $u_n \to u$ strongly in Y.

(iii) The mapping $\Psi' : Y \to Y^*$ is a homeomorphism.

Proof. (i) In general, when a functional $f : Y \to \mathbb{R}$ is of C^1-class, f is strictly convex if and only if $f' : Y \to Y^*$ is strictly monotone (cf. [29, Proposition 25.10]), that is,

$$\langle f'(u) - f'(v), u - v\rangle_{Y^*,Y} > 0 \text{ for all } u, v \in Y \text{ with } u \neq v.$$

From (A.4),

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle_{Y^*,Y} = \int_\Omega (\boldsymbol{a}(x, \boldsymbol{\nabla} u(x) - \boldsymbol{a}(x, \boldsymbol{\nabla} v(x))) \cdot (\boldsymbol{\nabla} u(x) - \boldsymbol{\nabla} v(x))dx > 0$$

for all $u, v \in Y$ with $u \neq v$, so Φ' is strictly monotone in Y, so Φ is strictly convex. The function $\widehat{M}$ is strictly monotone increasing and convex. Hence for $u, v \in Y$ with $u \neq v$ and $\tau \in (0,1)$, since $\Phi((1-\tau)u+\tau v) < (1-\tau)\Phi(u)+\tau\Phi(v)$, we have

$$\widehat{M}(\Phi(1-\tau)u+\tau v)) < \widehat{M}((1-\tau)\Phi(u)+\tau\Phi(v)) \leq (1-\tau)\widehat{M}(\Phi(u))+\tau\widehat{M}(\Phi(v)),$$

so $\Psi((1-\tau)u + \tau v) < (1-\tau)\Psi(u) + \tau\Psi(v)$. Thus Ψ is strictly convex, so $\Psi'(\cdot) = M(\Phi(\cdot))\Phi'(\cdot)$ is strictly monotone in Y.

Since it follows from the Hölder inequality (Proposition 2.2) and Proposition 3.5 (i) that

$$\begin{aligned}
&|\langle \Psi'(u), v\rangle_{Y^*,Y}| \\
&\quad = M(\Phi(u)) \left| \int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u(x)) \cdot \boldsymbol{\nabla} v(x) dx \right| \\
&\quad \leq cM(\Phi(u)) \int_\Omega (h_0(x)|\boldsymbol{\nabla} v(x)| + h_1(x)|\boldsymbol{\nabla} u(x)|^{p(x)-1}|\boldsymbol{\nabla} v(x)|)dx \\
&\quad = cM(\Phi(u)) \int_\Omega (h_0(x)|\boldsymbol{\nabla} v(x)| \\
&\qquad + h_1(x)^{1/p'(x)}|\boldsymbol{\nabla} u(x)|^{p(x)-1} h_1(x)^{1/p(x)}|\boldsymbol{\nabla} v(x)|)dx \\
&\quad \leq 2cm_1\Phi(u)^{l-1}(\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|v\|_Y \\
&\qquad + \|h_1^{1/p'(\cdot)}|\boldsymbol{\nabla} u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)}\|h_1^{1/p(\cdot)}|\boldsymbol{\nabla} v|\|_{L^{p(\cdot)}(\Omega)} \\
&\quad = 2cm_1\Phi(u)^{l-1}(\|h_0\|_{L^{p'(\cdot)}(\Omega)} + \|h_1^{1/p'(\cdot)}|\boldsymbol{\nabla} u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)})\|v\|_Y
\end{aligned}$$

for all $v \in Y$. Hence we have

$$\|\Psi'(u)\|_{Y^*} \leq 2cm_1\Phi(u)^{l-1}(\|h_0\|_{L^{p'(\cdot)}(\Omega)} + \|h_1^{1/p'(\cdot)}|\boldsymbol{\nabla} u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)}).$$

Here we note that

$$\Phi(u)^{l-1} \leq c^{l-1}(2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_Y + \|u\|_Y^{p^+} \vee \|u\|_Y^{p^-})^{l-1}$$

and

$$\rho_{p'(\cdot)}(h_1^{1/p'(\cdot)}|\boldsymbol{\nabla} u|^{p(\cdot)-1}) = \int_\Omega h_1(x)|\boldsymbol{\nabla} u(x)|^{p(x)}dx \leq \|u\|_Y^{p^+} \vee \|u\|_Y^{p-}.$$

If $\|u\| \leq M$, then it is clear that there exists a constant $C(M) > 0$ such that $\|\Psi'(u)\|_{Y^*} \leq C(M)$, so Ψ' is bounded on every bounded subset of Y.

Let $\|u\|_Y > 1$. Then from (M), (A.3) Lemma 4.2 (i),

$$\begin{aligned}\langle \Psi'(u), u\rangle_{Y^*,Y} &= M(\Phi(u))\int_\Omega \boldsymbol{a}(x, \boldsymbol{\nabla} u(x) \cdot \boldsymbol{\nabla} u(x) dx \\ &\geq k_0 M(\Phi(u))\int_\Omega h_1(x)|\boldsymbol{\nabla} u(x)|^{p(x)} dx \\ &\geq \frac{k_0}{p^+} m_0 \|u\|_Y^{(l-1)p^-} \|u\|_Y^{p^-} \\ &= \frac{m_0 k_0}{p^+}\|u\|_Y^{lp^-}.\end{aligned}$$

Since $lp^- > 1$, this implies the coervivity of Ψ'.

(ii) Let $u_n \to u$ weakly in Y and $\limsup_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle_{Y^*,Y} \leq 0$. Since Ψ' is monotone from (i), $\langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle_{Y^*,Y} \geq 0$. Hence

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty}\langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle_{Y^*,Y} \\ &= \liminf_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle_{Y^*,Y} \\ &\leq \limsup_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle_{Y^*,Y} \leq 0.\end{aligned}$$

Therefore, we have $\lim_{n\to\infty} M(\Phi(u_n))\langle \Phi'(u_n), u_n - u\rangle_{Y^*,Y} = 0$. Since $u_n \to u$ weakly in Y, the sequence $\{\|u_n\|_Y\}$ is bounded. Hence

$$\lim_{n\to\infty} M(\Phi(u_n))\langle \Phi'(u), u_n - u\rangle_{Y^*,Y} = 0.$$

Therefore, we have

$$\lim_{n\to\infty} M(\Phi(u_n))\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle_{Y^*,Y} = 0.$$

Thereby, since $M(\Phi(u_n)) \geq 0$ and $\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle_{Y^*,Y} \geq 0$, we obtain that $\lim_{n\to\infty} M(\Phi(u_n)) = 0$ or $\lim_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle_{Y^*,Y} = 0$. When $M(\Phi(u_n)) \to 0$ as $n \to \infty$, we have $\Phi(u_n) \to 0 = \Phi(0)$. By Lemma 4.4 with $M \equiv 1$, $u_n \to 0$ strongly in Y (in this case we necessarily have $u = 0$). When

$$\lim_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle_{Y^*,Y} = \lim_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle_{Y^*,Y} = 0,$$

since Φ' is of (S_+)-type (cf. [1, Proposition 21 (ii)]), we have $u_n \to u$ strongly in Y.

(iii) Since Ψ' is strictly monotone from (i), Ψ' is injective. We show that $\Psi' : Y \to Y^*$ is surjective. Let $w \in Y^*$. Define a functional on Y by

$$\varphi(u) = \Psi(u) - \langle w, u\rangle_{Y^*,Y} \text{ for } u \in Y.$$

From (M) and Lemma 4.2 (i), for $\|u\|_Y > 1$, we see that

$$\varphi(u) \geq \widehat{M}(\Phi(u)) - \langle w, u\rangle_{Y^*,Y} \geq \left(\frac{k_0}{p^+}\right)^l \|u\|_Y^{lp^-} - \|w\|_{Y^*}\|u\|_Y.$$

Since $lp^- > 1$, φ is coercive. Since Ψ is sequentially weakly lower semicontinuous, φ is so. If we put $\gamma = \inf_{u\in Y}\varphi(u)(< \infty)$, then there exists a sequence $\{u_n\} \subset Y$ such that $\gamma = \lim_{n\to\infty}\varphi(u_n)$. Since φ is coercive, the sequence $\{u_n\}$ is bounded. Since Y is a reflexive Banach space, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u_0 \in Y$ such that $u_n \to u_0$ weakly in Y, so $\varphi(u_0) \leq \liminf_{n'\to\infty}\varphi(u_{n'}) = \gamma$. This implies that $\gamma > -\infty$ and u_0 is a minimizer of φ, so $\varphi'(u_0) = 0$, i.e., $\Psi'(u_0) = w$. Therefore, Ψ' has an inverse operator $(\Psi')^{-1} : Y^* \to Y$. We show that $(\Psi')^{-1}$ is continuous. Let $f_n \to f$ in Y^* as $n \to \infty$. Then there exist $u_n, u \in Y$ such that $\Psi'(u_n) = f_n$ and $\Psi'(u) = f$. Then $\{u_n\}$ is bounded in Y. Indeed, if $\{u_n\}$ is unbounded, then there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\|u_{n'}\|_Y \to \infty$ as $n' \to \infty$. Hence

$$\langle\Psi'(u_{n'}), u_{n'}\rangle_{Y^*,Y} = \langle f_{n'}, u_{n'}\rangle_{Y^*,Y} \leq \|f_{n'}\|_{Y^*}\|u_{n'}\|_Y \leq C\|u_{n'}\|_Y$$

for some constant $C > 0$. This contradict the coerciveness of Ψ'.

Since Y is a reflexive Banach space, there exist a subsequence (still denoted by $\{u_{n'}\}$) and $u_0 \in Y$ such that $u_{n'} \to u_0$ weakly in Y. Hence

$$\begin{aligned}\lim_{n'\to\infty}\langle\Psi'(u_{n'}), u_{n'} - u_0\rangle_{Y^*,Y} &= \lim_{n'\to\infty}\langle\Psi'(u_{n'}) - \Psi'(u), u_{n'} - u_0\rangle_{Y^*,Y}\\ &= \lim_{n'\to\infty}\langle f_{n'} - f, u_{n'} - u_0\rangle_{Y^*,Y} = 0.\end{aligned}$$

Since Ψ' is of (S_+)-type, we see that $u_{n'} \to u_0$ strongly in Y. According to the continuity of Ψ', $\Psi'(u_{n'}) = f_{n'} \to f = \Psi'(u_0) = \Psi'(u)$, so we have $u_0 = u$ from the injectiveness of Ψ'. By the convergent principle (cf. [29, Theorem 10.13 (i)]), for full sequence $\{u_n\}$, $u_n \to u$ strongly in Y, that is, $(\Psi')^{-1}(f_n) \to (\Psi')^{-1}(f)$ as $n \to \infty$. □

• $J \in C^1(Y, \mathbb{R})$ and J has a compact derivative $J' : Y \to Y^*$. We prove this. Temporarily, we put $J(u) = J_{F_0}(u) + J_{G_0}(u)$ for $u \in Y$, where

$$J_{F_0}(u) = \int_\Omega F_0(x, u(x))dx \text{ and } J_{G_0}(u) = \int_{\Gamma_2} G_0(x, u(x))d\sigma.$$

By [2, Proposition 3.8], $J_{F_0}, J_{K_0} \in C^1(Y, \mathbb{R})$ and sequentially weakly continuous in Y. We show that $J'_{F_0} : Y \to Y^*$ is weakly-strongly continuous, that is, if $u_n \to u$ weakly in Y, then $J'_{F_0}(u_n) \to J'_{F_0}(u)$ strongly in Y^*. Let $u_n \to u$ weakly in Y. Then

$$\langle J'_{F_0}(u_n) - J_{F_0}(u), v\rangle_{Y^*,Y} = \int_\Omega (f_0(x, u_n(x)) - f_0(x, u(x)))v(x)dx \text{ for } v \in Y.$$

From Proposition 2.9 and (f_0), the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q_0(\cdot)}_{a_0(\cdot)}(\Omega)$ is compact. Since $Y \hookrightarrow X \hookrightarrow W^{1,p(\cdot)}(\Omega)$, there exists a constant $C > 0$ such that

$$\|v\|_{L^{q_0(\cdot)}_{a_0(\cdot)}(\Omega)} \le C\|v\|_Y \text{ for all } v \in Y.$$

By the Hölder inequality (Proposition 2.2), for any $v \in Y$,

$$\begin{aligned}&|\langle J'_{F_0}(u_n) - J'_{F_0}(u), v\rangle_{Y^*,Y}|\\ &\quad \le \int_\Omega a_0(x)^{-1/q_0(x)}|f_0(x,u_n(x)) - f_0(x,u(x))|a_0(x)^{1/q_0(x)}|v(x)|dx\\ &\quad \le 2\|a_0^{-1/q_0(\cdot)}|f_0(\cdot,u_n(\cdot)) - f_0(\cdot,u(\cdot))|\|_{L^{q'_0(\cdot)}(\Omega)}\|a_0^{1/q_0(\cdot)}|v(\cdot)|\|_{L^{q_0(\cdot)}(\Omega)}.\end{aligned}$$

Since

$$\|a_0^{1/q_0(\cdot)}v\|_{L^{q_0(\cdot)}(\Omega)} = \|v\|_{L^{q_0(\cdot)}_{a_0(\cdot)}(\Omega)} \le C\|v\|_Y,$$

we have

$$\|J'_{F_0}(u_n) - J'_{F_0}(u)\|_{Y^*} \le 2C\|a_0^{-1/q_0(\cdot)}|f_0(\cdot,u_n(\cdot)) - f_0(\cdot,u(\cdot))|\|_{L^{q'_0(\cdot)}(\Omega)}.$$

We want to show that $\|J'_{F_0}(u_n) - J'_{F_0}(u)\|_{Y^*} \to 0$ as $n \to \infty$. By Proposition 2.1 (iv), it suffices to show that

$$\rho_{q'_0(\cdot)}\left(a_0^{-1/q_0(\cdot)}f_0(\cdot,u_n(\cdot)) - a_0^{-1/q_0(\cdot)}f_0(\cdot,u(\cdot))\right) \to 0 \text{ as } n \to \infty. \tag{4.8}$$

We can see that

$$\begin{aligned}&\rho_{q'_0(\cdot)}\left(a_0^{-1/q_0(\cdot)}f_0(\cdot,u_n(\cdot)) - a_0^{-1/q_0(\cdot)}f_0(\cdot,u(\cdot))\right)\\ &\qquad\qquad = \int_\Omega a_0(x)^{-q'_0(x)/q_0(x)}|f_0(x,u_n(x)) - f_0(x,u(x))|^{q'_0(x)}dx.\end{aligned}$$

Since $u_n \to u$ weakly in Y and the embedding $Y \hookrightarrow L^{q_0(\cdot)}_{a_0(\cdot)}(\Omega)$ is compact, we can see that $u_n \to u$ strongly in $L^{q_0(\cdot)}_{a_0(\cdot)}(\Omega)$. From [3, Theorem A.1], there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $g \in L^{q_0(\cdot)}(\Omega)$ such that $a_0(x)^{1/q_0(x)}u_{n'}(x) \to a_0(x)^{1/q_0(x)}u(x)$ a.e. $x \in \Omega$ and $|a_0(x)^{1/q_0(x)}u_{n'}(x)| \le g(x)$ for a.e. $x \in \Omega$. Since $a_0(x) > 0$ and f_0 is a Carathéodory function, $f_0(x,u_{n'}(x)) \to f_0(x,u(x))$ a.e. $x \in \Omega$. From (f_0) and $a_0(x) \ge 1$, we have

$$\begin{aligned}&a_0(x)^{-q'_0(x)/q_0(x)}|f_0(x,u_{n'}(x) - f_0(x,u(x))|^{q'_0(x)}\\ &\quad \le C_1a_0(x)^{-q'_0(x)/q_0(x)}(2 + a_0(x)|u_{n'}(x)|^{q_0(x)-1} + a_0(x)|u(x)|^{q_0(x)-1})^{q'_0(x)}\\ &\quad \le C_2(a_0(x)^{-q'_0(x)/q_0(x)} + a_0(x)^{q'_0(x)-q'_0(x)/q_0(x)}(|u_{n'}(x)|^{q_0(x)} + |u(x)|^{q_0(x)}))\\ &\quad \le C_2(1 + a_0(x)(|u_{n'}(x)|^{q_0(x)} + |u(x)|^{q_0(x)}))\\ &\quad \le C_2(1 + 2g(x)^{q_0(x)}) \text{ for some positive constant } C_2.\end{aligned}$$

The last term is an integrable function in Ω independent of n'. Thus by the Lebesgue dominated convergence theorem, we have (4.8), so $\|J'_{F_0}(u_n) - J'_{F_0}(u)\|_{Y^*} \to 0$ as $n \to \infty$. Similarly, we can show that J'_{K_0} is also weakly-strongly continuous in Y. In particular, J'_{F_0} and J'_{K_0} are compact, so J' is also compact.

End of the proof of Theorem 3.7

Ψ has strictly local minimum at 0 and $\Psi(0) = J(0) = 0$. Since $\Psi(u) \geq 0$ for any $u \in Y$ and $\Psi(u) = 0$ if and only if $u = 0$, Ψ has a strictly local (global) minimum. By the definition (4.4) and (3.4) of J, it is clear that $J(0) = 0$.

Therefore, all the hypotheses of Theorem 4.1 hold. We derive that α defined by (4.1) satisfies $\alpha = 0$.

Fix $\varepsilon > 0$. From (3.6) and (3.7), there exist ρ_1 and ρ_2 with $0 < \rho_1 < 1 < \rho_2$ such that

$$\begin{aligned}
F_0(x,t) &\leq \varepsilon a_0(x)|t|^{lp^+} \text{ for a.e. } x \in \Omega \text{ and } t \in [-\rho_1, \rho_1], && (4.9)\\
F_0(x,t) &\leq \varepsilon a_0(x)|t|^{lp^-} \text{ for a.e. } x \in \Omega \text{ and } t \in \mathbb{R} \setminus [-\rho_2, \rho_2], && (4.10)\\
G_0(x,t) &\leq \varepsilon b_0(x)|t|^{lp^+} \text{ for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and } t \in [-\rho_1, \rho_1], && (4.11)\\
G_0(x,t) &\leq \varepsilon b_0(x)|t|^{lp^-} \text{ for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and } t \in \mathbb{R} \setminus [-\rho_2, \rho_2]. && (4.12)
\end{aligned}$$

From (3.5), we can choose $s \in \mathbb{R}$ such that $lp^+ < s$ and

$$lp^+ < s < \min\left\{\frac{\alpha_0^- - 1}{\alpha_0^-}\frac{Np^-}{N-p^-}, \frac{\beta_0^- - 1}{\beta^-}\frac{(N-1)p^-}{N-p^-}\right\} \text{ if } p^- < N.$$

We note that

$$lp^+ < s < \frac{\alpha_0(x) - 1}{\alpha_0(x)} p^*(x) \text{ for all } x \in \overline{\Omega}$$

and

$$lp^+ < s < \frac{\beta_0(x) - 1}{\beta_0(x)} p^\partial(x) \text{ for all } x \in \overline{\Gamma_2}.$$

For $\rho_1 \leq |t| \leq \rho_2$, from (f_0) and (g_0) and the fact that $a_0(x) \geq 1$ for a.e. $x \in \Omega$ and $b_0(x) \geq 1$ for σ-a.e. $x \in \Gamma_2$, there exists a constant $C = C(\rho_1, \rho_2) > 0$ such that

$$\begin{aligned}
F_0(x,t) &\leq c_1(|t| + a_0(x)|t|^{q_0(x)} \leq Ca_0(x)|t|^s \text{ for a.e. } x \in \Omega, && (4.13)\\
G_0(x,t) &\leq d_1(|t| + b_0(x)|t|^{r_0(x)} \leq Cb_0(x)|t|^s \text{ for } \sigma\text{-a.e. } x \in \Gamma_2. && (4.14)
\end{aligned}$$

Thus from (4.9)-(4.14), we have

$$F_0(x,t) \leq \varepsilon a_0(x)|t|^{lp^+} + Ca_0(x)|t|^s \text{ for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}, \quad (4.15)$$

$$G_0(x,t) \leq \varepsilon b_0(x)|t|^{lp^+} + Cb_0(x)|t|^s \text{ for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and all } t \in \mathbb{R}. \quad (4.16)$$

Since $Y \hookrightarrow L^{lp^+}_{a_0(\cdot)}(\Omega), L^s_{a_0(\cdot)}(\Omega), L^{lp^+}_{b_0(\cdot)}(\Gamma_2), L^s_{b_0(\cdot)}(\Gamma_2)$, we have

$$\|u\|_{L^{lp^+}_{a_0(\cdot)}(\Omega)} + \|u\|_{L^s_{a_0(\cdot)}(\Omega)} + \|u\|_{L^{lp^+}_{b_0(\cdot)}(\Gamma_2)} + \|u\|_{L^s_{b_0(\cdot)}(\Gamma_2)} \le C_1\|u\|_Y \quad (4.17)$$

for all $u \in Y$. By (4.15), (4.16) and (4.17),

$$J(u) = \int_\Omega F_0(x,u(x))dx + \int_{\Gamma_2} G_0(x,u(x))d\sigma \le C_1\varepsilon\|u\|_Y^{lp^+} + CC_1\|u\|_Y^s. \quad (4.18)$$

Moreover, it follows from (A.3) that

$$\Psi(u) = \widehat{M}\left(\int_\Omega A(x,\nabla u(x))dx\right) \ge \frac{m_0}{l}\left(\frac{k_0}{p^+}\right)^l \|u\|_Y^{lp^+} \text{ for } u \in Y \text{ with } \|u\|_Y < 1.$$

Since $s > lp^+$, we have

$$\limsup_{u\to 0}\frac{J(u)}{\Phi(u)} \le \frac{l}{m_0}\left(\frac{p^+}{k_0}\right)^l C_1\varepsilon. \quad (4.19)$$

On the other hand, for $|t| > \rho_2$, we have

$$F_0(x,t) \le C\rho_2^{q^+}a_0(x) + \varepsilon a_0(x)|t|^{lp^-} \text{ for a.e.} x \in \Omega,$$

and

$$G_0(x,t) \le C\rho_2^{r^+}b_0(x) + \varepsilon b_0(x)|t|^{lp^-} \text{ for } \sigma\text{-a.e.} x \in \Gamma_2.$$

For $|t| \le \rho_2$, $F_0(x,t) \le C\rho_2^{q_0^+}a_0(x)$ a.e. $x \in \Omega$ and $G_0(x,t) \le C\rho_2^{r_0^+}b_0(x)$ for σ-a.e. $x \in \Gamma_2$. Since $Y \hookrightarrow L^{lp^-}_{a_0(\cdot)}(\Omega), L^{lp^-}_{b_0(\cdot)}(\Gamma_2)$, we can see that

$$\begin{aligned}
J(u) &= \int_{\{x\in\Omega;|u(x)|\le\rho_2\}} F_0(x,u(x))dx + \int_{\{x\in\Omega;|u(x)|>\rho_2\}} F_0(x,u(x))dx \\
&\quad + \int_{\{x\in\Gamma_2;|u(x)|\le\rho_2\}} G_0(x,u(x))d\sigma + \int_{\{x\in\Gamma_2;|u(x)|>\rho_2\}} G_0(x,u(x))d\sigma \\
&\le C_1\int_\Omega a_0(x)dx + \varepsilon\int_\Omega a_0(x)|u(x)|^{lp^-}dx \\
&\quad + C_1\int_{\Gamma_2} b_0(x)d\sigma + \varepsilon\int_{\Gamma_2} b_0(x)|u(x)|^{lp^-}d\sigma \\
&\le C_1\|a_0\|_{L^1(\Omega)} + C_1\|b_0\|_{L^1(\Gamma_2)} + 2\varepsilon C_1\|u\|_Y^{lp^-}.
\end{aligned}$$

If $\|u\|_Y > 1$, then $\Psi(u) \ge \frac{n_0}{l}\left(\frac{k_0}{p^-}\right)^l \|u\|_Y^{lp-}$, so

$$\limsup_{\|u\|_Y\to\infty}\frac{J(u)}{\Psi(u)} \le 2\frac{l}{m_0}\left(\frac{p^-}{k_0}\right)^l C_1\varepsilon. \quad (4.20)$$

Since $\varepsilon > 0$ is arbitrary, it follows from (4.19) and (4.20) that

$$\max\left\{\limsup_{u\to 0}\frac{J(u)}{\Psi(u)}, \limsup_{\|u\|_Y\to\infty}\frac{J(u)}{\Psi(u)}\right\} \leq 0.$$

Thus we have $\alpha = 0$ in Theorem 4.1. By (3.4) and Remark 3.3, if we put $\theta = 1/\beta$, then the conclusion of Theorem 3.7 holds.

Corollary 4.6. Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ satisfying (1.2) and let $p \in C_+(\overline{\Omega})$. Assume that (A.1)-(A.5) hold, functions f_0 and g_0 satisfy (f_0) with $q_0^+ < lp^-$ and (g_0) with $r_0^+ < lp^-$, respectively, and (3.5) is satisfied. Moreover, assume that

$$\max\left\{\limsup_{t\to+0} ess\ sup_{x\in\Omega}\frac{f_0(x,t)}{a_0(x)|t|^{lp^+-1}}, \limsup_{t\to+0} ess\ sup_{x\in\Gamma_2}\frac{g_0(x,t)}{b_0(x)|t|^{lp^--1}}\right\} \leq 0, \tag{4.21}$$

$$\max\left\{\liminf_{t\to-0} ess\ inf_{x\in\Omega}\frac{f_0(x,t)}{a_0(x)|t|^{lp^+-1}}, \liminf_{t\to-0} ess\ inf_{x\in\Gamma_2}\frac{g_0(x,t)}{b_0(x)|t|^{lp^+-1}}\right\} \geq 0, \tag{4.22}$$

$$\max\left\{\limsup_{t\to+\infty} ess\ sup_{x\in\Omega}\frac{f_0(x,t)}{a_0(x)|t|^{lp^--1}}, \limsup_{t\to+\infty} ess\ sup_{x\in\Gamma_2}\frac{g_0(x,t)}{b_0(x)|t|^{lp^--1}}\right\} \leq 0, \tag{4.23}$$

$$\max\left\{\liminf_{t\to-\infty} ess\ inf_{x\in\Omega}\frac{f_0(x,t)}{a_0(x)|t|^{lp^--1}}, \liminf_{t\to-\infty} ess\ inf_{x\in\Gamma_2}\frac{g_0(x,t)}{b_0(x)|t|^{lp^--1}}\right\} \geq 0. \tag{4.24}$$

and there exists $\delta > 0$ such that $f_0(x,t) > 0$ for a.e. $x \in \Omega$ and $0 \leq t \leq \delta$.

Then the conclusion of Theorem 3.7 holds.

Proof. For any $\varepsilon > 0$, there exists $0 < \rho_1 < 1$ such that from (4.21)-(4.24),

$$\begin{aligned}
f_0(x,t) &\leq \varepsilon a_0(x)|t|^{lp^+-1} \text{ for a.e. } x \in \Omega \text{ and } t \in [0,\rho_1],\\
g_0(x,t) &\leq \varepsilon b_0(x)|t|^{lp^+-1} \text{ for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and } t \in [0,\rho_1],\\
f_0(x,t) &\geq -\varepsilon a_0(x)|t|^{lp^+-1} \text{ for a.e. } x \in \Omega \text{ and } t \in [-\rho_1,0],\\
g_0(x,t) &\geq -\varepsilon b_0(x)|t|^{lp^+-1} \text{ for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and } t \in [-\rho_1,0].
\end{aligned}$$

Then

$$F_0(x,t) = \int_0^t f_0(x,s)ds \leq \varepsilon a_0(x)|t|^{lp^+} \text{ for a.e. } x \in \Omega \text{ and } t \in [0,\rho_1],$$

and

$$F_0(x,t) = -\int_t^0 f_0(x,s)ds \le \int_{-|t|}^0 \varepsilon a_0(x)|s|^{lp^+-1}ds \le \varepsilon a_0(x)|t|^{lp^+}$$

for a.e. $x \in \Omega$ and $t \in [-\rho_1, 0]$. Hence

$$F_0(x,t) \le \varepsilon a_0(x)|t|^{lp^+} \text{ for a.e. } x \in \Omega \text{ and } t \in [-\rho_1, \rho_1]. \tag{4.25}$$

Similarly we have

$$G_0(x,t) \le \varepsilon b_0(x)|t|^{lp^+} \text{ for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and } t \in [-\rho_1, \rho_1]. \tag{4.26}$$

Therefore, we have

$$\max\left\{\limsup_{t\to 0} ess\ sup_{x\in\Omega}\frac{F_0(x,t)}{a_0(x)|t|^{lp^+}}, \limsup_{t\to 0} ess\ sup_{x\in\Gamma_2}\frac{G_0(x,t)}{b_0(x)|t|^{lp^+}}\right\} \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have (3.6).

For given $\varepsilon > 0$, we choose $\rho_2 > 1$ so that $2\max\{c_0\rho_2^{q_0^+-lp^-}, d_0\rho_2^{r_0^+-lp^-}\} < \varepsilon$. For $t > \rho_2$, from (4.23),

$$\begin{aligned} f_0(x,t) &\le \varepsilon a_0(x)|t|^{lp^--1} \text{ for a.e. } x \in \Omega \text{ and } t \in [\rho_2, \infty), \\ g_0(x,t) &\le \varepsilon b_0(x)|t|^{lp^--1} \text{ for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and } t \in [\rho_2, \infty). \end{aligned}$$

For $0 \le t \le \rho_2$, from (f_0) with $q_0^+ < lp^-$, we have

$$f_0(x,t) \le |f_0(x,t)| \le c_0(1+a_0(x)|t|^{q_0(x)-1}) \le c_0(1+a_0(x)\rho_2^{q_0^+-1}) \text{ for a.e. } x \in \Omega.$$

Since $a_0(x) \ge 1$ a.e. $x \in \Omega$ and $q_0^+ < lp^-$,

$$\begin{aligned} F_0(x,t) = \int_0^t f_0(x,s)ds &\le c_0(\rho_2 + a_0(x)\rho_2^{q_0^+}) \le 2c_0a_0(x)\rho_2^{q_0^+} \\ &= 2c_0a_0(x)\rho_2^{lp^-}\rho_2^{q_0^+-lp^-} \le \varepsilon a_0(x)\rho_2^{lp^-} \text{ for a.e. } x \in \Omega. \end{aligned}$$

Hence for $t \ge \rho_2$,

$$F_0(x,t) = \int_0^{\rho_2} f_0(x,s)ds + \int_{\rho_2}^t f_0(x,s)dx \le \varepsilon a_0(x)|t|^{lp^-}$$

for a.e. $x \in \Omega$ and $t \in [\rho_2, \infty)$.

Similarly for $t \ge \rho_2$, $G_0(x,t) \le 2\varepsilon b_0(x)|t|^{lp^-}$ for a.e. $x \in \Gamma_2$.

For $t \le -\rho_2$, from (4.24),

$$\begin{aligned} f_0(x,t) &\ge -\varepsilon|t|^{lp^-} \text{ for a.e. } x \in \Omega \text{ and } t \in (-\infty, -\rho_2], \\ g_0(x,t) &\ge -\varepsilon|t|^{lp^-} \text{ for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and } t \in (-\infty, -\rho_2]. \end{aligned}$$

Hence for $-\rho_2 \le t \le 0$, we have

$$-f_0(x,t) \le |f_0(x,t)| \le c_0(1 + a_0(x)\rho_2^{q_0^+ - 1}) \text{ for a.e. } x \in \Omega.$$

Thereby,

$$F_0(x,t) = \int_0^t f_0(x,s)ds = \int_{-|t|}^0 (-f_0(x,s))ds \le \int_{-|t|}^0 c_0(1 + a_0(x)\rho_2^{q_0^+ - 1})ds$$
$$\le c_0 a_0(x)\rho_2^{q_0^+} \le \varepsilon a_0(x)\rho_2^{lp^-} \text{ a.e. } x \in \Omega.$$

For $t \le -\rho_2$, we have

$$F_0(x,t) = \int_{-\rho_2}^0 (-f_0(x,s))ds + \int_t^{-\rho_2} (-f_0(x,s))ds \le 2\varepsilon a_0(x)|t|^{lp^-} \text{ a.e. } x \in \Omega.$$

Similarly for $t \le -\rho_2$, $G_0(x,t) \le 2\varepsilon b_0(x)|t|^{lp^-}$ for σ-a.e. $x \in \Gamma_2$. Thus for $|t| \ge \rho_2$, $F_0(x,t) \le 2\varepsilon a_0(x)|t|^{lp^-}$ for a.e. $x \in \Omega$ and $G_0(x,t) \le 2\varepsilon b_0(x)|t|^{lp^-}$ for σ-a.e. $x \in \Gamma_2$. Therefore, we have

$$\max\left\{\limsup_{|t|\to\infty} ess\, sup_{x\in\Omega} \frac{F_0(x,t)}{a_0(x)|t|^{lp^-}}, \limsup_{|t|\to\infty} ess\, sup_{x\in\Gamma_2} \frac{G_0(x,t)}{b_0(x)|t|^{lp^-}}\right\} \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have (3.7).

Moreover, since $f_0(x,t) > 0$ for a.e. $x \in \Omega$ and $0 \le t \le \delta$, we have $F_0(x,t) > 0$ for a.e. $x \in \Omega$ and $0 \le t \le \delta$. Thus all the hypotheses of Theorem 3.7 hold. This competes the proof. □

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